Exercise 1. Draw the tropical hypersurface defined by the homogeneous tropical polynomial
\[(4 \odot X_3) \oplus (1 \odot X_1X_2X_3) \oplus (4 \odot X_2^3) \oplus (1 \odot X_2X_3^2) \oplus (1 \odot X_2^2X_3) \oplus (6 \odot X_3^3) \].

How does the dual subdivision of the Newton polytope look like?

Exercise 2. Give an example of a point configuration and a subdivision which is not regular.

Exercise 3. What are the combinatorially distinct types of tropical conics in \(\mathbb{R}^3/\mathbb{R}1\)? By the way, what is a good definition for “combinatorially distinct” in this context?

Exercise 4. Let \(A \in \mathbb{R}^{2\times2}\) be a square matrix with two rows and columns. For which vectors \(b \in \mathbb{R}^2\) does the tropical linear system of equations \(A \odot x = b\) have a solution?

Exercise 5. Show that the field of complex Puiseux series \(\mathbb{C}\{\{t\}\}\) is not complete with respect to the product topology of \(\mathbb{R}^N\). Can you describe the completion?

Exercise 6. Let \(f\) and \(g\) be polynomials in \(\mathbb{C}\{\{t\}\}\[x_1^\pm, x_2^\pm, \ldots, x_d^\pm]\) such that \(g \in \langle f \rangle\). Show that their tropical hypersurfaces satisfy \(\mathcal{T}(g) \supseteq \mathcal{T}(f)\). When do have equality?

Exercise 7. Give a formula for the square root(s) of an arbitrary ordinary Puiseux series \(\sum_{k=m} a_k \cdot t^{k/N}\). Use this to show that the field \(\mathbb{R}\{\{t\}\}\) of ordinary real Puiseux series is real closed.

Exercise 8. Describe the tropicalizations of a few classical plane algebraic curves. Here are two examples.

1. The bivariate nonhomogeneous polynomial
\[x^3 - 3x^2 - y^2\]
gives the Tschirnhausen cubic as an affine curve.

2. Picking suitable coefficients \(b \neq c\) in the homogeneous polynomial
\[x(c^2y^2 - b^2z^2) + y(c^2x^2 - b^2x^2) + z(c^2z^2 - b^2y^2)\]
gives the Thomson cubic in the projective plane. What happens for various nonconstant choices of \(b\) and \(c\)?

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Problem 9. Construct and study tropical K3 surfaces via the methods from [3]. For this you probably want to use polymake [4]. Compare your observations with [1].

Problem 10. The “bottleneck degree” of an algebraic variety is the topic of [2]. Define and study the bottleneck degree of a plane tropical curve. What about higher-dimensional tropical hypersurfaces? Arbitrary tropical varieties?

References

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