Tropical Combinatorics

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Overview

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   The tropical semi-ring
   Polyhedral combinatorics
   Puiseux series

2 Tropical Convexity
   Tropical polytopes
   Covector decompositions
   Products of simplices and mixed subdivisions

3 Tropical Linear Programming
   Tropical polyhedra
   The interior point method for ordinary LPs
   The tropical central path
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Tropical Arithmetic

tropical semi-ring: \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\) where

\[x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y\]

Example

\[(3 \oplus 5) \odot 2 = 3 + 2 = 5 = \min(5, 7) = (3 \odot 2) \oplus (5 \odot 2)\]

History

- can be traced back (at least) to the 1960s
  - e.g., see monography [Cunningham-Green 1979]
- optimization, functional analysis, signal processing, . . .
- recent development (since 2002) initiated by Kapranov, Mikhalkin, Sturmfels, . . .
Tropical Polynomials

- read ordinary (Laurent) polynomial with real coefficients as function
- replace operations “+” and “·” by “⊕” and “⊙”

Example

\[ F(x) = (3 ⊙ x ⊙ 3) ⊕ (1 ⊙ x ⊙ 2) ⊕ (2 ⊙ x) ⊕ 4 \]
\[ = \min(3 + 3x, 1 + 2x, 2 + x, 4) \]

- tropical polynomial \( F \) vanishes at \( p \) \( \iff \) there are at least two terms where the minimum \( F(p) \) is attained

Example

\[ F(1) = \min(3 + 3, [1+2], [2+1], 4) = 3 \]
Tropical Hypersurfaces

- *tropical semi-module* \((\mathbb{R}^d, \oplus, \odot)\)
  - componentwise tropical addition
  - *tropical scalar multiplication*

- *tropical hypersurface* \(\mathcal{T}(F) :=\) vanishing locus of (multi-variate) tropical polynomial \(F\)

**Example**

\[
F(x) = (3 \odot x^3) \oplus (1 \odot x^2) \oplus (2 \odot x) \oplus 4
\]

\[\mathcal{T}(F) = \{-2, 1, 2\} \subset \mathbb{R}^1\]
Proposition

For a tropical polynomial $F : \mathbb{R}^d \to \mathbb{R}$ the dome

$$D(F) := \left\{ (p, s) \in \mathbb{R}^{d+1} \mid p \in \mathbb{R}^d, s \in \mathbb{R}, s \leq F(p) \right\}$$

is an unbounded convex polyhedron of dimension $d + 1$.

Corollary

The tropical hypersurface $T(F)$ coincides with the image of the codimension-2-skeleton of the polyhedron $D(F)$ in $\mathbb{R}^d$ under the orthogonal projection which omits the last coordinate.
The Extended Newton Polyhedron

- extended Newton polyhedron $\tilde{N}(F) = \text{convex hull of the support}$
  $\text{supp}(F)$ lifted by coefficients $+$ upwards ray

**Theorem**

*Tropical hypersurface $\mathcal{T}(F)$ is dual to the 1-coskeleton of $\tilde{N}(F)$.*
The Tropical Torus

tropical polynomial $F$ homogeneous of degree $\delta$ if for all $p \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$:

$$F(\lambda \odot p) = F(\lambda \cdot \mathbf{1} + p) = \lambda^{\odot \delta} \odot F(p) = \delta \cdot \lambda + F(p)$$

**Definition**

tropical $(d-1)$-torus $\mathbb{R}^d / \mathbb{R}^1$

map

$$(x_1, x_2, \ldots, x_d) + \mathbb{R}^1 = (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}^1$$

$$\mapsto (x_2 - x_1, \ldots, x_d - x_1)$$

defines homeomorphism $\mathbb{R}^d / \mathbb{R}^1 \approx \mathbb{R}^{d-1}$
Tropical Hyperplanes

\[ F(x) = (\alpha_1 \odot x_1) \oplus (\alpha_2 \odot x_2) \oplus (\alpha_3 \odot x_3) \] linear homogeneous

\[ \mathcal{T}(F) = -(\alpha_1, \alpha_2, \alpha_3) + (\mathbb{R}_{\geq 0}e_1 \cup \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_3) + \mathbb{R}1 \]

\[ = (0, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3) + (\mathbb{R}_{\geq 0}(-e_2 - e_3) \cup \mathbb{R}_{\geq 0}e_2 \cup \mathbb{R}_{\geq 0}e_3) \]
Tropical Conics

general tropical conic

\[(a_{200} \odot x_1^{2}) \oplus (a_{110} \odot x_1 \odot x_2) \oplus (a_{101} \odot x_1 \odot x_3) \oplus (a_{020} \odot x_2^{2}) \oplus (a_{011} \odot x_2 \odot x_3) \oplus (a_{002} \odot x_3^{2})\]

Example

\[(a_{200}, a_{110}, a_{101}, a_{020}, a_{011}, a_{002}) = (6, 5, 5, 6, 5, 7)\]
Max-Tropical Hyperplanes

duality between min and max:

\[ \max(-x, -y) = -\min(x, y) \]

Remark

\( \mathcal{T} \) is min-trop. hypersurface \( \iff \) \( -\mathcal{T} \) is max-trop. hypersurface
Puiseux series with complex coefficients:

\[
\mathbb{C}\{\{t\}\} = \left\{ \sum_{k=m}^{\infty} a_k \cdot t^{k/N} \bigg| m \in \mathbb{Z}, N \in \mathbb{N}^\times, a_k \in \mathbb{C} \right\}
\]

- Newton-Puiseux-Theorem: \( \mathbb{C}\{\{t\}\} \) is isomorphic to the algebraic closure of Laurent series \( \mathbb{C}((t)) \)
  - isomorphic to \( \mathbb{C} \) by [Steinitz 1910]
The Valuation Map

valuation map

\[ \text{val} : \mathbb{C}\{\{t\}\} \to \mathbb{Q} \cup \{\infty\} \]

maps Puiseux series \( \gamma(t) = \sum_{k=m}^{\infty} a_k \cdot t^{k/N} \) to lowest degree
\[ \min \{ k/N \mid k \in \mathbb{Z}, a_k \neq 0 \} \]; setting \( \text{val}(0) := \infty \)

\[
\begin{align*}
\text{val}(\gamma(t) + \delta(t)) & \geq \min\{\text{val}(\gamma(t)), \text{val}(\delta(t))\} \\
\text{val}(\gamma(t) \cdot \delta(t)) & = \text{val}(\gamma(t)) + \text{val}(\delta(t)).
\end{align*}
\]

Remark

inequality becomes equation if no cancellation occurs
A Lifting Theorem I

Theorem (Einsiedler, Kapranov & Lind 2006)

For \( f \in \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}] \) the tropical hypersurface \( T(\text{trop}(f)) \cap \mathbb{Q}^d \) (over the rationals) equals the set \( \text{val}(V(\langle f \rangle)) \).

“Tropical geometry is a piece-wise linear shadow of classical geometry.”
A Lifting Theorem II

Proof of easy inclusion "\( \mathcal{T}(\text{trop}(f)) \supseteq \text{val}(V(\langle f \rangle)) \)".

- let \( f = \sum_{i \in I} \gamma_i x^i \) for \( I \subset \mathbb{N}^d \) with tropicalization \( F \)
- consider zero \( u \in (\mathbb{K}^\times)^d \) of \( f \)
- for \( i \in I \) we have \( \text{val}(\gamma_i u^i) = \text{val}(\gamma_i) + \langle i, \text{val}(u) \rangle = \text{val}(\gamma_i) \odot \text{val}(u)^\odot_i \)
- minimum

\[
F(\text{val}(u)) = \bigoplus_{i \in I} \text{val}(\gamma_i) \odot \text{val}(u)^\odot_i
\]

attained at least twice since otherwise the terms \( \gamma_i u^i \) cannot cancel to yield zero

\[ \square \]
Example

Consider $f(x) = t^3x^3 - (t + t^4 + t^5)x^2 + (t^2 + t^3 + t^6)x - t^4$.
This factors as

$$f(x) = (x - t^{-2}) \cdot (x - t) \cdot (x - t^2) \cdot t^3.$$ 

The tropicalization $F = \text{trop}(f)$ reads

$$F(x) = (3 \circledast x^{\circ3}) \oplus (1 \circledast x^{\circ2}) \oplus (2 \circledast x) \oplus 4 = \min(3 + 3x, 1 + 2x, 2 + x, 4).$$

$$\mathcal{T}(F) = \{-2, 1, 2\}$$
Conclusion I

- tropicalization of (homogeneous) polynomial $F$
- tropical hypersurface $\mathcal{T}(F)$
  - codimension-2-skeleton of unbounded convex polyhedron
  - extended Newton polyhedron $\tilde{\mathcal{N}}(F)$
- tropical hypersurface = image of ordinary hypersurface under valuation map
for $x, y \in \mathbb{T}^d$ let

$$[x, y]_{\text{trop}} := \{ (\lambda \odot x) \oplus (\mu \odot y) \mid \lambda, \mu \in \mathbb{R} \}$$

- $S \subseteq \mathbb{T}^d$ tropically convex: $[x, y]_{\text{trop}} \subseteq S$ for all $x, y \in S$
- $S$ tropically convex $\Rightarrow \lambda \odot S = \lambda \mathbf{1} + S \subseteq S$ for all $\lambda \in \mathbb{R}$
  - consider tropically convex sets in $\text{TTP}^{d-1} = (\mathbb{T}^d \setminus \{\infty \mathbf{1}\})/\mathbb{R}\mathbf{1}$
  - recall: we identify

$$(x_0, x_1, \ldots, x_d) + \mathbb{R}\mathbf{1} = (0, x_1 - x_0, \ldots, x_d - x_0) + \mathbb{R}\mathbf{1}$$

with $(x_1 - x_0, \ldots, x_d - x_0)$
- tropical polytope := tropical convex hull of finitely many points in $\text{TTP}^{d-1} \supset \mathbb{R}^d/\mathbb{R}\mathbf{1} \approx \mathbb{R}^{d-1}$
Example: Tropical Line Segment in $\mathbb{R}^3/\mathbb{R}1$

\[\{(0, 2, 0), (0, -2, -2)\}_{trop}\]

\[= \{ \lambda \odot (0, 2, 0) \oplus \mu \odot (0, -2, -2) \mid \lambda, \mu \in \mathbb{R}\}\]

\[= \{ (\min(\lambda, \mu), \min(\lambda + 2, \mu - 2), \min(\lambda, \mu - 2)) \}\}

\[= \{ (\lambda, \lambda + 2, \lambda) \mid \lambda \leq \mu - 4\}\]

\[\cup \{ (\lambda, \mu - 2, \lambda) \mid \mu - 4 \leq \lambda \leq \mu - 2\}\]

\[\cup \{ (\lambda, \mu - 2, \mu - 2) \mid \mu - 2 \leq \lambda \leq \mu\}\]

\[\cup \{ (\mu, \mu - 2, \mu - 2) \mid \mu \leq \lambda\}\]

\[= \{ (0, \mu - \lambda - 2, 0) \mid 2 \leq \mu - \lambda \leq 4\}\]

\[\cup \{ (0, \mu - \lambda - 2, \mu - \lambda - 2) \mid 0 \leq \mu - \lambda \leq 2\}\]

Case Distinction

\[\lambda \in (-\infty, \mu - 4] \cup [\mu - 4, \mu - 2] \cup [\mu - 2, \mu] \cup [\mu, \infty)\]
The Running Example

\( n = 4, \ d = 3 \)
\( v_1 = (0, 1, 0)^T, \ v_2 = (0, 4, 1)^T, \ v_3 = (0, 3, 3)^T, \ v_4 = (0, 0, 2)^T \)
Covectors

consider $V \in \mathbb{T}^{d \times n}$ (and read columns as points in $\mathbb{TP}^{d-1}$)

**Definition**

covector of $p \in \mathbb{R}^d/\mathbb{R}1$ w.r.t. $V$ given by $T_V(p) = (T_1, T_2, \ldots, T_d)$ with

$$k \in T_i \iff i \in \text{argmin} \{ j \in [d] \mid v_{jk} - p_j \}$$

$$\iff i \in \text{argmax} \{ j \in [d] \mid p_j - v_{jk} \}$$

**Example**

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 1 & 3 & 2 \end{pmatrix} \quad T_V \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = (\{2, 3\}, \{1, 4\}, \emptyset)$$
Covector Decomposition of $\mathbb{R}^d/\mathbb{R}^1$ ... induced by max-tropical hyperplane arrangement $\mathcal{A}(V)$
Recall: Max-Tropical Hyperplanes

duality between min and max:

$$\max(-x, -y) = -\min(x, y)$$

Remark

\(\mathcal{T}\) is min-trop. hypersurface \(\iff -\mathcal{T}\) is max-trop. hypersurface
Theorem (Develin & Sturmfels 2004; Fink & Rincón 2015; J. & Loho 2016)

The covector decomposition $\mathcal{T}(V)$ of $\mathbb{R}^d$ induced by $V \in \mathbb{T}^{d \times n}$

1 is dual to a regular subdivision of

$$\text{conv}\left\{ (e_i, e_j) \in \mathbb{R}^d \times \mathbb{R}^n \mid v_{ij} \neq \infty \right\} ,$$

2 and it induces a polyhedral decomposition of $\text{tconv}(V)$. 
Covector Decomposition of Standard Example
Products of Simplices and Their Subpolytopes

- \( \text{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{R}^d/\mathbb{R}1 \) dual to regular subdivision of \( \Delta_{d-1} \times \Delta_{n-1} \) defined by lifting \( e_i \times e_j \) to height \( v_{ij} \)
  - general position \( \Longleftrightarrow \) triangulation
- Lifting vertices to \( \infty \) defines subpolytope (on remaining vertices)
- Extra feature from swapping factors \( \leadsto \) tconv(rows) \( \cong \) tconv(columns)

\[ \begin{array}{c}
\Delta_2 \times \Delta_1 \\
\text{tconv(2 points in } \mathbb{R}^3/\mathbb{R}1) \\
\end{array} \]
Mixed Subdivisions

• $P, Q$ : polytopes in $\mathbb{R}^d$
• $P + Q = \{ p + q \mid p \in P, q \in Q \}$ Minkowski sum
• Minkowski cell of $P + Q = \text{full-dimensional subpolytope which is Minkowski sum of subpolytopes of } P \text{ and } Q$

Definition

Polytopal subdivision of $P + Q$ into Minkowski cells is mixed if for any two of its cells $P' + Q'$ and $P'' + Q''$ the intersections $P' \cap P''$ and $Q' \cap Q''$ both are faces.

• fine = cannot be refined (as a mixed subdivision!)
• can be generalized to finitely many summands
Example With 4 Summands

fine mixed subdivision of \textit{dilated simplex} $\Delta_2 + \Delta_2 + \Delta_2 + \Delta_2 = 4\Delta_2$
Cayley Trick, General Form

- $e_1, e_2, \ldots, e_n$ : affine basis of $\mathbb{R}^{n-1}$
- $\phi_k : \mathbb{R}^d \rightarrow \mathbb{R}^{n-1} \times \mathbb{R}^d$ embedding $p \mapsto (e_k, p)$
- Cayley embedding of $P_1, P_2, \ldots, P_n$:

$$C(P_1, P_2, \ldots, P_n) = \text{conv} \bigcup_{i=1}^{n} \phi_i(P_i).$$

Theorem (Sturmfels 1994; Huber, Rambau & Santos 2000)

1. For any polyhedral subdivision of $C(P_1, P_2, \ldots, P_n)$ the intersection of its cells with $\{\frac{1}{n} \sum e_i\} \times \mathbb{R}^d$ yields a mixed subdivision of $\frac{1}{n} \sum P_i$.
2. This correspondence is a poset isomorphism from the subdivisions of $C(P_1, P_2, \ldots, P_n)$ to the mixed subdivisions of $\sum P_i$. Further, the coherent mixed subdivisions are bijectively mapped to the regular subdivisions.
Cayley Trick for Products of Simplices

- consider \( P_1 = P_2 = \cdots = P_n = \Delta_{d-1} = \text{conv}\{e_1, e_2, \ldots, e_d\} \)
- \( C(\Delta_{d-1}, \Delta_{d-1}, \ldots, \Delta_{d-1}) \cong \Delta_{d-1} \times \Delta_{n-1} \)

**Corollary**

1. For any polyhedral subdivision of \( \Delta_{d-1} \times \Delta_{n-1} \) the intersection of its cells with \( \left\{ \frac{1}{n} \sum e_i \right\} \times \mathbb{R}^d \) yields a mixed subdivision of \( \frac{1}{n} \cdot (n\Delta_{d-1}) \).
2. This correspondence is a poset isomorphism from the subdivisions of \( \Delta_{d-1} \times \Delta_{n-1} \) to the mixed subdivisions of \( n\Delta_{d-1} \). Further, the coherent mixed subdivisions are bijectively mapped to the regular subdivisions.
• (fine) covectors $\leadsto$ coarse covectors
  • replace sets $T_k$ by their cardinality
• coarse covectors of maximal cells = vertex coordinates of mixed subdivision
A Tropical Proof of the Cayley Trick . . .
for products of simplices

- point \( v_i \in \mathbb{T}^{d-1} \) = apex of unique max-tropical hyperplane \( H^\text{max}(v_i) \)
- homogeneous linear form \( h_i \in \mathbb{C}\{\{t\}\}[x_1, x_2, \ldots, x_d] \);
  \[
  h := h_1 \cdot h_2 \cdots h_n
  \]

**Proposition**
The tropical hypersurface defined by \( \text{trop}^\text{max}(h) \) is the union of the max-tropical hyperplanes in \( \mathcal{A}(V) \).

- dual subdivision of Newton polytope \( n\Delta_{d-1} \)

**Corollary**
Let \( p \in \mathbb{T}^{d-1} \setminus \mathcal{A}(V) \) be a generic point. Then its coarse covector \( t_V(p) \) equals the exponent of the monomial in \( h \) which defines the unique facet of \( \mathcal{D}(\text{trop}^\text{max}(h)) \) above \( p \).
Conclusion II

- configuration of \( n \) points in \( \mathbb{T}P^{d-1}_{\min} \) corresponds to arrangement of \( n \) tropical hyperplanes in \( \mathbb{T}P^{d-1}_{\max} \)
  - tropical polytope = union of bounded cells (for finite coordinates)
- covector decomposition dual to regular subdivision of subpolytope \( \Delta_{n-1} \times \Delta_{d-1} \)
- tropical proof of special case of Cayley Trick
What is a Tropical Linear Program?

An ordinary linear program is an optimization problem like

\[
\begin{align*}
\text{minimize} & \quad c^\top x \\
\text{s.t.} & \quad Ax \geq b \\
& \quad x \in \mathbb{R}^n
\end{align*}
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n \).

Definition

A tropical linear program \( \text{LP}(A, b, c) \) is an optimization problem like

\[
\begin{align*}
\text{minimize} & \quad c^\top \circ x \\
\text{s.t.} & \quad A^+ \circ x \oplus b^+ \geq A^- \circ x \oplus b^- \\
& \quad x \in \mathbb{T}^n
\end{align*}
\]

where \( A^\pm \in \mathbb{T}^{m \times n}, b^\pm \in \mathbb{T}^m, c \in \mathbb{T}^n \).
Min-max optimization over tropical polyhedra

Beware: now $\oplus = \max$

- feasible set defined by

$$A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^-$$

is a tropical polyhedron; denoted $P(A, b)$

- each defining inequality corresponds to a tropical half-space

- level sets have apices, located on the line $(-c) + \mathbb{R} 1$

- optimal solution(s) form tropical polyhedron, too

minimize $\max(-1 + x_1, x_2)$

$$\begin{cases}
\max(x_1 - 5, x_2 - 2) \geq 0 \\
0 \geq \max(x_1 - 8, x_2 - 6) \\
x_1 - 2 \geq \max(x_2 - 5, 0) \\
\max(x_2 - 4, 0) \geq x_1 - 7 \\
x_2 \geq 1
\end{cases}$$
Fact sheet: Tropical polyhedra

- can also be represented in terms of vertices and rays
- tropical polytopes special case of tropical polyhedron defined by homogeneous tropical inequalities $A^+ \circ x \geq A^- \circ x$
  - arbitrary tropical polyhedra can be homogenized
- tropical linear programming [Butković & Aminu 2008]
- tropical fractional linear programming [Gaubert, Katz & Sergeev 2012]
- tropical LP feasibility equivalent to mean payoff games [Akian, Gaubert & Gutermann 2012]
Main Lemma of Tropical Linear Programming
where $\mathbb{K}$ is some field of real Puiseux series

Let $\mathcal{P} = \{ x \in \mathbb{K}^n \mid Ax + b \geq 0 \}$ be contained in $\mathbb{K}^n_{\geq 0}$.

Lemma (Develin & Yu 2007; ABGJ 2015)

If tropicalization of $(A, b)$ is sign generic then

$$\text{val}(\mathcal{P}) = \{ x \in \text{trop}^n \mid A^+ \circ x \oplus b^+ \geq A^- \circ x \oplus b^- \} ,$$

where $(A^+ b^+) = \text{val}(A^+ b^+)$ and $(A^- b^-) = \text{val}(A^- b^-)$.

Moreover, for any $I \subset [m]$, we have:

$$\text{val} \left( \{ x \in \mathcal{P} \mid A_I x + b_I = 0 \} \right) = \{ x \in \text{val}(\mathcal{P}) \mid A_i^+ \circ x \oplus b_i^+ = A_i^- \circ x \oplus b_i^- \} .$$

where $(A_I b_I)$ submatrix of $(A b)$ formed by rows with indices in $I$. 
The Interior Point Method of Linear Programming
[von Neumann] [Karmarkar 1984]

- start at analytic center
- trace central path by solving auxiliary (non-linear) optimization problems via Newton’s method
- optimality characterized by Karush–Kuhn–Tucker conditions

- Karmarkar 1984: polynomial time algorithm
- method depends on barrier function
  - no STRONGLY polynomial time algorithm known for LP
  - Smale’s 9th problem
Fact Sheet: Interior Point Method

- method depends on barrier function
  - no STRONGLY polynomial time algorithm known
- Karmarkar 1984: polynomial time algorithm
  - Khachiyan 1979: ellipsoid method
- Nesterov & Nemirovski 1994: generalization to non-linear convex programming

Conjecture (Deza, Terlaky and Zinchenko (2008))

The total curvature of the central path is bounded by $O(n)$.

"Continuous Hirsch Conjecture"

- Dedieu, Malajovich & Shub 2005: true “on the average”
- De Loera, Sturmfels & Vinzant 2012: similar result
- disproved by Allamigeon, Benchimol, Gaubert & J. 2014+
Long and Winding Central Paths

Theorem (Allamigeon, Benchimol, Gaubert & J. 2014+)

There is a family of ordinary linear programs with $m = 3r + 4$ linear inequalities in $n = 2r + 2$ variables such that the total curvature of the central path is at least $\Omega(2^r)$.

• counter-example to the “Continuous Hirsch Conjecture” of Deza, Terlaky and Zinchenko (2008)

• Smale’s 9th problem
Interior Point Method: Our Setup

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, $\mu > 0$.

**primal** linear program:

\[
\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & Ax \leq b, \ x \geq 0, \ x \in \mathbb{R}^n \\
\end{aligned}
\]

\[\text{LP}(A, b, c)\]

**dual** linear program:

\[
\begin{aligned}
\text{maximize} \quad & -b^\top y \\
\text{subject to} \quad & -A^\top y \leq c, \ y \geq 0, \ y \in \mathbb{R}^m \\
\end{aligned}
\]

**associated logarithmic barrier problem**:

\[
\begin{aligned}
\text{minimize} \quad & \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\
\text{subject to} \quad & Ax + w = b, \ x > 0, \ w > 0 \\
\end{aligned}
\]
A System of Polynomial Equations

logarithmic barrier problem

minimize $\frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i)$

subject to $Ax + w = b$, $x > 0$, $w > 0$

for $\mu > 0$ has unique optimal solution $(x^{\mu}, w^{\mu})$ characterized by

$$Ax + w = b$$

$$-A^\top y + s = c$$

$$w_i y_i = \mu \quad \text{for all } i \in [m]$$

$$x_j s_j = \mu \quad \text{for all } j \in [n]$$

$x, w, y, s > 0$

That is, there uniquely exist $y^{\mu}$ and $s^{\mu}$ such that $(x^{\mu}, w^{\mu}, y^{\mu}, s^{\mu})$ is a solution ...
The Central Path

Definition

The *central path* is the image of the map

\[ C_{A,b,c} : \mathbb{R}_{>0} \rightarrow \mathbb{R}^{2m+2n}, \quad \mu \mapsto (x^{\mu}, w^{\mu}, y^{\mu}, s^{\mu}) . \]

- *primal central path* = projection onto \(x\)-coordinates
- *dual central path* = projection onto \(y\)-coordinates

Conjecture (Deza, Terlaky & Zinchenko 2008)

The total curvature of the primal central path is at most \(O(m)\).

- Dedieu, Malajovich & Shub 2005: \(O(n)\) holds on the average
- De Loera, Sturmfels & Vinzant 2012: similar result via matroid theory
A Simple Example . . .

Consider the Puiseux polyhedron \( \mathcal{P} \subset \mathbb{K}^2 \) defined by:

\[
\begin{align*}
    x_1 + x_2 & \leq 2 \\
    tx_1 & \leq 1 + t^2 x_2 \\
    tx_2 & \leq 1 + t^3 x_1 \\
    x_1 & \leq t^2 x_2 \\
    x_1, x_2 & \geq 0 .
\end{align*}
\] (1)

Then the set \( \text{val}(\mathcal{P}) \) is described by the tropical linear inequalities:

\[
\begin{align*}
    \max(x_1, x_2) & \leq 0 \\
    1 + x_1 & \leq \max(0, 2 + x_2) \\
    1 + x_2 & \leq \max(0, 3 + x_1) \\
    x_1 & \leq 2 + x_2 .
\end{align*}
\] (2)
... and Two of Its Primal Tropical Central Paths

\[
\begin{align*}
\min x_1 \\
\min tx_1 + x_2
\end{align*}
\]
A Family of Linear Programs

... with $2r + 2$ variables $u_0, v_0, u_1, v_1, \ldots, u_r, v_r$ and $3r + 4$ inequalities:

$$
\begin{align*}
\text{min} & \quad v_0 \\
\text{s.t.} & \quad u_0 \leq t \\
& \quad v_0 \leq t^2 \\
& \quad v_i \leq t^{1 - \frac{1}{2^i}} (u_{i-1} + v_{i-1}) \quad \text{for } i \in [r] \\
& \quad u_i \leq tu_{i-1} \quad \text{for } i \in [r] \\
& \quad u_i \leq tv_{i-1} \quad \text{for } i \in [r] \\
& \quad u_r \geq 0, \ v_r \geq 0
\end{align*}
$$

depending on a real parameter $t > 0$

primal central path has total curvature at least $\Omega(2^r)$ for $t \gg 0$
The Primal Tropical Central Paths of Our Examples
lifting a construction by Bezem, Nieuwenhuis and Rodríguez-Carbonell, 2008

total curvature $\Omega(2')$

- left: $r = 2$
- below: $r = 1$
Conclusion III

- Tropical geometry yields new results for classical linear programs.
- In specific situations, it is possible to derive metric information from tropicalization.
- Tropical linear programs are interesting from a computational complexity perspective.