OPTIMIZATION AND TROPICAL GEOMETRY: EXERCISES AND PROBLEMS 2

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Exercise 1. Draw the tropical hypersurface defined by the homogeneous tropical polynomial

$$(4 \odot X_1^3) \oplus (1 \odot X_1 X_2 X_3) \oplus (4 \odot X_2^3) \oplus (1 \odot X_2^2 X_3) \oplus (1 \odot X_2 X_3^2) \oplus (6 \odot X_3^3) \ .$$

How does the dual subdivision of the Newton polytope look like?

Exercise 2. Give an example of a point configuration and a subdivision which is not regular.

Exercise 3. What are the combinatorially distinct types of tropical conics in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$? By the way, what is a good definition for "combinatorially distinct" in this context?

Exercise 4. Let $A \in \mathbb{R}^{2\times 2}$ be a square matrix with two rows and columns. For which vectors $b \in \mathbb{R}^2$ does the tropical linear system of equations $A \odot x = b$ have a solution?

Exercise 5. Show that the field of complex Puiseux series $\mathbb{C}\{\{t\}\}$ is not complete with respect to the product topology of $\mathbb{R}^{\mathbb{N}}$. Can you describe the completion?

Exercise 6. Let f and g be polynomials in $\mathbb{C}\{\{t\}\}[x_1^{\pm}, x_2^{\pm}, \ldots, x_d^{\pm}]$ such that $g \in \langle f \rangle$. Show that their tropical hypersurfaces satisfy $\mathcal{T}(g) \supseteq \mathcal{T}(f)$. When do have equality?

Exercise 7. Give a formula for the square root(s) of an arbitrary ordinary Puiseux series $\sum_{k=m} a_k \cdot t^{k/N}$. Use this to show that the field $\mathbb{R}\{\{t\}\}$ of ordinary real Puiseux series is real closed.

Exercise 8. Describe the tropicalizations of a few classical plane algebraic curves. Here are two examples.

(1) The bivariate nonhomogeneous polynomial

$$x^3 - 3x^2 - y^2$$

gives the *Tschirnhausen cubic* as an affine curve.

(2) Picking suitable coefficients $b \neq c$ in the homogeneous polynomial

$$x(c^2y^2 - b^2z^2) + y(c^2z^2 - b^2x^2) + z(c^2x^2 - b^2y^2)$$

gives the *Thomson cubic* in the projective plane. What happens for various nonconstant choices of b and c?

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Problem 9. Construct and study tropical K3 surfaces via the methods from [3]. For this you probably want to use `polymake` [4]. Compare your observations with [1].

Problem 10. The "bottleneck degree" of an algebraic variety is the topic of [2]. Define and study the *bottleneck degree* of a plane tropical curve. What about higher-dimensional tropical hypersurfaces? Arbitrary tropical varieties?

References

1. Gabriele Balletti, Marta Panizzut, and Bernd Sturmfels, *K3 polytopes and their quartic surfaces*, 2018, Preprint `arXiv:1806.02236`.
2. Sandra Di Rocco, David Eklund, and Madeleine Weinstein, *The bottleneck degree of algebraic varieties*, 2019, Preprint `arXiv:1904.04502`.
3. Simion Filip, *Tropical dynamics of area-preserving maps*, 2019, Preprint `arXiv:1903.00794`.
4. Simon Hampe and Michael Joswig, *Tropical computations in* `polymake`, Algorithmic and experimental methods in algebra, geometry, and number theory (Gebhard Böckle, Wolfram Decker, and Gunter Malle, eds.), Springer, Cham, 2017, pp. 361–385. MR 3792732

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