Optimization and Tropical Geometry: **3. Tropical Linear Programming, MEAN-PAYOFF and Semi-Algebraic Sets**

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Tropical linear programming

An ordinary linear program is an optimization problem like

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{s.t.} & A x \geq b \\ & x \in \mathbb{R}^n \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Definition

A tropical linear program $\mathrm{LP}(A, b, c)$ is an optimization problem like

$$\begin{array}{ll} \text{minimize} & c^\top \odot x \\ \text{s.t.} & A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^- \\ & x \in \mathbb{T}^n \end{array}$$

where $A^\pm \in \mathbb{T}^{m \times n}$, $b^\pm \in \mathbb{T}^m$, $c \in \mathbb{T}^n$.

Min-max optimization over tropical polyhedra

here: $\oplus = \max$

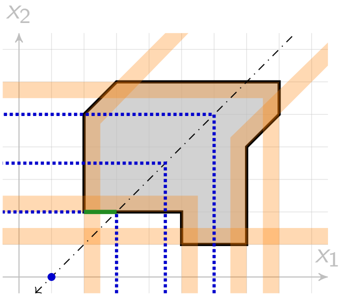
- feasible set defined by

$$A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^-$$

  is a tropical polyhedron; denoted $\mathcal{P}(A, b)$
- each defining inequality corresponds to a tropical half-space
- level sets have apices, located on the line $(-c) + \mathbb{R}\mathbf{1}$
- set of optimal solutions forms tropical polyhedron, too

$$\text{minimize } \max(-1 + x_1, x_2)$$

$$\text{s.t.} \begin{cases} \max(x_1 - 5, x_2 - 2) \geq 0 \\ 0 \geq \max(x_1 - 8, x_2 - 6) \\ x_1 - 2 \geq \max(x_2 - 5, 0) \\ \max(x_2 - 4, 0) \geq x_1 - 7 \\ x_2 \geq 1 \end{cases}$$



Ordered fields and semirings

here: $\oplus = \min$

A field $\mathbb{F}$ is ordered if exists total ordering $\leq$ on $\mathbb{F}$ such that for $a, b, c \in \mathbb{F}$:

(i) if $a \leq b$ then $a + c \leq b + c$,

(ii) if $0 \leq a$ and $0 \leq b$ then $0 \leq a \cdot b$.

- every ordered field has characteristic zero

A semiring $(S, \oplus, \odot)$ is ordered if (i) holds.

- example: set $\mathbb{F}_{\geq 0}$ of nonnegative elements of any ordered field $\mathbb{F}$

Proposition

On the set of nonnegative real Puiseux series the valuation map induces a homomorphism

$$\mathrm{ord} \ : \ \mathbb{R}\{\{t\}\}_{\geq 0} \longrightarrow \mathbb{T}$$

of semirings, which reverses the ordering.

The main lemma of tropical linear programming

again: $\oplus = \max$, "reverse" Puiseux series $\Longrightarrow$ ord preserves ordering

Let $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \geq 0\}$ be contained in positive orthant of $\mathbb{K}^n$.

Lemma (Develin & Yu 2007; ABGJ 2015)

If tropicalization of $(\boldsymbol{A}, \boldsymbol{b})$ is sign generic then

$$\operatorname{ord}(\mathcal{P}) = \left\{x \in \mathbb{T}^n \,\middle|\, A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^- \right\} \ ,$$

where $(A^+ \ b^+) = \operatorname{ord}(\boldsymbol{A}^+ \boldsymbol{b}^+)$ *and* $(A^- \ b^-) = \operatorname{ord}(\boldsymbol{A}^- \ \boldsymbol{b}^-)$.

Moreover, for any $I \subset [m]$, we have:

$$\operatorname{ord}\big(\{\boldsymbol{x} \in \mathcal{P} \mid \boldsymbol{A}_I \boldsymbol{x} + \boldsymbol{b}_I = 0\}\big) = \left\{x \in \operatorname{ord}(\mathcal{P}) \,\middle|\, A_I^+ \odot x \oplus b_I^+ = A_I^- \odot x \oplus b_I^- \right\} \ .$$

where $(\boldsymbol{A}_I \ \boldsymbol{b}_I)$ submatrix of $(\boldsymbol{A} \ \boldsymbol{b})$ formed by rows with indices in I.

A simple example

Consider the Puiseux polyhedron $\boldsymbol{\mathcal{P}} \subset \mathbb{K}^2$ defined by:

$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\ t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\ t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\ \boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \ . \end{aligned} \tag{1}$$

Then the set $\operatorname{ord}(\boldsymbol{\mathcal{P}})$ is described by the tropical linear inequalities:

$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \ . \end{aligned} \tag{2}$$

The feasibility problem of tropical linear programming

Let $A, B \in \mathbb{T}^{m \times n}$.

Then the set $\{x \in \mathbb{TP}^{d-1} \colon A \odot x \leq B \odot x\}$ is a tropical cone.

Definition (Tropical linear programming feasibility)

Does there exist $x \in \mathbb{TP}^{d}$ such that $A \odot x \leq B \odot x$?

Theorem (Akian, Gaubert & Guterman 2012)

The feasibility problem of tropical linear programming lies in the complexity classes NP *and* $\mathrm{co}\text{-}\mathrm{NP}$.

Constructing a class of bipartite digraphs

Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{T}_{\max}^{m \times n}$ such that each column of A contains a finite coefficient, and each row of B contains a finite coefficient.

This defines a weighted bipartite digraph $D(A, B)$ on $m + n$ nodes:

- m row nodes and n column nodes
- weights

$$\omega(i,j) \ = \ \begin{cases} a_{j,i} & \text{if } i \text{ is a column (and } j \text{ is a row),} \\ b_{i,j} & \text{if } i \text{ is a row (and } j \text{ is a column).} \end{cases}$$

Example

The matrices

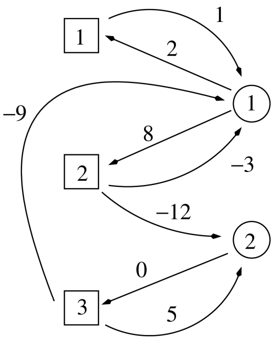
$$A = \begin{pmatrix} 2 & -\infty \\ 8 & -\infty \\ -\infty & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -\infty \\ -3 & -12 \\ -9 & 5 \end{pmatrix}$$

correspond to the system of tropical linear inequalities:

$$\begin{aligned} 2 + x_1 &\leq 1 + x_1 \\ 8 + x_1 &\leq \max(-3 + x_1, -12 + x_2) \\ x_2 &\leq \max(-9 + x_1, 5 + x_2) \end{aligned}$$

and the bipartite digraph $D(A, B)$:

[Example 2.3 from Akian, Gaubert & Guterman (2012)]

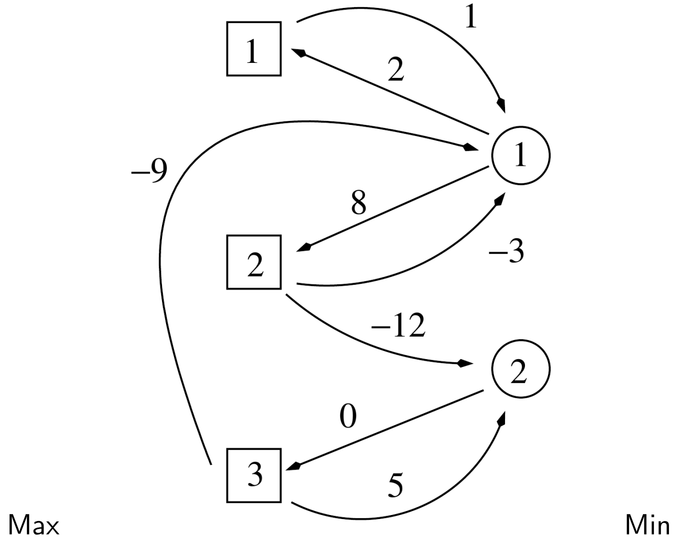


Mean-payoff games

The pair of matrices (A, B) defines an infinite game for two players, Max and Min.

- “board” = bipartite digraph $D(A, B)$
- players alternatingly move one token from a node to an adjacent node
- Max plays on the rows, Min play on the columns
  - $\omega(i, j) = b_{ij}$ = payment from Min to Max if i is a row (and j is a column)
  - $\omega(i, j) = a_{ji}$ = payment from Max to Min if i is a column (and j is a row)
- value of one (infinite) match = mean payoff

An example play



The Shapley operator

Given $A = (a_{ij}), B = (b_{ij}) \in \mathbb{T}_{\max}^{m \times n}$ define the Shapley operator $S = S_{A,B} : \mathbb{T}_{\max}^n \to \mathbb{T}_{\max}^n$ by

$$S(x) \ = \ A^{\#} \odot^{\min} (B \odot^{\max} x) \, ,$$

where $A^{\#} = -A^{\top}$.

- convention: $(+\infty) + (-\infty) = +\infty$
- observe: $A \odot^{\max} x \leq y \iff x \leq A^{\#} \odot^{\min} y$

Exercise (essential properties of S):

1. S is order-preserving, i.e., $x \leq y$ implies $S(x) \leq S(y)$ for all $x, y \in \mathbb{T}_{\max}^n$;
2. S is additively homogeneous, i.e., $S(\lambda \mathbf{1} + x) = \lambda \mathbf{1} + S(x)$ for all $\lambda \in \mathbb{T}_{\max}$ and $x \in \mathbb{T}_{\max}^n$;
3. S is continuous.

MEAN-PAYOFF

The Shapley operator describes the payoff of two subsequent half-moves.

Definition

The limit

$$\chi(S) := \lim_{N\to\infty} S^N(0)/N$$

is the called the value of the mean-payoff game for $D(A, B)$.

- Ehrenfeucht & Mycielski 1979: the value $\chi(S)$ is finite
- Zwick & Paterson 1996: the decision problem MEAN-PAYOFF, which asks whether $\chi(S)$ is positive, lies in NP and $\mathrm{co\text{-}NP}$

The feasibility problem of tropical linear programming

Theorem (Akian, Gaubert & Guterman 2012)

The system of homogeneous linear tropical inequalities $A \odot x \leq B \odot x$ has a solution $x \in \mathbb{TP}^d$ if and only if all initial states are winning for Max if and only if $\chi(S) \geq 0$.

Proof.

We have $A \odot^{\max} x \leq B \odot^{\max} x$ if and only if $x \leq S(x)$.

Assume that there exists $x \neq -\infty$ with $A \odot^{\max} x \leq B \odot^{\max} x$. By the essential properties of S the value $\chi(S)$ exists, and it is nonnegative because of $x \leq S(x)$.

Conversely, assume that Max has one winning state (or, equivalently, all states are winning). That is, there exists $x' \neq -\infty$ such that $\lim_{N\to\infty} S^N(x')/N \geq 0$. Then $x' \leq S(x')$ and $A \odot^{\max} x' \leq B \odot^{\max} x'$. □

Corollary

The feasibility problem of tropical linear programming lies in the complexity classes NP and $\mathrm{co\text{-}NP}$.

Tropicalizing the simplex method

Theorem (ABGJ 2015)

There is a tropical simplex method satifying:

- *Under certain nondegenericty assumption, the algorithm terminates and returns an optimal solution for any tropical pivoting rule.*
- *Every iteration (pivoting and computing reduced costs) can be done in $O(n(m+n))$ arithmetic operations over $\mathbb{T}$.*
- *Moreover, the algorithm traces the image under the valuation map of the path followed by the classical simplex algorithm applied to any lift to real Puiseux series, with a compatible pivoting rule.*

Theorem (ABGJ 2014)

Suppose there is a strongly-polynomial combinatorial pivoting strategy for the classical simplex method. Then each tropical linear program can be solved in strongly polynomial time.

On the interior point method

Theorem (Allamigeon, Benchimol, Gaubert & J. 2018)

There is a family, $\mathbf{LW}_r(t)$, of linear programs in $2r$ variables with $3r+1$ constraints, depending on $t>1$, such the number of iterations of any primal-dual path-following interior point algorithm with a log-barrier function which iterates in the wide neighborhood of the central path is exponential in r for $t \gg 0$.

Theorem (Allamigeon, Benchimol, Gaubert & J. 2018)

On the same family of LPs the total curvature of the central path is in $\Omega(2^r)$ for $t \gg 0$.

Recall: simple example . . .

Consider the Puiseux polyhedron $\boldsymbol{\mathcal{P}} \subset \mathbb{K}^2$ defined by:

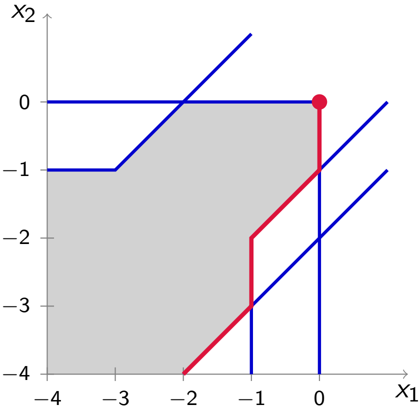
$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\ t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\ t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\ \boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \ . \end{aligned} \tag{3}$$

Then the set $\mathrm{ord}(\boldsymbol{\mathcal{P}})$ is described by the tropical linear inequalities:

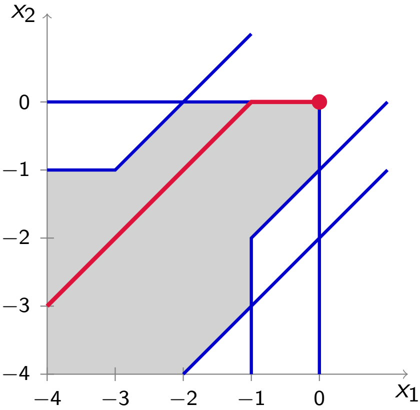
$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \ . \end{aligned} \tag{4}$$

. . . and two of its primal tropical central paths

- tropical central path = ord(Puiseux central path)



$\min \boldsymbol{x}_1$



$\min t\boldsymbol{x}_1 + \boldsymbol{x}_2$

The Linear Programs $\mathbf{LW}_r(t)\mathbf{LW}_r^{\epsilon}(t)$ ...

$$
\begin{aligned}
\text{minimize} \quad & x_1 \\
\text{subject to} \quad & x_1 \le t^2 \\
& x_2 \le t \\
& \left.\begin{aligned} & x_{2j+1} \le t\, x_{2j-1}\,,\ x_{2j+1} \le t\, x_{2j} \\ & x_{2j+2} \le t^{1-1/2^j}(x_{2j-1} + x_{2j}) \end{aligned}\right] \ 1 \le j < r \\
& x_{2r-1} \ge 0\,,\ x_{2r} \ge 0\epsilon
\end{aligned}
$$

for $r \ge 1$ and $t \gg 0$
and $1 \gg \epsilon \ge 0$

... have long and winding central paths.

References

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