Tropical Combinatorics

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Overview

1. Tropical Hypersurfaces
   - The tropical semi-ring
   - Polyhedral combinatorics
   - Puiseux series
2. Tropical Convexity
   - Tropical polytopes
   - Covector decompositions
   - Products of simplices and mixed subdivisions
3. Tropical Linear Programming
   - Tropical polyhedra
   - The interior point method for ordinary LPs
   - The tropical central path
   - Long and winding central paths

Tropical Arithmetic

tropical semi-ring: $(\underbrace{\mathbb{R} \cup \{\infty\}}_{\mathbb{T}_{\min}}, \oplus, \odot)$ where

$$x \oplus y \;:=\; \min(x, y) \quad \text{and} \quad x \odot y \;:=\; x + y$$

Example

$$(3 \oplus 5) \odot 2 \;=\; 3 + 2 \;=\; 5 \;=\; \min(5, 7) = (3 \odot 2) \oplus (5 \odot 2)$$

History

- can be traced back (at least) to the 1960s
  - e.g., see monography [Cunningham-Green 1979]
- optimization, functional analysis, signal processing, ...
- recent development (since 2002) initiated by Kapranov, Mikhalkin, Sturmfels, ...

Tropical Polynomials

- read ordinary (Laurent) polynomial with real coefficients as function
- replace operations "$+$" and "$\cdot$" by "$\oplus$" and "$\odot$"

Example

$$F(x) = (3 \odot x^{\odot 3}) \oplus (1 \odot x^{\odot 2}) \oplus (2 \odot x) \oplus 4$$
$$= \min(3 + 3x, 1 + 2x, 2 + x, 4)$$

- tropical polynomial F vanishes at p $:\Leftrightarrow$ there are at least two terms where the minimum $F(p)$ is attained

Example

$$F(1) \;=\; \min(3 + 3, \boxed{1+2}, \boxed{2+1}, 4) \;=\; 3$$

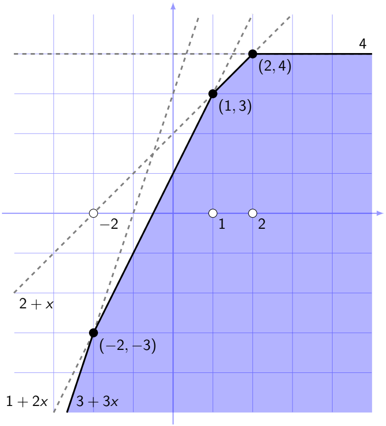
Tropical Hypersurfaces

- *tropical semi-module* $(\mathbb{R}^d, \oplus, \odot)$
  - componentwise tropical addition
  - *tropical scalar multiplication*
- tropical hypersurface $\mathcal{T}(F)$:= vanishing locus of (multi-variate) tropical polynomial F

Example

$F(x) = (3 \odot x^{\odot 3}) \oplus (1 \odot x^{\odot 2}) \oplus (2 \odot x) \oplus 4$

$\mathcal{T}(F) = \{-2, 1, 2\} \subset \mathbb{R}^1$



Polyhedral Combinatorics

Proposition

For a tropical polynomial $F : \mathbb{R}^d \to \mathbb{R}$ the dome

$$\mathcal{D}(F) \;:=\; \left\{ (p,s) \in \mathbb{R}^{d+1} \;\middle|\; p \in \mathbb{R}^d,\, s \in \mathbb{R},\, s \leq F(p) \right\}$$

is an unbounded convex polyhedron of dimension $d+1$.

Corollary

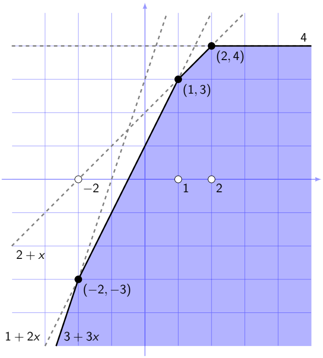
The tropical hypersurface $\mathcal{T}(F)$ coincides with the image of the codimension-2-skeleton of the polyhedron $\mathcal{D}(F)$ in $\mathbb{R}^d$ under the orthogonal projection which omits the last coordinate.

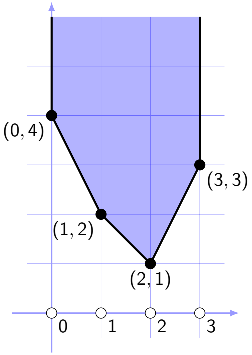
The Extended Newton Polyhedron

- extended Newton polyhedron $\widetilde{\mathcal{N}}(F)$ = convex hull of the support supp(F) lifted by coefficients + upwards ray

Theorem

Tropical hypersurface $\mathcal{T}(F)$ is dual to the 1-coskeleton of $\widetilde{\mathcal{N}}(F)$.





The Tropical Torus

tropical polynomial F *homogeneous of degree* δ if for all $p \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$:

$$F(\lambda \odot p) \;=\; F(\lambda \cdot \mathbf{1} + p) \;=\; \lambda^{\odot \delta} \odot F(p) \;=\; \delta \cdot \lambda + F(p)$$

Definition

tropical $(d-1)$-torus $\mathbb{R}^d / \mathbb{R}\mathbf{1}$

map

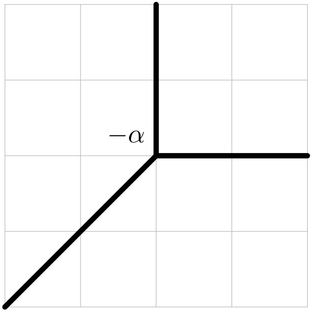
$$\begin{aligned} (x_1, x_2, \ldots, x_d) + \mathbb{R}\mathbf{1} &= (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}\mathbf{1} \\ &\mapsto (x_2 - x_1, \ldots, x_d - x_1) \end{aligned}$$

defines homeomorphism $\mathbb{R}^d / \mathbb{R}\mathbf{1} \approx \mathbb{R}^{d-1}$

Tropical Hyperplanes

$F(x) = (\alpha_1 \odot x_1) \oplus (\alpha_2 \odot x_2) \oplus (\alpha_3 \odot x_3)$ linear homogeneous

$$\begin{aligned} \mathcal{T}(F) &= -(\alpha_1, \alpha_2, \alpha_3) + (\mathbb{R}_{\geq 0} e_1 \cup \mathbb{R}_{\geq 0} e_2 \cup \mathbb{R}_{\geq 0} e_3) + \mathbb{R}\mathbf{1} \\ &= (0, \alpha_1 - \alpha_2, \alpha_1 - \alpha_3) + (\mathbb{R}_{\geq 0}(-e_2 - e_3) \cup \mathbb{R}_{\geq 0} e_2 \cup \mathbb{R}_{\geq 0} e_3) \end{aligned}$$



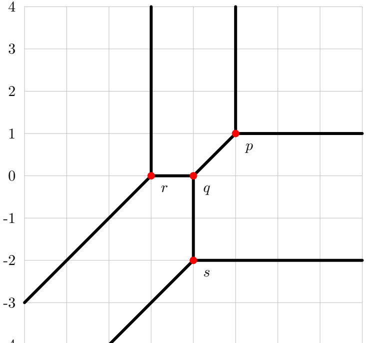
Tropical Conics

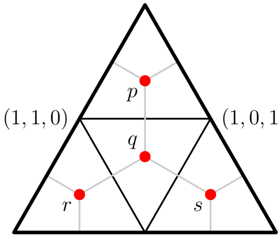
general tropical conic

$$(a_{200} \odot x_1^{\odot 2}) \oplus (a_{110} \odot x_1 \odot x_2) \oplus (a_{101} \odot x_1 \odot x_3) \oplus (a_{020} \odot x_2^{\odot 2}) \oplus (a_{011} \odot x_2 \odot x_3) \oplus (a_{002} \odot x_3^{\odot 2})$$

Example

$$(a_{200}, a_{110}, a_{101}, a_{020}, a_{011}, a_{002}) = (6, 5, 5, 6, 5, 7)$$





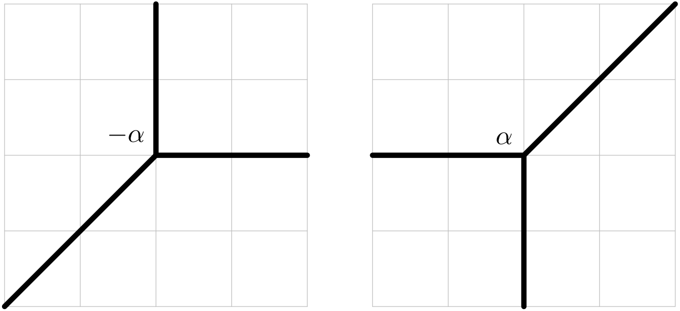
Max-Tropical Hyperplanes

duality between min and max:

$$\max(-x,-y) \;=\; -\min(x,y)$$

Remark

$\mathcal{T}$ is min-trop. hypersurface $\iff -\mathcal{T}$ is max-trop. hypersurface



min/max

Fields of Puiseux Series

Puiseux series with complex coefficients:

$$\mathbb{C}\{\{t\}\} \;=\; \left\{ \sum_{k=m}^{\infty} a_k \cdot t^{k/N} \;\middle|\; m \in \mathbb{Z}, N \in \mathbb{N}^{\times}, a_k \in \mathbb{C} \right\}$$

- Newton-Puiseux-Theorem: $\mathbb{C}\{\{t\}\}$ isomorphic to algebraic closure of *Laurent series* $\mathbb{C}((t))$
  - isomorphic to $\mathbb{C}$ by [Steinitz 1910]

The Valuation Map

valuation map

$$\text{val} : \mathbb{C}\{\{t\}\} \to \mathbb{Q} \cup \{\infty\}$$

maps Puiseux series $\gamma(t) = \sum_{k=m}^{\infty} a_k \cdot t^{k/N}$ to lowest degree $\min\{k/N \mid k \in \mathbb{Z},\, a_k \neq 0\}$; setting $\text{val}(0) := \infty$

$$\begin{aligned} \text{val}(\gamma(t) + \delta(t)) &\geq \min\{\text{val}(\gamma(t)), \text{val}(\delta(t))\} \\ \text{val}(\gamma(t) \cdot \delta(t)) &= \text{val}(\gamma(t)) + \text{val}(\delta(t)) \,. \end{aligned}$$

Remark

inequality becomes equation if no cancellation occurs

A Lifting Theorem I

Theorem (Einsiedler, Kapranov & Lind 2006)

For $f \in \mathbb{K}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}]$ *the tropical hypersurface* $\mathcal{T}(\mathrm{trop}(f)) \cap \mathbb{Q}^d$ *(over the rationals) equals the set* $\mathrm{val}(V(\langle f \rangle))$.

"Tropical geometry is a piece-wise linear shadow of classical geometry."

A Lifting Theorem II

Proof of easy inclusion "$\mathcal{T}(\mathrm{trop}(f)) \supseteq \mathrm{val}(V(\langle f \rangle))$".

- let $f = \sum_{i \in I} \gamma_i x^i$ for $I \subset \mathbb{N}^d$ with tropicalization F
- consider zero $u \in (\mathbb{K}^\times)^d$ of f
- for $i \in I$ we have $\mathrm{val}(\gamma_i u^i) = \mathrm{val}(\gamma_i) + \langle i, \mathrm{val}(u) \rangle = \mathrm{val}(\gamma_i) \odot \mathrm{val}(u)^{\odot i}$
- minimum

$$F(\mathrm{val}(u)) \;=\; \bigoplus_{i \in I} \mathrm{val}(\gamma_i) \odot \mathrm{val}(u)^{\odot i}$$

  attained at least twice since otherwise the terms $\gamma_i u^i$ cannot cancel to yield zero

□

Example

Consider $f(x) = t^3x^3 - (t + t^4 + t^5)x^2 + (t^2 + t^3 + t^6)x - t^4$.
This factors as

$$f(x) \; = \; (x - t^{-2}) \cdot (x - t) \cdot (x - t^2) \cdot t^3 \; .$$

The tropicalization $F = \mathrm{trop}(f)$ reads

$$\begin{aligned} F(x) \; &= \; (3 \odot x^{\odot 3}) \oplus (1 \odot x^{\odot 2}) \oplus (2 \odot x) \oplus 4 \\ &= \; \min(3 + 3x, 1 + 2x, 2 + x, 4) \; . \end{aligned}$$

$$\mathcal{T}(F) \; = \; \{-2, 1, 2\}$$

Conclusion I

- tropicalization of (homogeneous) polynomial F
- tropical hypersurface $\mathcal{T}(F)$
  - codimension-2-skeleton of unbounded convex polyhedron
  - extended Newton polyhedron $\widetilde{\mathcal{N}}(F)$
- tropical hypersurface = image of ordinary hypersurface under valuation map

Tropical Convexity

[Zimmermann 1977] [Develin & Sturmfels 2004] [J. & Loho 2016] ...

for $x, y \in \mathbb{T}^d$ let

$$[x, y]_{\text{trop}} := \{(\lambda \odot x) \oplus (\mu \odot y) \mid \lambda, \mu \in \mathbb{R}\}$$

- $S \subseteq \mathbb{T}^d$ tropically convex: $[x, y]_{\text{trop}} \subseteq S$ for all $x, y \in S$
- S tropically convex $\Rightarrow \lambda \odot S = \lambda \mathbf{1} + S \subseteq S$ for all $\lambda \in \mathbb{R}$
  - consider tropically convex sets in $\mathbb{TP}^{d-1} = (\mathbb{T}^d \setminus \{\infty \mathbf{1}\})/\mathbb{R}\mathbf{1}$
  - recall: we identify

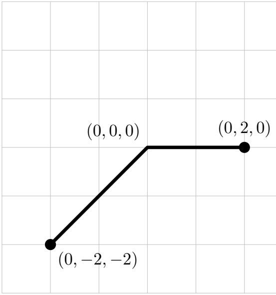
$$(x_0, x_1, \ldots, x_d) + \mathbb{R}\mathbf{1} = (0, x_1 - x_0, \ldots, x_d - x_0) + \mathbb{R}\mathbf{1}$$

  with $(x_1 - x_0, \ldots, x_d - x_0)$

- tropical polytope := *tropical convex hull* of finitely many points in $\mathbb{TP}^{d-1} \supset \mathbb{R}^d/\mathbb{R}\mathbf{1} \approx \mathbb{R}^{d-1}$

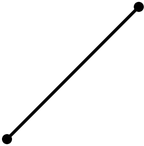
Example: Tropical Line Segment in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$

$$
\begin{aligned}
&[(0,2,0),(0,-2,-2)]_{\text{trop}} \\
&= \{\,\lambda \odot (0,2,0) \oplus \mu \odot (0,-2,-2) \mid \lambda,\mu \in \mathbb{R}\} \\
&= \{(\min(\lambda,\mu), \min(\lambda+2,\mu-2), \min(\lambda,\mu-2))\} \\
&= \{\,(\lambda,\lambda+2,\lambda) \mid \lambda \leq \mu - 4\} \\
&\quad \cup \{\,(\lambda,\mu-2,\lambda) \mid \mu-4 \leq \lambda \leq \mu-2\} \\
&\quad \cup \{\,(\lambda,\mu-2,\mu-2) \mid \mu-2 \leq \lambda \leq \mu\} \\
&\quad \cup \{\,(\mu,\mu-2,\mu-2) \mid \mu \leq \lambda\} \\
&= \{\,(0,\mu-\lambda-2,0) \mid 2 \leq \mu-\lambda \leq 4\} \\
&\quad \cup \{\,(0,\mu-\lambda-2,\mu-\lambda-2) \mid 0 \leq \mu-\lambda \leq 2\}
\end{aligned}
$$



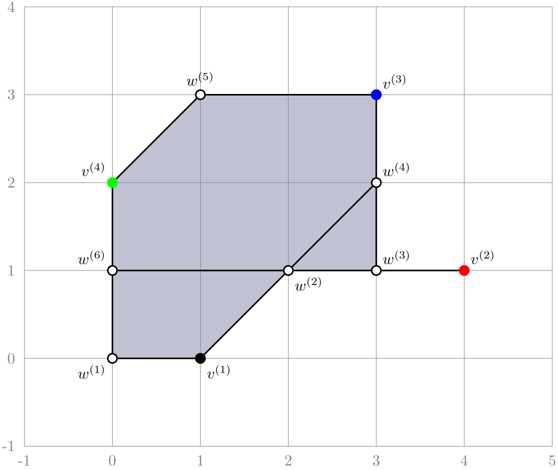
Case Distinction

$\lambda \in (-\infty, \mu-4] \cup [\mu-4, \mu-2] \cup [\mu-2, \mu] \cup [\mu, \infty)$



The Running Example

$n = 4,\ d = 3$
$v_1 = (0, 1, 0)^\top,\ v_2 = (0, 4, 1)^\top,\ v_3 = (0, 3, 3)^\top,\ v_4 = (0, 0, 2)^\top$



Covectors

consider $V \in \mathbb{T}^{d \times n}$ (and read columns as points in $\mathbb{TP}^{d-1}$)

Definition

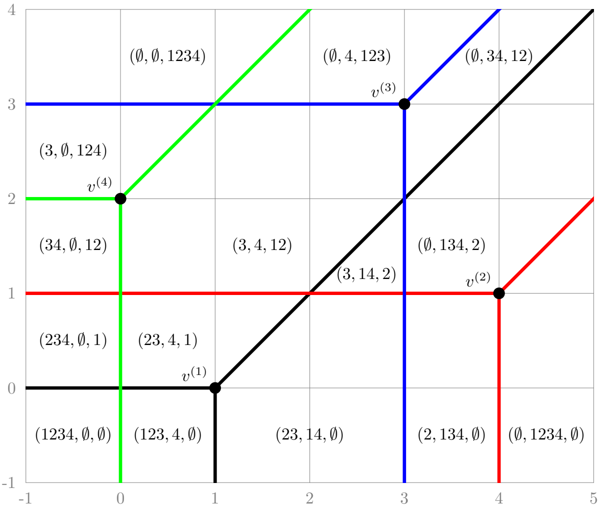
covector of $p \in \mathbb{R}^d / \mathbb{R}\mathbf{1}$ w.r.t. V given by $T_V(p) = (T_1, T_2, \dots, T_d)$ with

$$\begin{aligned} k \in T_i \quad &\iff \quad i \in \operatorname{argmin}\{j \in [d] \mid v_{jk} - p_j\} \\ &\iff \quad i \in \operatorname{argmax}\{j \in [d] \mid p_j - v_{jk}\} \end{aligned}$$

Example

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 \\ 0 & 1 & 3 & 2 \end{pmatrix} \qquad T_V \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = (\{2,3\}, \{1,4\}, \emptyset)$$

Covector Decomposition of $\mathbb{R}^d/\mathbb{R}\mathbf{1}$



... induced by max-tropical hyperplane arrangement $\mathfrak{A}(V)$

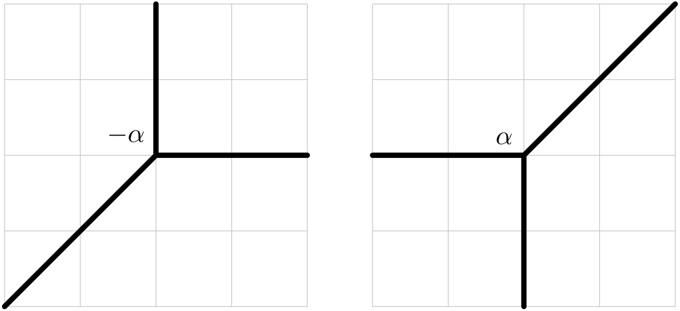
Recall: Max-Tropical Hyperplanes

duality between min and max:

$$\max(-x, -y) \; = \; -\min(x, y)$$

Remark

$\mathcal{T}$ is min-trop. hypersurface $\Leftrightarrow -\mathcal{T}$ is max-trop. hypersurface



min/max

Structure Theorem of Tropical Convexity

Theorem (Develin & Sturmfels 2004;
Fink & Rincón 2015; J. & Loho 2016)

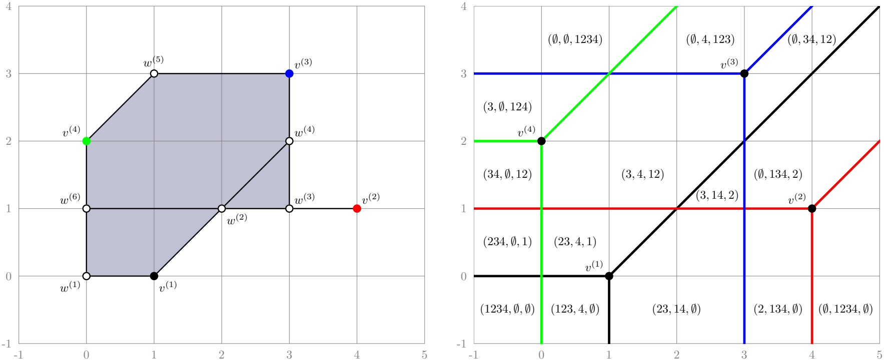
The covector decomposition $\mathcal{T}(V)$ *of* $\mathbb{R}^d$ *induced by* $V \in \mathbb{T}^{d \times n}$

1. *is dual to a regular subdivision of*

$$\operatorname{conv}\left\{ (e_i, e_j) \in \mathbb{R}^d \times \mathbb{R}^n \;\middle|\; v_{ij} \neq \infty \right\} ,$$

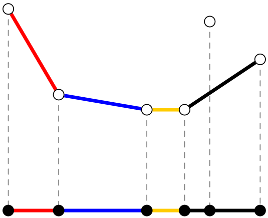
2. *and it induces a polyhedral decomposition of* $\operatorname{tconv}(V)$.

Covector Decomposition of Standard Example

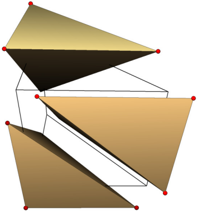


Products of Simplices and Their Subpolytopes

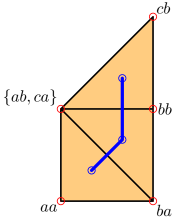
- $\operatorname{tconv}\{v_1, \ldots, v_n\} \subset \mathbb{R}^d/\mathbb{R}\mathbf{1}$ dual to regular subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ defined by lifting $e_i \times e_j$ to height v_{ij}
  - *general position* $\longleftrightarrow$ triangulation
- lifting vertices to ∞ defines subpolytope (on remaining vertices)
- extra feature from swapping factors $\rightsquigarrow$ tconv(rows) $\cong$ tconv(columns)



recall: regular subdivision



$\Delta_2 \times \Delta_1$



tconv(2 points in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$)

Mixed Subdivisions

- P, Q : polytopes in $\mathbb{R}^d$
- $P + Q = \{p + q \mid p \in P, \ q \in Q\}$ *Minkowski sum*
- *Minkowski cell* of $P + Q$ = full-dimensional subpolytope which is Minkowski sum of subpolytopes of P and Q

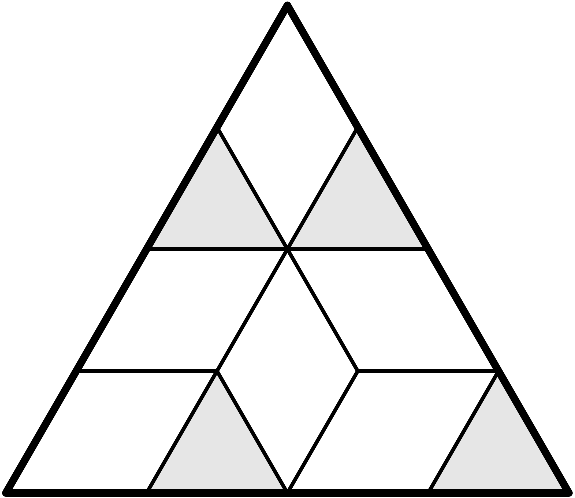
Definition

Polytopal subdivision of $P + Q$ into Minkowski cells is mixed if for any two of its cells $P' + Q'$ and $P'' + Q''$ the intersections $P' \cap P''$ and $Q' \cap Q''$ both are faces.

- fine = cannot be refined (as a mixed subdivision!)
- can be generalized to finitely many summands

Example With 4 Summands

fine mixed subdivision of *dilated simplex* $\Delta_2 + \Delta_2 + \Delta_2 + \Delta_2 = 4\Delta_2$



Cayley Trick, General Form

- $e_1, e_2, \ldots, e_n$: affine basis of $\mathbb{R}^{n-1}$
- $\phi_k : \mathbb{R}^d \to \mathbb{R}^{n-1} \times \mathbb{R}^d$ embedding $p \mapsto (e_k, p)$
- *Cayley embedding* of $P_1, P_2, \ldots, P_n$:

$$\mathcal{C}(P_1, P_2, \ldots, P_n) \;=\; \operatorname{conv} \bigcup_{i=1}^{n} \phi_i(P_i)\,.$$

Theorem (Sturmfels 1994; Huber, Rambau & Santos 2000)

1. *For any polyhedral subdivision of $\mathcal{C}(P_1, P_2, \ldots, P_n)$ the intersection of its cells with $\{\frac{1}{n}\sum e_i\} \times \mathbb{R}^d$ yields a mixed subdivision of $\frac{1}{n}\sum P_i$.*
2. *This correspondence is a poset isomorphism from the subdivisions of $\mathcal{C}(P_1, P_2, \ldots, P_n)$ to the mixed subdivisions of $\sum P_i$. Further, the coherent mixed subdivisions are bijectively mapped to the regular subdivisions.*

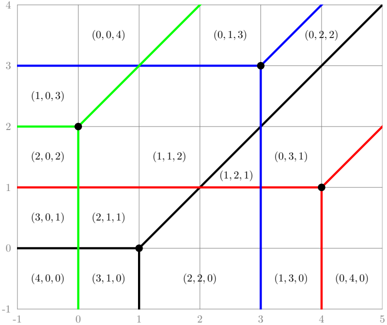
Cayley Trick for Products of Simplices

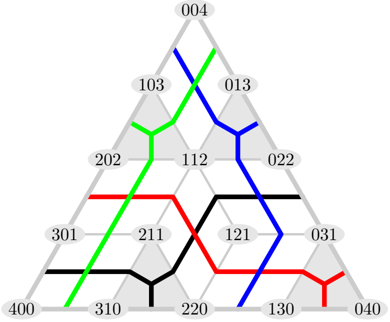
- consider $P_1 = P_2 = \cdots = P_n = \Delta_{d-1} = \text{conv}\{e_1, e_2, \ldots, e_d\}$
- $\mathcal{C}(\underbrace{\Delta_{d-1}, \Delta_{d-1}, \ldots, \Delta_{d-1}}_{n}) \cong \Delta_{d-1} \times \Delta_{n-1}$

Corollary

1. *For any polyhedral subdivision of $\Delta_{d-1} \times \Delta_{n-1}$ the intersection of its cells with $\{\frac{1}{n}\sum e_i\} \times \mathbb{R}^d$ yields a mixed subdivision of $\frac{1}{n} \cdot (n\Delta_{d-1})$.*
2. *This correspondence is a poset isomorphism from the subdivisions of $\Delta_{d-1} \times \Delta_{n-1}$ to the mixed subdivisions of $n\Delta_{d-1}$. Further, the coherent mixed subdivisions are bijectively mapped to the regular subdivisions.*

Back to Standard Example





- (fine) covectors $\rightsquigarrow$ coarse covectors
    - replace sets T_k by their cardinality
- coarse covectors of maximal cells = vertex coordinates of mixed subdivision

A Tropical Proof of the Cayley Trick ...

for products of simplices

- point $v_i \in \mathbb{T}^{d-1}$ = apex of unique max-tropical hyperplane $H^{\max}(v_i)$
- homogeneous linear form $h_i \in \mathbb{C}\{\{t\}\}[x_1, x_x, \ldots, x_d]$;

$$h \ := \ h_1 \cdot h_2 \cdots h_n$$

Proposition

The tropical hypersurface defined by $\mathrm{trop}^{\max}(h)$ is the union of the max-tropical hyperplanes in $\mathfrak{A}(V)$.

- dual subdivision of Newton polytope $n\Delta_{d-1}$

Corollary

Let $p \in \mathbb{T}^{d-1} \setminus \mathfrak{A}(V)$ be a generic point. Then its coarse covector $\mathbf{t}_V(p)$ equals the exponent of the monomial in h which defines the unique facet of $\mathcal{D}(\mathrm{trop}^{\max}(h))$ above p.

Conclusion II

- configuration of n points in $\mathbb{TP}^{d-1}_{\min}$ corresponds to arrangement of n tropical hyperplanes in $\mathbb{TP}^{d-1}_{\max}$
  - tropical polytope = union of bounded cells (for finite coordinates)
- covector decomposition dual to regular subdivision of subpolytope $\Delta_{n-1} \times \Delta_{d-1}$
- tropical proof of special case of Cayley Trick

What is a Tropical Linear Program?

An *ordinary linear program* is an optimization problem like

$$\begin{array}{ll} \text{minimize} & c^\top x \\ \text{s.t.} & Ax \geq b \\ & x \in \mathbb{R}^n \end{array}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$.

Definition

A tropical linear program $\mathrm{LP}(A, b, c)$ is an optimization problem like

$$\begin{array}{ll} \text{minimize} & c^\top \odot x \\ \text{s.t.} & A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^- \\ & x \in \mathbb{T}^n \end{array}$$

where $A^\pm \in \mathbb{T}^{m \times n}$, $b^\pm \in \mathbb{T}^m$, $c \in \mathbb{T}^n$.

Min-max optimization over tropical polyhedra

Beware: now $\oplus = \max$

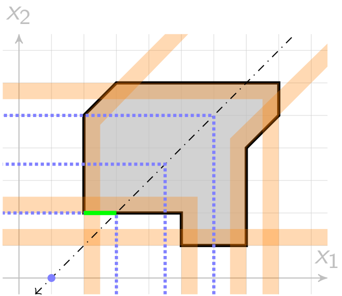
- feasible set defined by

$$A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^-$$

  is a tropical polyhedron; denoted $\mathcal{P}(A, b)$
- each defining inequality corresponds to a tropical half-space
- level sets have *apices*, located on the line $(-c) + \mathbb{R}\mathbf{1}$
- optimal solution(s) form tropical polyhedron, too

$$\text{minimize } \max(-1 + x_1, x_2)$$

$$\text{s.t.} \begin{cases} \max(x_1 - 5, x_2 - 2) \geq 0 \\ 0 \geq \max(x_1 - 8, x_2 - 6) \\ x_1 - 2 \geq \max(x_2 - 5, 0) \\ \max(x_2 - 4, 0) \geq x_1 - 7 \\ x_2 \geq 1 \end{cases}$$



Fact sheet: Tropical polyhedra

- can also be represented in terms of *vertices* and *rays*
  [Gaubert 1992] [J. 2005] [Gaubert & Katz 2011], . . .
- tropical polytopes special case of tropical polyhedron defined by homogeneous tropical inequalities $A^+ \odot x \geq A^- \odot x$
  - arbitrary tropical polyhedra can be homogenized
- tropical linear programming [Butković & Aminu 2008]
- tropical fractional linear programming
  [Gaubert, Katz & Sergeev 2012]
- tropical LP feasibility equivalent to mean payoff games
  [Akian, Gaubert & Gutermann 2012]

Main Lemma of Tropical Linear Programming

where $\mathbb{K}$ is some field of real Puiseux series

Let $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{K}^n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \geq 0\}$ be contained in $\mathbb{K}^n_{\geq 0}$.

Lemma (Develin & Yu 2007; ABGJ 2015)

If tropicalization of $(\boldsymbol{A}, \boldsymbol{b})$ *is sign generic then*

$$\operatorname{val}(\mathcal{P}) = \{x \in \operatorname{trop}^n \mid A^+ \odot x \oplus b^+ \geq A^- \odot x \oplus b^-\} ,$$

where $(A^+ \; b^+) = \operatorname{val}(\boldsymbol{A}^+ \boldsymbol{b}^+)$ *and* $(A^- \; b^-) = \operatorname{val}(\boldsymbol{A}^- \; \boldsymbol{b}^-)$.

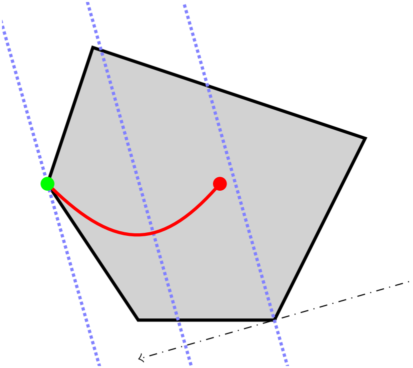
Moreover, for any $I \subset [m]$, we have:

$$\operatorname{val}(\{\boldsymbol{x} \in \mathcal{P} \mid \boldsymbol{A}_I \boldsymbol{x} + \boldsymbol{b}_I = 0\}) = \{x \in \operatorname{val}(\mathcal{P}) \mid A_I^+ \odot x \oplus b_I^+ = A_I^- \odot x \oplus b_I^-\} .$$

where $(\boldsymbol{A}_I \; \boldsymbol{b}_I)$ submatrix of $(\boldsymbol{A} \; \boldsymbol{b})$ formed by rows with indices in I.

The Interior Point Method of Linear Programming

[von Neumann] [Karmarkar 1984]



- start at analytic center
- trace central path by solving auxiliary (non-linear) optimization problems via Newton's method
- optimality characterized by Karush–Kuhn–Tucker conditions

- Karmarkar 1984: polynomial time algorithm
- method depends on *barrier function*
  - no STRONGLY polynomial time algorithm known for LP
  - Smale's 9th problem

Fact Sheet: Interior Point Method

- method depends on barrier function
  - no STRONGLY polynomial time algorithm known
- Karmarkar 1984: polynomial time algorithm
  - Khachiyan 1979: ellipsoid method
- Nesterov & Nemirovski 1994: generalization to non-linear convex programming

Conjecture (Deza, Terlaky and Zinchenko (2008))

The total curvature of the central path is bounded by $O(n)$.

"Continuous Hirsch Conjecture"

- Dedieu, Malajovich & Shub 2005: true "on the average"
- De Loera, Sturmfels & Vinzant 2012: similar result
- disproved by Allamigeon, Benchimol, Gaubert & J. 2014+

Long and Winding Central Paths

Theorem (Allamigeon, Benchimol, Gaubert & J. 2014+)

There is a family of ordinary linear programs with $m = 3r + 4$ linear inequalities in $n = 2r + 2$ variables such that the total curvature of the central path is at least $\Omega(2^r)$.

- counter-example to the "Continuous Hirsch Conjecture" of Deza, Terlaky and Zinchenko (2008)
- Smale's 9th problem

Interior Point Method: Our Setup

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, $\mu > 0$.

primal linear program:

assume bounded w/ non-empty interior

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \le b,\ x \ge 0,\ x \in \mathbb{R}^n \end{aligned} \qquad \text{LP}(A, b, c)$$

dual linear program:

$$\begin{aligned} \text{maximize} \quad & -b^\top y \\ \text{subject to} \quad & -A^\top y \le c,\ y \ge 0,\ y \in \mathbb{R}^m \end{aligned}$$

associated *logarithmic barrier problem*:

$$\begin{aligned} \text{minimize} \quad & \frac{c^\top x}{\mu} - \sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\ \text{subject to} \quad & Ax + w = b,\ x > 0, w > 0 \end{aligned}$$

A System of Polynomial Equations

logarithmic barrier problem

$$\begin{array}{ll} \text{minimize} & \dfrac{c^\top x}{\mu} - \displaystyle\sum_{j=1}^{n} \log(x_j) - \sum_{i=1}^{m} \log(w_i) \\ \text{subject to} & Ax + w = b,\ x > 0, w > 0 \end{array}$$

for $\mu > 0$ has *unique optimal solution* (x^μ, w^μ) characterized by

$$\begin{aligned} Ax + w &= b \\ -A^\top y + s &= c \\ w_i y_i &= \mu \quad \text{for all } i \in [m] \\ x_j s_j &= \mu \quad \text{for all } j \in [n] \\ x, w, y, s &> 0 \end{aligned}$$

That is, there uniquely exist y^μ and s^μ such that $(x^\mu, w^\mu, y^\mu, s^\mu)$ is a solution ...

The Central Path

Definition

The *central path* is the image of the map

$$\mathcal{C}_{A,b,c} : \mathbb{R}_{>0} \to \mathbb{R}^{2m+2n}, \quad \mu \mapsto (x^\mu, w^\mu, y^\mu, s^\mu).$$

- primal central path = projection onto x-coordinates
- *dual central path* = projection onto y-coordinates

Conjecture (Deza, Terlaky & Zinchenko 2008)

The total curvature of the primal central path is at most $O(m)$.

- Dedieu, Malajovich & Shub 2005: $O(n)$ holds on the average
- De Loera, Sturmfels & Vinzant 2012:
  similar result via matroid theory

A Simple Example . . .

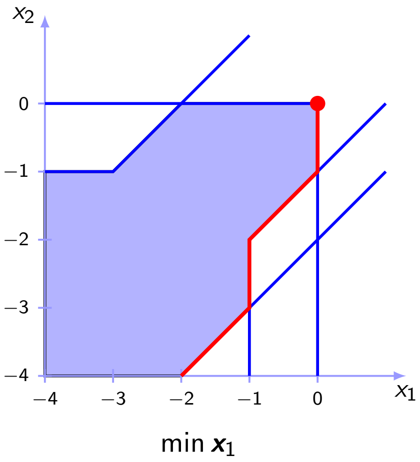
Consider the Puiseux polyhedron $\boldsymbol{\mathcal{P}} \subset \mathbb{K}^2$ defined by:

$$\begin{aligned} \boldsymbol{x}_1 + \boldsymbol{x}_2 &\leq 2 \\ t\boldsymbol{x}_1 &\leq 1 + t^2\boldsymbol{x}_2 \\ t\boldsymbol{x}_2 &\leq 1 + t^3\boldsymbol{x}_1 \\ \boldsymbol{x}_1 &\leq t^2\boldsymbol{x}_2 \\ \boldsymbol{x}_1, \boldsymbol{x}_2 &\geq 0 \ . \end{aligned} \tag{1}$$

Then the set $\mathrm{val}(\boldsymbol{\mathcal{P}})$ is described by the tropical linear inequalities:

$$\begin{aligned} \max(x_1, x_2) &\leq 0 \\ 1 + x_1 &\leq \max(0, 2 + x_2) \\ 1 + x_2 &\leq \max(0, 3 + x_1) \\ x_1 &\leq 2 + x_2 \ . \end{aligned} \tag{2}$$

. . . and Two of Its Primal Tropical Central Paths



A Family of Linear Programs

... with $2r+2$ variables $\boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{u}_1, \boldsymbol{v}_1, \ldots, \boldsymbol{u}_r, \boldsymbol{v}_r$ and $3r+4$ inequalities:

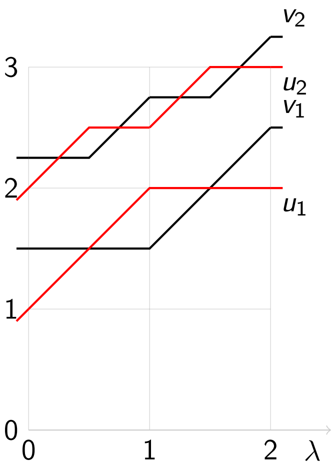
$$
\begin{array}{rll}
\min & \boldsymbol{v}_0 & \\
\text{s.t.} & \boldsymbol{u}_0 \leq t & \\
& \boldsymbol{v}_0 \leq t^2 & \\
& \boldsymbol{v}_i \leq t^{1-\frac{1}{2^i}}(\boldsymbol{u}_{i-1} + \boldsymbol{v}_{i-1}) & \text{for } i \in [r] \\
& \boldsymbol{u}_i \leq t\,\boldsymbol{u}_{i-1} & \text{for } i \in [r] \\
& \boldsymbol{u}_i \leq t\,\boldsymbol{v}_{i-1} & \text{for } i \in [r] \\
& \boldsymbol{u}_r \geq 0,\ \boldsymbol{v}_r \geq 0 &
\end{array}
$$

depending on a real parameter $t > 0$

primal central path has total curvature at least $\Omega(2^r)$ for $t \gg 0$

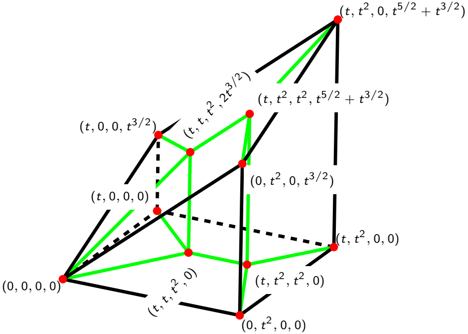
The Primal Tropical Central Paths of Our Examples

lifting a construction by Bezem, Nieuwenhuis and Rodríguez-Carbonell, 2008



total curvature $\Omega(2^r)$

- left: $r = 2$
- below: $r = 1$



Conclusion III

- tropical geometry yields new results for classical linear programs
- in specific situations possible to derive metric information from tropicalization
- tropical linear programs are interesting from a computational complexity perspective