Optimization and Tropical Geometry:
2. Tropical hypersurfaces

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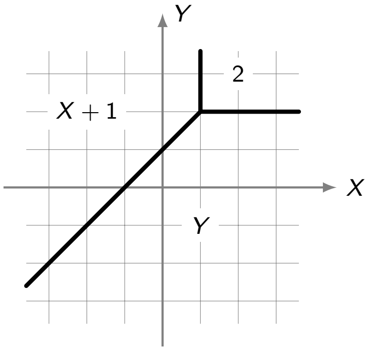
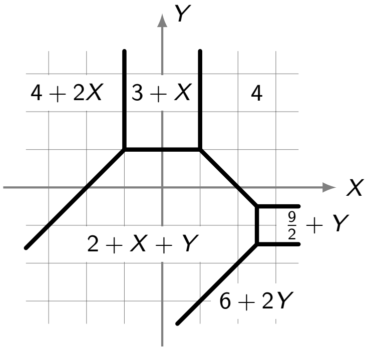
Recall: Regions of linearity of tropical polynomials



$\min(2, 1 + X, Y)$



$\min(4, 3 + X, 4 + 2X, 2 + X + Y,$
$6 + 2Y, \frac{9}{2} + Y)$

Definition

The tropical hypersurface of a k-variate tropical polynomial F is the set of points x in $\mathbb{R}^k$ where the minimum in the evaluation $F(x)$ is attained at least twice.

The dome of a tropical polynomial

Proposition

For a d-variate tropical polynomial F the set

$$\mathcal{D}(F) \;:=\; \left\{(x,s)\in\mathbb{R}^{d+1} \,\middle|\, x\in\mathbb{R}^d,\, s\in\mathbb{R},\, s\le F(x)\right\}$$

is an unbounded convex polyhedron of dimension $d+1$.

We call $\mathcal{D}(F)$ the dome of F.

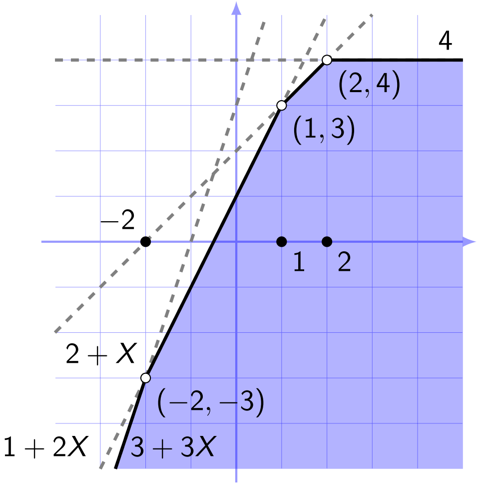
Corollary

The tropical hypersurface $\mathcal{T}(F)$ coincides with the image of the codimension-2-skeleton of its dome $\mathcal{D}(F)$ in $\mathbb{R}^d$ under the orthogonal projection which omits the last coordinate.

Recall: A univariate tropical polynomial

Example

$F(X) = (3 \odot X^3) \oplus (1 \odot X^2) \oplus (2 \odot X) \oplus 4$



Polytopal Subdivisions

Definition

A polyhedral complex is a finite collection of polyhedra which is closed with respect to taking faces and intersections.

- polytopal = cells are polytopes
- simplicial complex or triangulation = cells are simplices

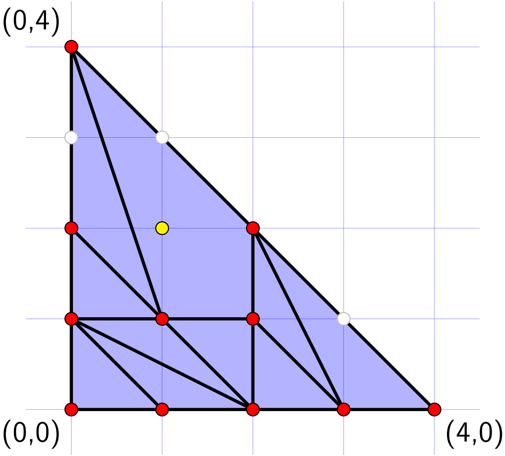
Definition

A polytopal subdivision of a finite point set $P \subset \mathbb{R}^d$ is

1. a polytopal complex $\mathcal{C}$ with $\bigcup\{C \colon C \in \mathcal{C}\} = \operatorname{conv} P$ such that
2. the vertices of $\mathcal{C}$ form a subset of P.

Example

$A = \{(u, v) \in \mathbb{Z}^2 \colon u \geq 0\,,\ v \geq 0\,,\ u + v \leq 4\}$



- conv $A = 4 \cdot \Delta_2$
- subdivision of A with ten maximal cells: nine triangles and one quadrangle
- the points $(0, 3)$, $(1, 2)$,$(1, 3)$ and $(3, 1)$ do not occur as vertices

Regular Subdivisions

Consider a height function $\omega : A \to \mathbb{R}$. Yields unbounded polyhedron

$$U(A, \omega) \;=\; \operatorname{conv}\{(u, \omega(u)) \mid u \in A\} + \operatorname{pos}\{e_{d+1}\} \tag{1}$$

in $\mathbb{R}^{d+1}$.

- lower face of $U(A, \omega)$ has outward pointing normal vector h pointing downward, i.e., satisfying $\langle h, e_{d+1} \rangle < 0$

Definition

Projecting lower faces of $U(A, \omega)$ by omitting the last coordinate yields the regular subdivision $\Sigma(A, \omega)$ of A induced by ω.

Example (Delaunay subdivision)

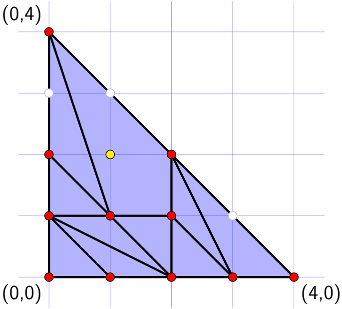
Let $A \subset \mathbb{R}^d$ be arbitrary, and take $\omega : A \to \mathbb{R}$, $p \mapsto \|p\|^2$.

Example

The height function ω:

$$\begin{array}{ccccc} (0,0)\mapsto 8 & (1,0)\mapsto 4 & (0,1)\mapsto 2 & (2,0)\mapsto 1 & (1,1)\mapsto 0 \\ (0,2)\mapsto 1 & (3,0)\mapsto 2 & (2,1)\mapsto 0 & (1,2)\mapsto 0 & (0,3)\mapsto 4 \\ (4,0)\mapsto 8 & (3,1)\mapsto 4 & (2,2)\mapsto 0 & (1,3)\mapsto 2 & (0,4)\mapsto 0 \end{array}$$

induces …



The dual subdivision of a tropical hypersurface

Let F be a d-variate tropical polynomial.

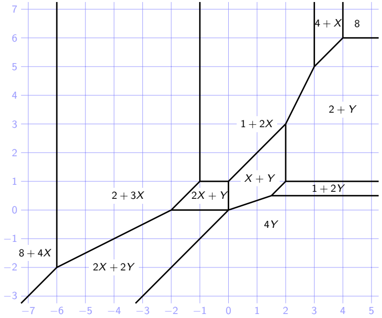
- dual subdivision $\mathcal{S}(F)$ = regular subdivision of exponent vectors induced by coefficients of F
- in this case $U(A, \omega) =: \widetilde{\mathcal{N}}(F)$ extended Newton polyhedron

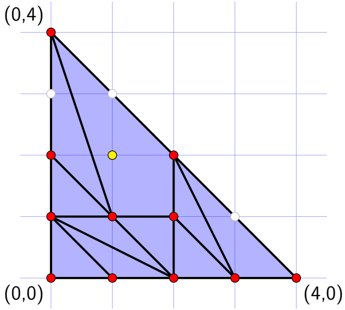
Theorem

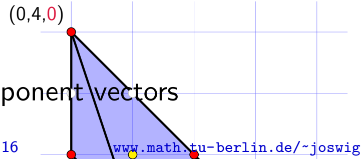
1. *There is an inclusion reversing bijection between the faces of $\mathcal{T}(F)$ and the cells of the dual subdivision $\mathcal{S}(F)$ of $\operatorname{supp} F$. In this way k-dimensional cells of $\mathcal{T}(F)$ are mapped to $(d-k)$-dimensional cells of $\mathcal{S}(F)$.*
2. *The latter cells are obtained from orthogonally projecting the bounded $(d-k)$-dimensional faces of $\widetilde{\mathcal{N}}(F)$ by omitting the last coordinate.*
3. *The vertices of $\widetilde{\mathcal{N}}(F)$ or, equivalently, the 0-dimensional cells of $\mathcal{S}(F)$ bijectively correspond to the facets of $\mathcal{D}(F)$ and thus also to the regions of $\mathcal{T}(F)$.*

Example and its homogenization

$$F = \min(8 + 4Z, 4 + X + 3Z, 2 + Y + 3Z, 1 + 2X + 2Z, X + Y + 2Z,$$
$$1 + 2Y + 2Z, 2 + 3X + Z, 2X + Y + Z, X + 2Y + Z,$$
$$4 + 3Y + Z, 8 + 4X, 4 + 3X + Y, 2X + 2Y, 2 + X + 3Y, 4Y)$$







Definition

Newton polytope $\mathcal{N}(F)$:= convex hull of exponent vectors

Puiseux series

The set of Puiseux series with coefficients in some field $\mathbb{F}$ is defined as

$$\mathbb{F}\{\{t\}\} \;=\; \left\{ \sum_{k=m}^{\infty} a_k \cdot t^{k/N} \;\middle|\; m \in \mathbb{Z},\, N \in \mathbb{N}^{\times},\, a_k \in \mathbb{F} \right\} \;.$$

- formal series, i.e., convergence does not matter unless stated otherwise
- convolution product:

$$\Big(\sum_{p=\ell}^{\infty} a_p \cdot t^{p/N}\Big) \cdot \Big(\sum_{q=m}^{\infty} b_q \cdot t^{q/N}\Big) \;=\; \sum_{r=\ell+m}^{\infty} \Big(\sum_{s=\ell}^{r-m} a_s b_{r-s}\Big) \cdot t^{r/N}$$

- Puiseux series $\mathbb{F}\{\{t\}\}$ form a field

Often we abbreviate $\mathbb{K} = \mathbb{F}\{\{t\}\}$.

The valuation map

The map

$$\mathrm{ord} : \mathbb{F}\{\{t\}\} \to \mathbb{Q} \cup \{\infty\}$$

sends a Puiseux series $\gamma(t) = \sum_{k=m}^{\infty} a_k \cdot t^{k/N}$ to its order, i.e., the lowest degree $\min\{k/N \colon k \in \mathbb{Z},\ a_k \neq 0\}$ occurring with a nonzero coefficient.

By definition $\mathrm{ord}(0)$ is set to ∞.

We have

$$\begin{aligned} \mathrm{ord}\big(\gamma(t) + \delta(t)\big) &\geq \min\{\mathrm{ord}\big(\gamma(t)\big), \mathrm{ord}\big(\delta(t)\big)\} \quad \text{as well as} \\ \mathrm{ord}\big(\gamma(t) \cdot \delta(t)\big) &= \mathrm{ord}\big(\gamma(t)\big) + \mathrm{ord}\big(\delta(t)\big) \ . \end{aligned}$$

By applying the valuation map $\mathrm{ord} : \mathbb{K} \to \mathbb{Q} \cup \{\infty\}$ componentwise one obtains a map

$$\mathrm{ord} : \mathbb{K}^d \to (\mathbb{Q} \cup \{\infty\})^d \ .$$

Restricting this map to the algebraic torus $(\mathbb{K}^\times)^d$ yields values in the rational vector space $\mathbb{Q}^d$.

Tropicalization

The Laurent polynomial

$$f(x_1, \ldots, x_d) \;=\; \gamma(t) x_1^{u_1} x_2^{u_2} \ldots x_d^{u_d} + \delta(t) x_1^{v_1} x_2^{v_2} \ldots x_d^{v_d} + \ldots$$

in $\mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}]$ has the tropicalization

$$\begin{aligned} \operatorname{trop}(f)(x_1, \ldots, x_d) \;:=\; & \operatorname{ord}\bigl(\gamma(t)\bigr) \odot x_1^{\odot u_1} x_2^{\odot u_2} \ldots x_d^{\odot u_d} \\ & \oplus \operatorname{ord}\bigl(\delta(t)\bigr) \odot x_1^{\odot v_1} x_2^{\odot v_2} \ldots x_d^{\odot v_d} \oplus \ldots \quad , \end{aligned}$$

which is a tropical polynomial.

The fundamental theorem of tropical geometry

Let $\mathbb{K} = \mathbb{F}\{\{t\}\}$.

Theorem (Kapranov 2000; Einsiedler, Kapranov & Lind 2006)

For a polynomial $f \in \mathbb{K}[x_1^{\pm}, x_2^{\pm}, \dots, x_d^{\pm}]$ *the tropical hypersurface* $\mathcal{T}(\operatorname{trop}(f))$ *equals the topological closure of the set* $\operatorname{ord}(V(f))$ *in* $\mathbb{R}^d$*.*

- base case $d = 1$ is Newton–Puiseux theorem:
  - if $\mathbb{F}$ is algebraically closed of characteristic zero, then $\mathbb{F}\{\{t\}\}$ is algebraically closed, too

Example

$$f = x_1 + x_2 - t^2 \xrightarrow{\text{trop}} \operatorname{trop}(f) = \min(X_1, X_2, 2)$$

$$V(f) = \{(p, t^2 - p) \in \mathbb{K}^2\} \xrightarrow{\text{ord}} \mathcal{T}(\operatorname{trop}(f))$$

What is $\operatorname{ord}(p, t^2 - p)$?

Let $p = a_k t^{\alpha_k} + a_{k+1} t^{\alpha_{k+1}} + \ldots$ with $\operatorname{ord}(p) = \alpha_k$.

Case $\alpha_k < 2$:

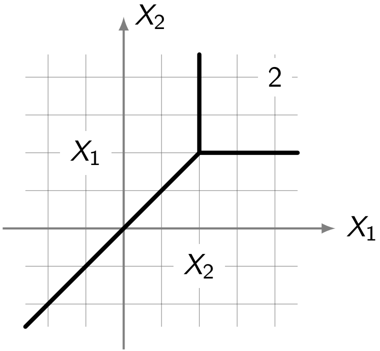
$$\operatorname{ord}(p, t^2 - p) = (\alpha_k, \alpha_k)$$

Case $\alpha_k > 2$:

$$\operatorname{ord}(p, t^2 - p) = (\alpha_k, 2)$$

Case $\alpha_k = 2$:

$$\operatorname{ord}(p, t^2 - p) = \begin{cases} (2, 2) \\ (2, \alpha_{k+\epsilon}) \end{cases}$$



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