POLYHEDRAL COMPUTATIONS WITH polymake

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Abstract. This is part of the "Hands-on Course on Geometric Software" of the Intensive Research Program on Discrete, Combinatorial and Computational Geometry, http://dccg.upc.edu/irp2018/, Barcelona, April 30 - May 4, 2018.

1. First Constructions

Don't forget: `help`, `apropos` and `F1` are your friends! Warning: On some operating systems/window managers the `F1` key may be taken by some other function. In this case, e.g., `Polymake::User::help_key='_k2';` will let you choose `F2` as a replacement.

Exercise 1.1. Define a triangle in terms of its vertices and compute the facets.

Exercise 1.2. Solve a linear program.

Exercise 1.3. Construct a simplicial polytope and verify that its dual is simple.

Exercise 1.4. Compute the Voronoi diagram of some point configuration spanning 4-space and compute the number of maximal cells.

Exercise 1.5. Start out with a regular 3-cube and remove one of its facets; verify that the resulting polyhedron is unbounded. Compute the f-vector and explain the result. This might need some browsing through polymake documentation on the web.

Exercise 1.6. Construct the product of a pentagon and a heptagon. Draw two distinct Schlegel diagrams.

Exercise 1.7. Check out the effect of `script "analyze_this.pl";` followed by `analyze_this(dodecahed`

2. The Second Step

Exercise 2.1. Which Johnson polytope(s) has/have the maximal number of facets? How many of the facets are triangles, quadrangles, pentagons etc? How many maximal flags (i.e., incident triplets of vertices, edges, facets) exist?

Exercise 2.2. Produce 100 random polytopes (in any way you like) and determine the average f-vector.

Exercise 2.3. What is the operation on polytopes which is dual to the product? Construct a dual pair of polytopes and verify.

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Exercise 2.4. Let σ be a rational polyhedral cone. *Gordan's Lemma* says that the induced semigroup algebra $\mathbb{C}[\sigma \cap \mathbb{Z}^n]$ is finitely generated. Construct an example where the minimal number of generators exceeds the number of rays of σ.

Exercise 2.5. Pick your favorite 3-polytope. Produce 64 random rotations and visualize them arranged in a $4 \times 4 \times 4$-grid.

Exercise 2.6. Take some simplicial polytope in dimension six and verify that the homology of the boundary complex fits a 5-sphere. Hint: construct an object of type `SimplicialComplex`, which is part of the `topaz` application.

Exercise 2.7. Compute the h-vector of the simplicial complex Δ on four vertices with facets $\{1, 2\}$ and $\{2, 3, 4\}$. Describe its Stanley-Reisner ring $\mathbb{Q}[\Delta]$. Warning: pay attention to the numbering of the vertices; `polymake` always starts with 0.

Exercise 2.8. How many distinct combinatorial types of cubical 3-polytopes can you produce?

Exercise 2.9. Verify that the 3-dimensional permutahedron is combinatorially equivalent to the secondary polytope for the prism over the 3-simplex.

Exercise 2.10. Can you compute all triangulations of a convex hexagon? What about the triangulations of a prism over that hexagon?

Exercise 2.11. How many combinatorial types of matroids of rank 4 on 8 elements are there? What is the minimal and the maximal number of circuits which occurs? What is the average, and how does the distribution look like? Warning: this requires access to the `polyDB`; while the code is part of the polymake distribution, it requires the Perl module `MongoDB`, and this may be a bit subtle to install; cf. `help "install_mongodb";`.

3. Towards Research Questions

Exercise 3.1. A simple polytope is *even* if each 2-face has an even number of vertices.

Construct even simple polytopes (of arbitrary dimension). Can you do more than full truncations or zonotopes?

This is related to Barnette's Conjecture: The vertex-edge graph of every even simple 3-polytope admits a Hamiltonian cycle.

Exercise 3.2. The tropical polynomial

$$F(x, y) \;=\; \min\bigl(x,\ 2x,\ 2y,\ x + 2y,\ 2x + y\bigr)$$

defines a plane tropical cubic $C = \mathcal{T}(F)$. Show a picture. Warning: `polymake` requires a homogeneous tropical polynomial! What is the set of tropical curves of the same *type*? Hint: discuss "type" and look at the regular subdivision of the monomials (seen as points in the plane) which is induced by the coefficients.

This is a question about moduli spaces of tropical curves.

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