Optimization and Tropical Geometry:
1. Shortest paths and the Hungarian method

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Shortest path problems

Let Γ be a finite directed graph on n nodes, equipped with real arc weights.

Three variants:

- s–t shortest path problem:
  - the source node s and the target node t are fixed
- single source or single target shortest path problem:
  - either s or t are fixed, and the other nodes vary arbitrarily
- all-pairs shortest path

It matters whether or not we restrict to positive weights only.

$\rightarrow$ Schrijver CO/A, Chapters 6,7,8

Tropical arithmetic

For $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ we call $(\mathbb{T}, \min, +)$ the tropical semiring. Often we abbreviate $\oplus = \min$ and $\odot = +$.

Example

$3 \odot (4 \oplus 5) = 3 + \min(4, 5) = 7 = \min(3 + 4, 3 + 5) = (3 \odot 4) \oplus (3 \odot 5)$.

Instead of general graph Γ on n nodes with real arc weights consider complete directed graph $\widetilde{K}_n$ with arc weights in $\mathbb{T}$.

Then the directed adjaceny matrix $D = (d_{uv})_{u,v}$ with

$$d : [n] \times [n] \to \mathbb{T}\,, \ (u, v) \mapsto d_{uv}$$

encodes the graph Γ.

There is a natural way to define tropical matrix multiplication.

Powers of tropical matrices

Let Γ be given by its directed adjacency matrix $D \in \mathbb{T}^{n \times n}$.

Naive algorithm for all-pairs shortest path:

1. compute the tropical matrix power $D^{\odot(n-1)}$
2. there is a negative cycle if and only if $D^{\odot(n-1)}$ has a negative entry on the diagonal
3. otherwise the coefficient of $D^{\odot(n-1)}$ at (u, v) is the length of a shortest u–v path

overall cost $= O(n^4)$

Kleene stars

Definition (Kleene star)

$$D^* := I \oplus D \oplus D^{\odot 2} \oplus \cdots \oplus D^{\odot(n-1)} \oplus \cdots \quad ,$$

where $I = D^{\odot 0}$ is the tropical identity matrix, with coefficients 0 on the diagonal and ∞ otherwise.

Well defined if sequence converges. Then $D^* = D^{\odot(n-1)}$.

Floyd–Warshall algorithm (1962)

Idea: reduce the complexity of computing D^* to $O(n^3)$ via dynamic programming.

Measure weight of a shortest path from u to v with all intermediate nodes restricted to the set $\{1, 2, \ldots, r\}$, which is

$$d_{uv}^{(r)} = \begin{cases} d_{uv} & \text{if } r = 0 \\ \min\left(d_{uv}^{(r-1)}, \, d_{ur}^{(r-1)} + d_{rv}^{(r-1)}\right) & \text{if } r \geq 1 \end{cases} . \tag{1}$$

That is, in the nontrivial step of the computation we check if going through the new node r gives an advantage.

- correctness follows from the fact that $(\mathbb{T}, \oplus, \odot)$ is a semiring, equipped with a total ordering

Floyd–Warshall algorithm (1962), continued

We set $D^{(r)} = \left(d_{uv}^{(r)}\right)_{u,v}$.

- with $D^{(r-1)}$ known the computation of a single coefficient $d_{uv}^{(r)}$ requires only constant time
- negative cycle exists if and only if some diagonal coefficient of $D^{(n)}$ is negative
- otherwise we have $D^{(n)} = D^{\odot(n-1)} = D^*$
- overall cost $= O(n^3)$

In general, the matrix $D^{(r)}$ is distinct from any tropical power $D^{\odot k}$.

Fact: optimal complexity known for arbitrary arc weights.

Tropical polynomials

We can consider (multivariate) tropical polynomials like

$$4 \oplus 3X \oplus 4X^2 \oplus 2XY \oplus 6Y^2 \oplus \tfrac{9}{2}Y$$
$$= \min(4, 3+X, 4+2x, 2+X+Y, 6+2Y, \tfrac{9}{2}+Y) \ .$$

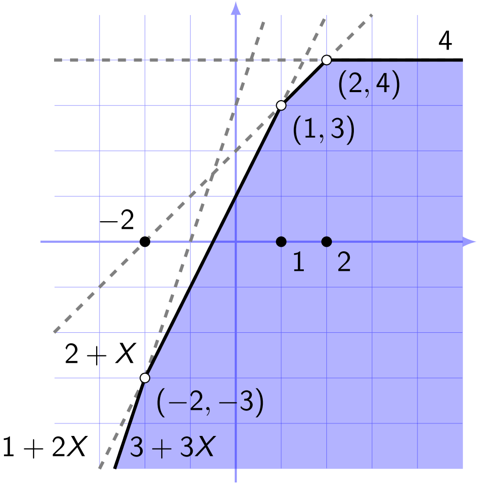
They can be added and multiplied tropically, to obtain another semiring, which we write as $\mathbb{T}[X, Y]$.

- via tropical evaluation a k-variate tropical polynomial $F \in \mathbb{T}[X_1, \ldots, X_k]$ defines a (continuous) piecewise linear map from $\mathbb{R}^k$ to $\mathbb{R}$
- evaluation defines partial ordering of tropical polynomials

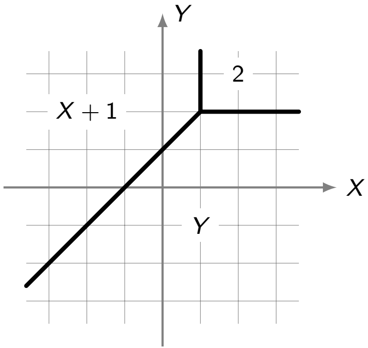
A univariate tropical polynomial

Example

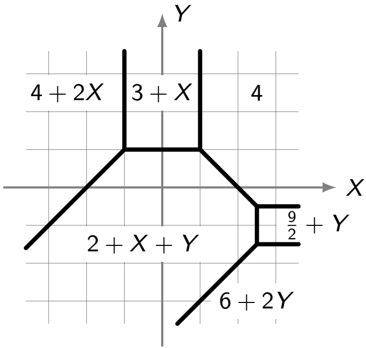
$F(X) = (3 \odot X^3) \oplus (1 \odot X^2) \oplus (2 \odot X) \oplus 4$



Regions of linearity of tropical polynomials



$\min(2, 1 + X, Y)$



$\min(4, 3 + X, 4 + 2X, 2 + X + Y,$
$6 + 2Y, \frac{9}{2} + Y)$

Definition

The tropical hypersurface of a k-variate tropical polynomial F is the set of points x in $\mathbb{R}^k$ where the minimum in the evaluation $F(x)$ is attained at least twice.

Parameterized all-pairs shortest paths

Proposition

The solution to the all-pairs shortest paths problem of a directed graph with n nodes and weighted adjacency matrix

$$D \in \mathbb{T}[X_1, \ldots, X_k]$$

is a polyhedral decomposition of $\mathbb{R}^k$ induced by up to n^2 tropical polynomials corresponding to the nonconstant coefficients of $D^{\odot(n-1)}$. On each polyhedral cell the lengths of all shortest paths are linear functions in the k parameters.

First algorithm: (parameterized) Floyd–Warshall

Example

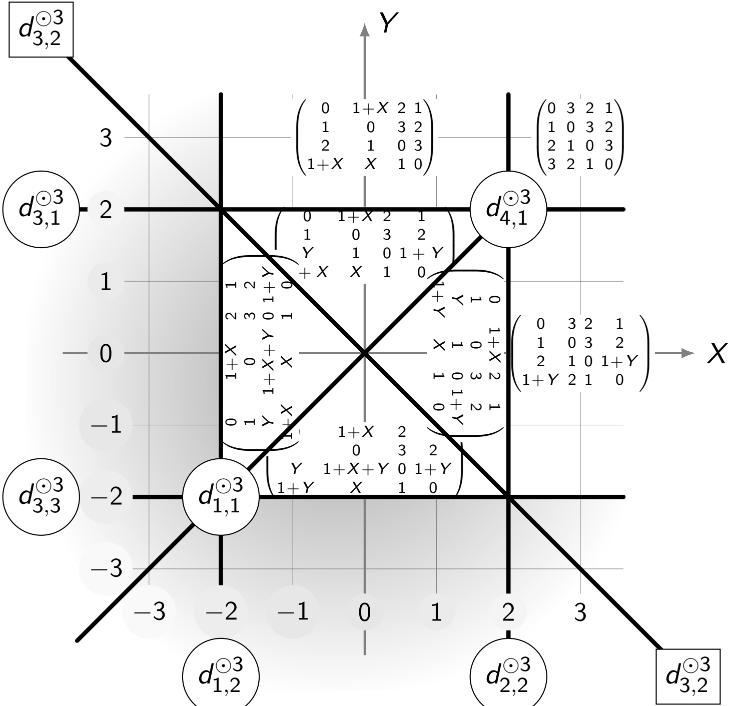
Consider the directed graph Γ on four nodes with the weighted adjacency matrix

$$D = \begin{pmatrix} 0 & \infty & \infty & 1 \\ 1 & 0 & \infty & \infty \\ Y & 1 & 0 & \infty \\ \infty & X & 1 & 0 \end{pmatrix} , \tag{2}$$

whose coefficients lie in the semiring $\mathbb{T}[X, Y]$ of bivariate tropical polynomials.

Then

$$D^{\odot 3} = \begin{pmatrix} \min(2+X, 2+Y, 0) & \min(1+X, 3) & 2 & 1 \\ 1 & \min(2+X, 0) & 3 & 2 \\ \min(Y, 2) & \min(1+X+Y, 1) & \min(2+Y, 0) & \min(1+Y, 3) \\ \min(1+X, 1+Y, 3) & \min(X, 2) & 1 & \min(2+X, 2+Y, 0) \end{pmatrix} .$$


$d_{3,2}^{\odot 3}$
Y
X
$d_{3,1}^{\odot 3}$
$d_{4,1}^{\odot 3}$
$d_{3,3}^{\odot 3}$
$d_{1,1}^{\odot 3}$
$d_{1,2}^{\odot 3}$
$d_{2,2}^{\odot 3}$
$d_{3,2}^{\odot 3}$

Parameterized all-pairs shortest paths, continued

Theorem (J. & Schröter 2019+)

Let $D \in \mathbb{T}[x_1, \ldots, x_k]^{n \times n}$ be the weighted adjacency matrix of a directed graph on n nodes.
Suppose that D has separated variables.
Then, between any pair of nodes, there are at most 2^k pairwise incomparable shortest paths. Moreover, the Kleene star D^*, which encodes all parameterized shortest paths, can be computed in $O(k \cdot 2^k \cdot n^3)$ time, if it exists.

- separated variables: each coefficient of D involves a constant plus at most one of the k indeterminates

Sketch of proof

- Assume no negative cycles.
- Then there is at least one shortest path between any two nodes (possibly of infinite length).
- In each shortest path each arc occurs at most once. By our assumption this means that the total weight is $\lambda + x_{i_1} + \cdots + x_{i_\ell}$ for $\lambda \in \mathbb{T}$ and $x_{i_1} + \cdots + x_{i_\ell}$ is a multilinear tropical monomial, i.e., each indeterminate occurs with multiplicity zero or one. There are 2^k distinct multilinear monomials, and hence this bounds the number of incomparable shortest paths between any two nodes.
- Use Floyd–Warshall.
- The tropical multiplication, i.e., ordinary sum, of two multilinear monomials takes linear time in the number of indeterminates, which is at most k.

The linear assignment problem

Problem

Given 4 soccer players and 4 positions, what is the best formation?

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- assignment = choice of coefficients, one per column/row

$$\begin{aligned} \text{best} &= \min_{\omega \in \mathrm{Sym}(4)} a_{1,\omega(1)} + a_{2,\omega(2)} + a_{3,\omega(3)} + a_{4,\omega(4)} \\ &= \bigoplus_{\omega \in \mathrm{Sym}(4)} a_{1,\omega(1)} \odot a_{2,\omega(2)} \odot a_{3,\omega(3)} \odot a_{4,\omega(4)} \end{aligned}$$

Definition

The tropical determinant is the multivariate homogeneous tropical polynomial arising from Leibniz' rule (ignoring the signs).

Hungarian method (Kuhn 1955; Munkres 1957)

Input: matrix $A \in \mathbb{T}^{n \times n}$
Output: minimum weight maximal matching in $B(A)$

$\mu \leftarrow \emptyset$
repeat
  $U_\mu \leftarrow$ nodes in $[n]$ not covered by μ
  $W_\mu \leftarrow$ nodes in $[n']$ not covered by μ
  $B_\mu \leftarrow$ directed graph with node set $[n] \sqcup [n']$,
    edges with weights induced by A, directed from $[n]$ to $[n']$,
    except for those in μ, which are reversed and
    get negated weights
  if there is a path from U_μ to W_μ in B_μ then
    $\pi \leftarrow$ edge set of shortest one among these
    $\mu \leftarrow \mu \triangle \pi$
until there is no path from U_μ to W_μ
return μ

overall cost: $O(n^3)$

Tropical eigenvalues

Let $D = (d_{ij})$ be a $n \times n$-matrix with coefficients in the tropical semiring $\mathbb{T}$.

Definition

A vector $x \in \mathbb{T}^n \setminus \{\infty\}$ is a tropical eigenvector for D with respect to the tropical eigenvalue $\lambda \in \mathbb{R}$ if

$$D \odot x \; = \; \lambda \odot x \; .$$

If x is a tropical eigenvector with respect to the tropical eigenvalue λ then this definition amounts to requiring

$$(d_{u,1} \odot x_1) \oplus (d_{u,2} \odot x_2) \oplus \cdots \oplus (d_{u,n} \odot x_n) \; = \; \lambda \odot x_u \quad \text{for all } u \in [n] \; . \tag{3}$$

This yields as a consequence $\lambda + x_u \leq d_{u,v} + x_v$ and thus

$$x_u - x_v \; \leq \; d_{u,v} - \lambda \quad \text{for all } u, v \in [n] \; . \tag{4}$$

Cycle means

For a directed path $\pi = ((u_0, u_1), \ldots, (u_{k-1}, u_k))$ in $\Gamma = \Gamma(D)$, i.e, for $d_{u_0 u_1}, \ldots, d_{u_{k-1} u_k}$ finite, the number

$$c(\pi) \; := \; \frac{1}{k}(d_{u_0 u_1} + \cdots + d_{u_{k-1} u_k})$$

is the mean weight of π.

If π is a cycle, i.e., for $u_0 = u_k$, then $c(\pi)$ is also called the cycle mean of π.

Lemma

Let λ be a tropical eigenvalue of D, and let ζ be a cycle in $\Gamma(D)$. Then we have $\lambda \leq c(\zeta)$.

Minimum cycle mean

The minimum cycle mean of $\Gamma = \Gamma(D)$ is

$$\lambda(D) \ := \ \min\{c(\zeta) \mid \zeta \text{ directed cycle in } \Gamma\} \ . \tag{5}$$

- $\lambda(D) \geq 0$ if and only if "weighted digraph polyhedron" $Q(D)$ is not empty
- $\lambda(A) = \infty$ if Γ is acyclic

Now let Γ be strongly connected.

Proposition

If $\lambda(D) = 0$ then each column of D^* which is contained in a zero weight cycle is a tropical eigenvector of D for the tropical eigenvalue zero.

Theorem

The minimum cycle mean $\lambda(D)$ is the only tropical eigenvalue of D.

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