Museums, Triangles and Algebraic Curves

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1 Guarding a Museum

2 Algebraic Curves

3 What’s the Connection?
The Museum
A simple polygon is a closed sequence of finitely many line segments without self-crossings.

- allows to distinguish between inside and outside
- (polygonal version of) Jordan’s Curve Theorem
The Museum Is a Simple Polygon
The Restricted View of a Guard
Subdividing Into Triangles
Coloring the Vertices (Triangle by Triangle)
Choosing One Color
The Art Gallery Theorem

Theorem (Vašek Chvátal 1975; Steve Fisk 1978)

For a museum (or art gallery) which is modelled as a simple polygon with \( n \) vertices at most \( \lfloor \frac{n}{3} \rfloor \) guards suffice.

Here \( n = 23 \), \( \lfloor \frac{n}{3} \rfloor = 7 \), and 5 guards suffice.
Three Colors for Vertices & Two Colors for Triangles
Example: Neil’s Parabola

Wanted: all points \((x, y)\) such that

\[ x^3 - y^2 = 0 \]

- \(x = 0, y = 0:\)
  \[ 0^3 - 0^2 = 0 - 0 = 0 \]

- \(x = 1, y = -1:\)
  \[ 1^3 - (-1)^2 = 1 - 1 = 0 \]

- \(x = \sqrt[3]{3} \approx 1.4422, y = \sqrt{3} \approx 1.7321:\)
  \[ (\sqrt[3]{3})^3 - (\sqrt{3})^2 = 3 - 3 = 0 \]
standard parabola

\[ x^2 - y \]

circle

\[ x^2 + y^2 - 1 \]

Neil’s parabola

\[ x^3 - y^2 \]

deltoid

\[ (x^2 + y^2)^2 + 18(x^2 + y^2) - 27 \]

cardiod

\[ (x^2 + y^2 - 1)^2 - 4((x - 1)^2 + y^2) \]
Newton: Curves, Lexicon Technicum, London (1710)

Source: Google books
The Recipe
How to create a curve from a (special kind of) triangulation

• take a convex lattice polygon $P$
• triangulate, using all interior lattice points
• suppose that these vertices can be 3-colored such that the two vertices of each edge receive distinct colors
• replace lattice point $(i, j)$ by $x^i y^j$
• pick one real number per color
  • e.g., -12, 7, -30
• add up to form polynomial
  \[-12(1 + x^3 + xy + y^3) + 7(x + x^2 y + y^2) - 30(x^2 + y + xy^2)\]
Two Curves with Three Points of Intersection

\[-12(1 + x^3 + xy + y^3) + 7(x + x^2y + y^2) - 30(x^2 + y + xy^2) = 0\]

\[490(1 + x^3 + xy + y^3) - 890(x + x^2y + y^2) + 20(x^2 + y + xy^2) = 0\]

number of white triangles - number of black triangles = 3

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Lower Bounds for the Number of Points of Intersection

Theorem (Soprunova & Sottile, 2006)

Let $C$ and $D$ be two curves constructed from a triangulation $\Delta$ according to the recipe. Then $C$ and $D$ intersect in at least $\sigma(\Delta)$ many points, ... provided that certain additional conditions are satisfied.

$$\sigma(\Delta) = |\#(\text{white triangles}) - \#(\text{black triangles})|$$
Chek Lap Kok Floor Pattern

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Chocolate
What Comes Next?

triangulation in 3 dimensions

secondary fan of $C_4 \ast C_5$


The “Additional Conditions”

- \( P \subset \mathbb{R}^{d}_{\geq 0} \): lattice \( d \)-polytope with \( N \) lattice points, \( \Delta \) induced by height function \( \lambda \) generic coefficients

\[
\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [t^v \mid v \in P \cap \mathbb{Z}^d],
\]

- toric variety \( X_P = (\text{Zariski}) \) closure of image
- real part \( Y_P = X_P \cap \mathbb{RP}^{N-1} \), lift \( Y_P^+ \) to \( S^{N-1} \) must be oriented
- \( s \)-deformation \( s.Y_P \) (for \( s \in (0,1] \)) = closure of the image of

\[
s.\phi_P : (\mathbb{C}^\times)^d \to \mathbb{CP}^{N-1} : t \mapsto [s^{\lambda(v)} t^v \mid v \in P \cap \mathbb{Z}^d]
\]

- Wronski projection

\[
\mathbb{CP}^{N-1} \setminus E \to \mathbb{CP}^d
\]

\[
\pi : [x_v \mid v \in P \cap \mathbb{Z}^d] \mapsto [\sum_{v \in c^{-1}(i)} x_v \mid i = 0, 1, \ldots, d]
\]

must avoid

\[
E = \left\{ x \in \mathbb{CP}^{N-1} \mid \sum_{v \in c^{-1}(i)} x_v = 0 \quad \text{for} \quad i = 0, 1, \ldots, d \right\}
\]