Optimization and Tropical Geometry:
4. Product-Mix Auctions

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Product-mix auctions

- $m$ bidders ("agents") compete for combinations of several goods
- $n$ types of indivisible goods; good bundle = point in $\mathbb{Z}^n$
  - buyers and sellers play the same role
- valuation $u^j : A^j \to \mathbb{R}$ for agent $j \in [m]$, where $A^j \subseteq \mathbb{Z}^n$

The Minkowski sum

$$ A = \sum_{j=1}^{m} A^j = \left\{ \sum_{j=1}^{m} a^j \mid a^j \in A^j \text{ for } j \in [m] \right\} $$

comprises all combinations of good bundles for these agents.
Demand sets and aggregate demand

Now, the auctioneer fixes a price \( p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \), and the agent wants to maximize her profit.

The corresponding bundles form the demand set

\[
D_{uj}(p) := \arg \max_{a \in A^i} \{ u_j(a) - p \cdot a \}
\]

The aggregate valuation function \( U : A \rightarrow \mathbb{R}^n \) is the maximum total valuation taken over all ways to partition each bundle \( a \in A \):

\[
U(a) := \max \left\{ \sum_{j=1}^{m} u_j(a_j) \left| a_j \in A_j \text{ and } \sum a_j = a \right. \right\}
\]

The aggregate demand at \( p \) is

\[
D_U(p) := \arg \max_{a \in A} \{ U(a) - p \cdot a \}
\]

Then \( D_U(p) = \sum D_{uj}(p) \in \sum A^i = A \).
Competitive equilibrium

Let $a \in \mathbb{Z}^n$ be a bundle.

**Definition**
We say that competitive equilibrium exists at $a$ if there is a price $p \in \mathbb{R}^n$ such that $a \in D_U(p)$.

- in this case the price $p$ is chosen such that all agents simultaneously receive a bundle which maximizes their profit
Tropical hypersurfaces and their union

The valuation function of agent $j$ defines the $n$-variate max-tropical polynomial

$$F_j(X) := \max_{a \in A^j} u^j(a)X^a$$

The tropical hypersurface $\mathcal{T}(F_j)$ is the set of prices where the agent is indifferent with at least two bundles.

The aggregate valuation corresponds to the product

$$F(X) := F_1(X) \odot F_2(X) \odot \cdots \odot F_m(X)$$

and the union

$$\mathcal{T}(F) = \mathcal{T}(F_1) \cup \mathcal{T}(F_2) \cup \cdots \cup \mathcal{T}(F_m)$$
Example

We consider $m = 2$ agents and $n = 2$ goods.

$$F_1(X, Y) = \max(0, 3 + Y, 5 + 2Y, 9 + X + 2Y)$$
$$F_2(X, Y) = \max(0, 1 + X, 1 + Y)$$
Proposition

The diagram

\[
\begin{array}{ccc}
V(f) & \rightarrow & V(f \cdot g) \leftrightarrow V(g) \\
\downarrow \text{ord} & & \downarrow \text{ord} & \downarrow \text{ord} \\
T(F) & \rightarrow & T(F \circ G) \leftrightarrow T(G) \\
\downarrow \text{id} \times F & & \downarrow \text{id} \times (F \circ G) & \downarrow \text{id} \times G \\
\partial D(F) & \circ G & \rightarrow & \partial D(F \circ G) \leftarrow \partial D(G) \\
\end{array}
\]

commutes. The map \( \circ G \) sends a point \((w, s) \in \mathbb{R}^{d+1}\) to \((w, s + G(w))\), and \( \circ F \) is similarly defined.

The unmarked horizontal arrows are embeddings of subsets.
When does competitive equilibrium exist?

Simple cases

1. if all valuations $u^j : A^j \to \mathbb{R}$ are constant:
   - competitive equilibrium exists at $a \in \text{conv}(A) \cap \mathbb{Z}^n$ if and only if $a \in \sum A^j = A$

2. if $n = 1$ and the valuation is not constant: checking if competitive equilibrium exists at a given $a \in A$ equivalent to SUBSET-SUM, which is $\text{NP}$-complete

Exercise

For $m = n = 2$ there are point sets $A^1, A^2 \subset \mathbb{Z}^2$ such that no competitive equilibrium exists, no matter which utility functions are used.
Lemma

Let $u^1, u^2, \ldots, u^m$ be valuation functions of $m$ agents on supports $A^1, A^2, \ldots, A^m \subset \mathbb{Z}^n$.

Further let $A = \sum A^i$ and $U : A \to \mathbb{R}$ be the aggregate valuation.

Then

1. aggregate tropical polynomial $F(X) = F_1(X) \circ F_2(X) \circ \cdots \circ F_m(X)$, where $F_j(X)$ tropical polynomial associated with $u^j$;
2. $\mathcal{T}(F) = \bigcup \mathcal{T}(F^j)$;
3. $D_U(p) = \sum D_{u^i}(p)$ for any price $p \in \mathbb{R}^n$;
4. competitive equilibrium exists at $a \in \mathbb{Z}^n$ if and only $(a, p \cdot a)$ lies in the boundary of the dome of $F$. 
The unimodularity theorem

- nonzero vector $d \in \mathbb{Z}^n$ primitive $\iff \gcd(d_1, \ldots, d_n) = 1$
- set $D \subset \mathbb{Z}$ unimodular $\iff$ for each $\mathbb{R}$-basis in $D$ the $\mathbb{Z}$-linear span is $\mathbb{Z}^n$

Let $D \subset \mathbb{Z}^n$ be primitive and $A^j \subset \mathbb{Z}^n$ arbitrary.

Definition

A valuation $u_j : A^j \to \mathbb{R}$ is of demand type $D$ if every edge of the subdivision dual to the tropical hypersurface induced by $u$ is parallel to some vector in $D$.

Theorem (Baldwin & Klemperer 2012+; Danilov, Koshevey & Murota 2001; Tran & Yu 2015+)

Every collection of concave valuation functions $\{u^j : j \in [m]\}$ of demand type $D$ has competitive equilibrium if and only if $D$ is unimodular.
References


