Optimization and Tropical Geometry:
4. Product-Mix Auctions

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Product-mix auctions

- m bidders ("agents") compete for combinations of several goods
- n types of indivisible goods; good bundle = point in $\mathbb{Z}^n$
  - buyers and sellers play the same role
- valuation $u^j : A^j \to \mathbb{R}$ for agent $j \in [m]$, where $A^j \subseteq \mathbb{Z}^n$

The Minkowski sum

$$A = \sum_{j=1}^{m} A^j = \left\{ \sum_{j=1}^{m} a^j \;\middle|\; a^j \in A^j \text{ for } j \in [m] \right\}$$

comprises all combinations of good bundles for these agents.

Demand sets and aggregate demand

Now, the auctioneer fixes a price $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, and the agent wants to maximize her profit.

The corresponding bundles form the demand set

$$D_{u^j}(p) \;:=\; \arg\max_{a \in A^j} \big\{ u_j(a) - p \cdot a \big\}$$

The aggregate valuation function $U : A \to \mathbb{R}^n$ is the maximum total valuation taken over all ways to partition each bundle $a \in A$:

$$U(a) \;:=\; \max \left\{ \sum_{j=1}^{m} u_j(a_j) \;\middle|\; a_j \in A_j \text{ and } \sum a_j = a \right\}$$

The aggregate demand at p is

$$D_U(p) \;:=\; \arg\max_{a \in A} \big\{ U(a) - p \cdot a \big\}$$

Then $D_U(p) = \sum D_{u^j}(p) \in \sum A^j = A$.

Competitive equilibrium

Let $a \in \mathbb{Z}^n$ be a bundle.

Definition

We say that competitive equilibrium exists at a if there is a price $p \in \mathbb{R}^n$ such that $a \in D_U(p)$.

- in this case the price p is chosen such that all agents simultaneously receive a bundle which maximizes their profit

Tropical hypersurfaces and their union

The valuation function of agent j defines the n-variate max-tropical polynomial

$$F_j(X) \;:=\; \max_{a \in A^j} u^j(a) X^a$$

The tropical hypersurface $\mathcal{T}(F_j)$ is the set of prices where the agent is indifferent with at least two bundles.

The aggregate valuation corresponds to the product

$$F(X) \;:=\; F_1(X) \odot F_2(X) \odot \cdots \odot F_m(X)$$

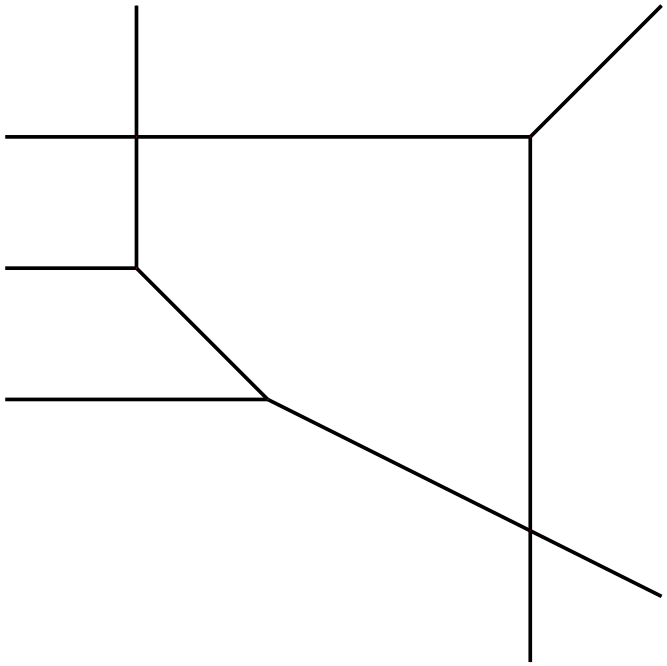
and the union

$$\mathcal{T}(F) = \mathcal{T}(F_1) \cup \mathcal{T}(F_2) \cup \cdots \cup \mathcal{T}(F_m)$$

Example

We consider $m = 2$ agents and $n = 2$ goods.

$$\begin{aligned} F_1(X, Y) &= \max(0, 3 + Y, 5 + 2Y, 9 + X + 2Y) \\ F_2(X, Y) &= \max(0, 1 + X, 1 + Y) \end{aligned}$$



$\mathcal{T}(F_1 \odot F_2)$

$\uparrow$ `polymake` [ipynb]

Proposition

The diagram

$$
\begin{tikzcd}
V(f) \arrow[r] \arrow[d, "\mathrm{ord}"] & V(f \cdot g) \arrow[d, "\mathrm{ord}"] & V(g) \arrow[l] \arrow[d, "\mathrm{ord}"] \\
\mathcal{T}(F) \arrow[r] \arrow[d, "\mathrm{id} \times F"] & \mathcal{T}(F \odot G) \arrow[d, "\mathrm{id} \times (F \odot G)"] & \mathcal{T}(G) \arrow[l] \arrow[d, "\mathrm{id} \times G"] \\
\partial\mathcal{D}(F) \arrow[r, "\odot G"] & \partial\mathcal{D}(F \odot G) & \partial\mathcal{D}(G) \arrow[l, "\odot F"']
\end{tikzcd}
$$

commutes. The map $\odot G$ sends a point $(w, s) \in \mathbb{R}^{d+1}$ to $(w, s + G(w))$, and $\odot F$ is similarly defined.

The unmarked horizontal arrows are embeddings of subsets.

When does competitive equilibrium exist?

Simple cases

1. if all valuations $u^j : A^j \to \mathbb{R}$ are constant:
   - ▷ competitive equilibrium exists at $a \in \mathrm{conv}(A) \cap \mathbb{Z}^n$ if and only if $a \in \sum A^j = A$
2. if $n = 1$ and the valuation is not constant: checking if competitive equilibrium exists at a given $a \in A$ equivalent to SUBSET-SUM, which is NP-complete

Exercise

For $m = n = 2$ there are point sets $A^1, A^2 \subset \mathbb{Z}^2$ such that no competitive equilibrium exists, no matter which utility functions are used.

Lemma

Let $u^1, u^2, \ldots, u^m$ be valuation functions of m agents on supports $A^1, A^2, \ldots, A^m \subset \mathbb{Z}^n$.

Further let $A = \sum A^j$ and $U : A \to \mathbb{R}$ be the aggregate valuation.

Then

1. aggregate tropical polynomial $F(X) = F_1(X) \odot F_2(X) \odot \cdots \odot F_m(X)$, where $F_j(X)$ tropical polynomial associated with u^j;
2. $\mathcal{T}(F) = \bigcup \mathcal{T}(F^j)$;
3. $D_U(p) = \sum D_{u^j}(p)$ for any price $p \in \mathbb{R}^n$;
4. competitive equilibrium exists at $a \in \mathbb{Z}^n$ if and only $(a, p \cdot a)$ lies in the boundary of the dome of F.

The unimodularity theorem

- nonzero vector $d \in \mathbb{Z}^n$ primitive $:\iff \gcd(d_1, \ldots, d_n) = 1$
- set $D \subset \mathbb{Z}$ unimodular
  $:\iff$ for each $\mathbb{R}$-basis in D the $\mathbb{Z}$-linear span is $\mathbb{Z}^n$

Let $D \subset \mathbb{Z}^n$ be primitive and $A^j \subset \mathbb{Z}^n$ arbitrary.

Definition

A valuation $u_j : A^j \to \mathbb{R}$ is of demand type D if every edge of the subdivision dual to the the tropical hypersurface induced by u is parallel to some vector in D.

Theorem (Baldwin & Klemperer 2012+; Danilov, Koshevey & Murota 2001; Tran & Yu 2015+)

Every collection of concave valuation functions $\{u^j : j \in [m]\}$ of demand type D has competitive equilibrium if and only if D is unimodular.

References

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- Vladimir Danilov, Gleb Koshevoy, and Kazuo Murota, Discrete convexity and equilibria in economies with indivisible goods and money, Math. Social Sci. **41** (2001), no. 3, 251–273. MR 1818477
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