# OPTIMIZATION AND TROPICAL GEOMETRY: EXERCISES AND PROBLEMS 1

MICHAEL JOSWIG

**Exercise 1.** Let $F : \mathbb{R}^d \to \mathbb{R}$ be a tropical polynomial. Show that for any fixed vector $y \in \mathbb{R}^d$ the map

$$F_y : \mathbb{R}^d \to \mathbb{R}\,,\ x \mapsto F(x+y)$$

is a tropical polynomial map, too. How are the tropical hypersurfaces $\mathcal{T}(F)$ and $\mathcal{T}(F_y)$ related?

**Exercise 2.** Consider the directed graph $\Gamma$ on four nodes with the weighted adjacency matrix

$$D \;=\; \begin{pmatrix} 0 & \infty & \infty & 1 \\ 1 & 0 & \infty & \infty \\ y & 1 & 0 & \infty \\ \infty & x & 1 & 0 \end{pmatrix} \quad ,$$

whose coefficients lie in the semiring $\mathbb{T}[x, y]$ of bivariate tropical polynomials. Find all labeled types of shortest-path trees directed to the nodes 1 and 2 (i.e., corresponding to the first two columns of $D$) and their dependence on $x$ and $y$.

**Exercise 3.** Let $k \geq 1$ be a fixed integer. Construct a directed adjacency matrix $D \in \mathbb{T}[X_1, \ldots, X_k]^{n\times n}$, with separated variables, such that the number of labeled shortest-path trees (directed to some node) is maximized.

The *tropical moment map* of degree $n-1$ is defined as

$$(1)\quad m_{n-1} \,:\, \mathbb{R} \to \mathbb{R}^n \,:\, t \mapsto (t^{\odot 0}, t^{\odot 1}, \ldots, t^{\odot (d-1)}) = (0, t, 2t, \ldots, (n-1)t) \ .$$

**Exercise 4** (Tropical Vandermonde determinant)**.** Pick $n$ real numbers $t_1 < t_2 < \cdots < t_n$ in an ascending ordering. Let $A$ be the $n\times n$-matrix whose rows are formed by the vectors $m_{n-1}(t_1), m_{n-1}(t_2), \ldots, m_{n-1}(t_n)$. Compute $\operatorname{tdet}(A)$, and decide if $A$ is tropically regular or singular.

The latter notions are naturally derived from the notion of vanishing of a tropical polynomial. That is, for a *tropically singular* matrix the minimum in the Leibniz expression is attained at least twice. Otherwise that matrix is called *tropically regular*.

***

*Date*: 15 April 2019.

**Problem 5.** Employ the tropical geometry techniques from [2] to study parameterized flow problems. Also consult [1]. Specifically,

(a) Find and prove a parameterized version of the max-flow min-cut theorem.
(b) What are the consequences does this have for parameterized minimum or maximum matching problems?
(c) What can you say about the optimal complexity of parameterized flow algorithms for a fixed number of parameters?
(d) What about $b$-transshipment?

## References


1. Giorgio Gallo, Michael D. Grigoriadis, and Robert E. Tarjan, *A fast parametric maximum flow algorithm and applications*, SIAM J. Comput. **18** (1989), no. 1, 30–55. MR 978165
2. Michael Joswig and Benjamin Schröter, *The tropical geometry of shortest paths*, 2019, Preprint `arXiv:1904.01082`.


(Michael Joswig) Institut für Mathematik, TU Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany

*Email address*: `joswig@math.tu-berlin.de`