Optimization and Tropical Geometry:

1. Shortest paths and the Hungarian method

Michael Joswig

TU Berlin

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partially joint w/ Benjamin Schröter
Shortest path problems

Let $\Gamma$ be a finite directed graph on $n$ nodes, equipped with real arc weights.

Three variants:

- $s$–$t$ shortest path problem:
  - the source node $s$ and the target node $t$ are fixed
- single source or single target shortest path problem:
  - either $s$ or $t$ are fixed, and the other nodes vary arbitrarily
- all-pairs shortest path

It matters whether or not we restrict to positive weights only.

$\rightarrow$ Schrijver CO/A, Chapters 6,7,8
Tropical arithmetic

For $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ we call $(\mathbb{T}, \min, +)$ the tropical semiring. Often we abbreviate $\oplus = \min$ and $\odot = +$.

Example

$$3 \odot (4 \oplus 5) = 3 + \min(4, 5) = 7 = \min(3 + 4, 3 + 5) = (3 \odot 4) \oplus (3 \odot 5).$$

Instead of general graph $\Gamma$ on $n$ nodes with real arc weights consider complete directed graph $\tilde{K}_n$ with arc weights in $\mathbb{T}$.

Then the directed adjacency matrix $D = (d_{uv})_{u,v}$ with

$$d : [n] \times [n] \to \mathbb{T}, \ (u, v) \mapsto d_{uv}$$

encodes the graph $\Gamma$.

There is a natural way to define tropical matrix multiplication.
Powers of tropical matrices

Let $\Gamma$ be given by its directed adjacency matrix $D \in \mathbb{T}^{n \times n}$.

Naive algorithm for all-pairs shortest path:

1. compute the tropical matrix power $D \odot (n-1)$
2. there is a negative cycle if and only if $D \odot (n-1)$ has a negative entry on the diagonal
3. otherwise the coefficient of $D \odot (n-1)$ at $(u, v)$ is the length of a shortest $u-v$ path

overall cost $= O(n^4)$
Kleene stars

Definition (Kleene star)

\[ D^* := I \oplus D \oplus D \odot^2 \oplus \cdots \oplus D \odot^{(n-1)} \oplus \cdots, \]

where \( I = D \odot^0 \) is the tropical identity matrix, with coefficients 0 on the diagonal and \( \infty \) otherwise.

Well defined if sequence converges. Then \( D^* = D \odot^{(n-1)} \).
Floyd–Warshall algorithm (1962)

Idea: reduce the complexity of computing \( D^* \) to \( O(n^3) \) via dynamic programming.

Measure weight of a shortest path from \( u \) to \( v \) with all intermediate nodes restricted to the set \( \{1, 2, \ldots, r\} \), which is

\[
d_{uv}^{(r)} = \begin{cases} 
  d_{uv} & \text{if } r = 0 \\
  \min \left( d_{uv}^{(r-1)}, d_{ur}^{(r-1)} + d_{rv}^{(r-1)} \right) & \text{if } r \geq 1
\end{cases}
\]

That is, in the nontrivial step of the computation we check if going through the new node \( r \) gives an advantage.

- correctness follows from the fact that \((\mathbb{T}, \oplus, \odot)\) is a semiring, equipped with a total ordering
We set $D^{(r)} = \left( d^{(r)}_{uv} \right)_{u,v}$.

- with $D^{(r-1)}$ known the computation of a single coefficient $d^{(r)}_{uv}$ requires only constant time
- negative cycle exists if and only if some diagonal coefficient of $D^{(n)}$ is negative
- otherwise we have $D^{(n)} = D^{\odot(n-1)} = D^*$
- overall cost $= O(n^3)$

In general, the matrix $D^{(r)}$ is distinct from any tropical power $D^{\odot k}$.

Fact: optimal complexity known for arbitrary arc weights.
Tropical polynomials

We can consider (multivariate) tropical polynomials like

\[ 4 \oplus 3X \oplus 4X^2 \oplus 2XY \oplus 6Y^2 \oplus \frac{9}{2}Y \]

\[ = \min(4, 3 + X, 4 + 2x, 2 + X + Y, 6 + 2Y, \frac{9}{2} + Y) . \]

They can be added and multiplied tropically, to obtain another semiring, which we write as \( \mathbb{T}[X, Y] \).

- via tropical evaluation a \( k \)-variate tropical polynomial \( F \in \mathbb{T}[X_1, \ldots, X_k] \) defines a (continuous) piecewise linear map from \( \mathbb{R}^k \) to \( \mathbb{R} \)
- evaluation defines partial ordering of tropical polynomials
A univariate tropical polynomial

Example

\[ F(X) = (3 \odot X^3) \oplus (1 \odot X^2) \oplus (2 \odot X) \oplus 4 \]
Regions of linearity of tropical polynomials

\[ \text{min}(2, 1 + X, Y) \]

\[ \text{min}(4, 3 + X, 4 + 2X, 2 + X + Y, 6 + 2Y, \frac{9}{2} + Y) \]

**Definition**

The *tropical hypersurface* of a $k$-variate tropical polynomial $F$ is the set of points $x$ in $\mathbb{R}^k$ where the minimum in the evaluation $F(x)$ is attained at least twice.
Parameterized all-pairs shortest paths

Proposition

The solution to the all-pairs shortest paths problem of a directed graph with $n$ nodes and weighted adjacency matrix

$$D \in \mathbb{T}[X_1, \ldots, X_k]$$

is a polyhedral decomposition of $\mathbb{R}^k$ induced by up to $n^2$ tropical polynomials corresponding to the nonconstant coefficients of $D^\odot(n-1)$. On each polyhedral cell the lengths of all shortest paths are linear functions in the $k$ parameters.

First algorithm: (parameterized) Floyd–Warshall
Consider the directed graph $\Gamma$ on four nodes with the weighted adjacency matrix

\[
D = \begin{pmatrix}
0 & \infty & \infty & 1 \\
1 & 0 & \infty & \infty \\
\gamma & 1 & 0 & \infty \\
\infty & \chi & 1 & 0 \\
\end{pmatrix}, \tag{2}
\]

whose coefficients lie in the semiring $\mathbb{T}[X, Y]$ of bivariate tropical polynomials.

Then

\[
D^{\odot 3} = \begin{pmatrix}
\min(2 + X, 2 + Y, 0) & \min(1 + X, 3) & 2 & 1 \\
1 & \min(2 + X, 0) & 3 & 2 \\
\min(Y, 2) & \min(1 + X + Y, 1) & \min(2 + Y, 0) & \min(1 + Y, 3) \\
\min(1 + X, 1 + Y, 3) & \min(X, 2) & 1 & \min(2 + X, 2 + Y, 0) \\
\end{pmatrix}.
\]
Parameterized all-pairs shortest paths, continued

Theorem (J. & Schröter 2019+)

Let $D \in \mathbb{T}[x_1, \ldots, x_k]^{n \times n}$ be the weighted adjacency matrix of a directed graph on $n$ nodes. Suppose that $D$ has separated variables. Then, between any pair of nodes, there are at most $2^k$ pairwise incomparable shortest paths. Moreover, the Kleene star $D^*$, which encodes all parameterized shortest paths, can be computed in $O(k \cdot 2^k \cdot n^3)$ time, if it exists.

▶ separated variables: each coefficient of $D$ involves a constant plus at most one of the $k$ indeterminates
Sketch of proof

- Assume no negative cycles.
- Then there is at least one shortest path between any two nodes (possibly of infinite length).
- In each shortest path each arc occurs at most once. By our assumption this means that the total weight is $\lambda + x_{i_1} + \cdots + x_{i_\ell}$ for $\lambda \in \mathbb{T}$ and $x_{i_1} + \cdots + x_{i_\ell}$ is a multilinear tropical monomial, i.e., each indeterminate occurs with multiplicity zero or one. There are $2^k$ distinct multilinear monomials, and hence this bounds the number of incomparable shortest paths between any two nodes.
- Use Floyd–Warshall.
- The tropical multiplication, i.e., ordinary sum, of two multilinear monomials takes linear time in the number of indeterminates, which is at most $k$. 
The linear assignment problem

Problem

Given 4 soccer players and 4 positions, what is the best formation?

\[
A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[\text{assignment} = \text{choice of coefficients, one per column/row}\]

\[
\text{best} = \min_{\omega \in \text{Sym}(4)} a_{1,\omega(1)} + a_{2,\omega(2)} + a_{3,\omega(3)} + a_{4,\omega(4)}
\]

\[= \bigoplus_{\omega \in \text{Sym}(4)} a_{1,\omega(1)} \odot a_{2,\omega(2)} \odot a_{3,\omega(3)} \odot a_{4,\omega(4)}\]

Definition

The tropical determinant is the multivariate homogeneous tropical polynomial arising from Leibniz’ rule (ignoring the signs).
Hungarian method (Kuhn 1955; Munkres 1957)

Input: matrix \( A \in \mathbb{T}^{n \times n} \)
Output: minimum weight maximal matching in \( B(A) \)

\[ \mu \leftarrow \emptyset \]

repeat

\[ U_\mu \leftarrow \text{nodes in } [n] \text{ not covered by } \mu \]
\[ W_\mu \leftarrow \text{nodes in } [n'] \text{ not covered by } \mu \]
\[ B_\mu \leftarrow \text{directed graph with node set } [n] \sqcup [n'] , \]
\[ \text{edges with weights induced by } A, \text{ directed from } [n] \text{ to } [n'] , \]
\[ \text{except for those in } \mu, \text{ which are reversed and} \]
\[ \text{get negated weights} \]

if there is a path from \( U_\mu \) to \( W_\mu \) in \( B_\mu \) then

\[ \pi \leftarrow \text{edge set of shortest one among these} \]

\[ \mu \leftarrow \mu \triangle \pi \]

until there is no path from \( U_\mu \) to \( W_\mu \)

return \( \mu \)

overall cost: \( O(n^3) \)
Tropical eigenvalues

Let \( D = (d_{ij}) \) be a \( n \times n \)-matrix with coefficients in the tropical semiring \( \mathbb{T} \).

**Definition**

A vector \( x \in \mathbb{T}^n \setminus \{\infty\} \) is a tropical eigenvector for \( D \) with respect to the tropical eigenvalue \( \lambda \in \mathbb{R} \) if

\[
D \odot x = \lambda \odot x .
\]

If \( x \) is a tropical eigenvector with respect to the tropical eigenvalue \( \lambda \) then this definition amounts to requiring

\[
(d_{u,1} \odot x_1) \oplus (d_{u,2} \odot x_2) \oplus \cdots \oplus (d_{u,n} \odot x_n) = \lambda \odot x_u \quad \text{for all } u \in [n] .
\] (3)

This yields as a consequence \( \lambda + x_u \leq d_{u,v} + x_v \) and thus

\[
 x_u - x_v \leq d_{u,v} - \lambda \quad \text{for all } u, v \in [n] .
\] (4)
Cycle means

For a directed path \( \pi = ((u_0, u_1), \ldots, (u_{k-1}, u_k)) \) in \( \Gamma = \Gamma(D) \), i.e., for 
\( d_{u_0u_1}, \ldots, d_{u_{k-1}u_k} \) finite, the number 
\[
c(\pi) := \frac{1}{k} (d_{u_0u_1} + \cdots + d_{u_{k-1}u_k})
\]
is the mean weight of \( \pi \).

If \( \pi \) is a cycle, i.e., for \( u_0 = u_k \), then \( c(\pi) \) is also called the cycle mean of \( \pi \).

**Lemma**

*Let \( \lambda \) be a tropical eigenvalue of \( D \), and let \( \zeta \) be a cycle in \( \Gamma(D) \). Then we have \( \lambda \leq c(\zeta) \).*
Minimum cycle mean

The minimum cycle mean of $\Gamma = \Gamma(D)$ is

$$\lambda(D) := \min \{ c(\zeta) | \zeta \text{ directed cycle in } \Gamma \}.$$ (5)

- $\lambda(D) \geq 0$ if and only if “weighted digraph polyhedron” $Q(D)$ is not empty
- $\lambda(A) = \infty$ if $\Gamma$ is acyclic

Now let $\Gamma$ be strongly connected.

**Proposition**

*If $\lambda(D) = 0$ then each column of $D^*$ which is contained in a zero weight cycle is a tropical eigenvector of $D$ for the tropical eigenvalue zero.*

**Theorem**

*The minimum cycle mean $\lambda(D)$ is the only tropical eigenvalue of $D$.***
References


