Chapter 1
Tropical Hypersurfaces

The tropical semiring is the triplet \((\mathbb{R} \cup \{\infty\}, \oplus, \odot)\) with
\[
x \oplus y := \min(x, y) \quad \text{and} \quad x \odot y := x + y.
\]

Its algebraic properties are somewhat modest, if one compares it with rings and fields, which are more common in most areas of mathematics. Still the distributive laws hold, and as an extra catch the new addition is idempotent, i.e., \(x \oplus x = x\) holds for all \(x \in \mathbb{R}\). The special element \(\infty\) serves is neutral with respect to the tropical addition. Notice that there are no additive inverses. We abbreviate \(T = \mathbb{R} \cup \{\infty\}\).

An elementary approach to tropical geometry is to study a multivariate polynomial with coefficients in \(T\) and its evaluation map. This leads to the definition of tropical hypersurfaces. As their key property tropical hyperfaces can be described in terms of polyhedral geometry. This is the topic of this lecture.

1.1 Tropical Polynomials

The tropical addition can be extended to vectors of real numbers by taking the minimum component-wise. This way we obtain a map
\[
\oplus : T^d \times T^d \to T^d.
\]

Defining the tropical scalar multiplication as
\[
\odot : T \times T^d \to T^d : (\lambda, x) \mapsto \lambda \odot x := \begin{pmatrix} \lambda \odot x_1 \\ \lambda \odot x_2 \\ \vdots \\ \lambda \odot x_d \end{pmatrix} = \begin{pmatrix} \lambda + x_1 \\ \lambda + x_2 \\ \vdots \\ \lambda + x_d \end{pmatrix} = \lambda \mathbf{1} + x
\]

we obtain the tropical semimodule \((T^d, \oplus, \odot)\); here \(\mathbf{1}\) denotes the all ones vector.
When we replace the ordinary arithmetic operations by their tropical counterparts we can create tropical polynomials such as

\[(3 \odot x^3) \oplus (1 \odot x^2) \oplus (2 \odot x) \oplus 4 = \min(3 + 3x, 1 + 2x, 2 + x, 4). \tag{1.1}\]

We may view this as a map from \(\mathbb{R}\) to \(\mathbb{R}\) via substituting \(x\), and we adopt the notational convention

\[u \odot a = u \odot u \odot \cdots \odot u \text{ \(a\) times} \quad = u + u + \cdots + u \text{ \(a\) times} = a \cdot u.\]

The four linear functions \(3 + 3x, 1 + 2x, 2 + x\), and (the constant function which is identically) \(4\) from Equation (1.1) are sketched in Figure 1.1 together with their pointwise minimum. The evaluation yields a piecewise-linear function, where the number of linear pieces is bounded by the number of monomials. Moreover, this piecewise-linear function is concave, i.e., the set of points below the minimum is convex.

Sometimes, just as for classical polynomials over a field of characteristic zero, it is legitimate to identify a tropical polynomial with the function defined by its evaluation. Formally, a classical or tropical polynomial is a map which assigns a coefficient to each exponent. In this way terms which do not contribute to the evaluation are kept nonetheless, and this is often useful. Therefore, the concept which we adopt for a tropical polynomial is this more formal one. However, we often abuse the same notion also for the function obtained from substituting indeterminates by real numbers.

There is a systematic way of turning an ordinary polynomial (with coefficients in a special field) into a tropical one. This will justify the naïve looking approach above, but we will postpone this discussion to the subsequent Chapter 2.

The same extends to the multivariate case. Again each monomial defines one linear inequality. A \(d\)-variate tropical polynomial \(F\) is a map from a finite set \(S \subset \mathbb{Z}^d\) to the reals (or infinity). If the exponent \(u \in S\) is mapped to the coefficient \(a_u\), then \(F\) induces the evaluation function which send \(x \in \mathbb{R}^d\) to the real number

\[F(x) = \bigoplus_{u \in S} a_u \odot x_1^{u_1} \odot x_2^{u_2} \odot \cdots \odot x_d^{u_d} = \min \{a_u + u_1 x_1 + u_2 x_2 + \cdots + u_d x_d \mid u \in S\} = \min \{a_u + \langle u, x \rangle \mid u \in S\}.\]

The support \(\text{supp}(F)\) is the set of exponents \(u\) for which \(a_u \neq \infty\). In the sequel we will always assume that \(F\) has non-empty support, i.e., \(F\) takes real values only. Notice that, by admitting arbitrary integer exponents our tropical polynomials are, in fact, tropical analogs of Laurent polynomials.

**Proposition 1.1.** For a \(d\)-variate tropical polynomial \(F\) the set

\[\mathcal{D}(F) := \left\{(p, s) \in \mathbb{R}^{d+1} \mid p \in \mathbb{R}^d, s \in \mathbb{R}, s \leq F(p)\right\}\]
is an unbounded convex polyhedron of dimension $d + 1$.

We call $\mathcal{D}(F)$ the **dome** of $F$.

**Proof.** By construction, the set $\mathcal{D}(F)$ is the intersection of finitely many affine half-spaces in $\mathbb{R}^{d+1}$ and hence a convex polyhedron. Choose some $p \in \mathbb{R}^d$ and $s < F(p)$. Then $(p, s)$ is in $\mathcal{D}(F)$, which shows that $\mathcal{D}(F)$ is not empty. Further, by choice of $s$ the point $(p, s)$ has a positive distance from each facet, which is defined by some monomial of $F$. Hence for some $\rho > 0$ the closed ball of radius $\rho$ around $(p, s)$ is entirely contained in $\mathcal{D}(F)$. This proves that the polyhedron is full-dimensional. As $s$ can be chosen arbitrarily small (and negative), we also see that $\mathcal{D}(F)$ cannot be bounded.

That $\mathcal{D}(F)$ is a full-dimensional unbounded convex polyhedron is a consequence of the fact that the outer facet normal vectors do not positively span the entire space $\mathbb{R}^{d+1}$. Indeed, all of them point *upward*, which we take as a synonym for the positive $e_{d+1}$-direction. The proof above just spells out the details in an elementary way.
Notation 1.2. Whenever it comes to computing with explicit coordinates one has to choose whether vectors are rows or columns. Throughout this book we take the following approach: Generally, vectors denoting points are columns. This entails that matrices act on the left, and linear forms are given as row vectors. This probably agrees with the majority. However, if there is sufficient hope that confusion can be avoided, we take the freedom to also write coordinate vectors describing individual points as rows without explicitly marking that such a vector ought to be transposed.

Classical algebraic geometry studies (affine) algebraic varieties, which are sets which a given collection of polynomials vanishes on, i.e., sets where the polynomials attain the value zero. For the tropical evaluation of a polynomial the value zero is not of any particular relevance (nor any other specific value). Instead the following definition turns out to be fruitful.

Definition 1.3. A \(d\)-variate tropical polynomial \(F\) vanishes at \(p \in \mathbb{R}^d\) if the minimum

\[
F(p) = \bigoplus_{u \in S} a_u \odot p_1^{\odot u_1} \odot p_2^{\odot u_2} \odot \cdots \odot p_d^{\odot u_d}
\]

is attained at least twice. The vanishing locus

\[
\mathcal{T}(F) := \left\{ p \in \mathbb{R}^d \mid F \text{ vanishes at } p \right\}
\]

is the tropical hypersurface defined by \(F\).

The justification that this, indeed, is the proper notion of “vanishing” is deferred to Chapter 2. The \(k\)-skeleton of a polyhedral complex is the polyhedral subcomplex of faces of dimension at most \(k\). The \(k\)-coskeleton is the set of faces of dimension at least \(k\) with the induced partial ordering induced by containment.

Corollary 1.4. The tropical hypersurface \(\mathcal{T}(F)\) coincides with the image of the codimension-2-skeleton of its dome \(\mathcal{D}(F)\) in \(\mathbb{R}^d\) under the orthogonal projection which omits the last coordinate.

**Proof.** Each facet of the polyhedron \(\mathcal{D}(F)\) from Proposition 1.1 corresponds to one of the tropical monomials \(x_1^{\odot u_1} \odot x_2^{\odot u_2} \odot \cdots \odot x_d^{\odot u_d}\). Vanishing means that at least two terms must cancel. This implies that \(F\) vanishes at \(p\) if and only if the point \((p, F(p))\) in the boundary \(\partial \mathcal{D}(F)\) is contained in at least two facets. \(\square\)

This says that a tropical hypersurface of a \(d\)-variate tropical polynomial is a \((d-1)\)-dimensional polyhedral complex. Notice that \(F\) vanishes at \(p\) if and only if the piecewise-linear hypersurface \(\partial \mathcal{D}(F)\) is not differentiable at \((p, F(p))\).

Example 1.5. For the tropical polynomial \(F(x) = (3 \odot x^{\odot 3}) \oplus (1 \odot x^{\odot 2}) \oplus (2 \odot x) \oplus 4\) from the beginning of this section the dome \(\mathcal{D}(F)\) is 2-dimensional. Its 0-skeleton consists of the three vertices, \((-2, -3), (1, 3),\) and \((2, 4)\). The tropical hypersurface \(\mathcal{T}(F)\) is the set \((-2, 1, 2)\).
Another way to read Corollary [1.4] is to say that the relative interiors of the facets of the dome $\mathscr{D}(F)$ vertically project onto the connected components of the complement of the tropical hypersurface $\mathscr{T}(F)$ in $\mathbb{R}^d$. We call the topological closure of one such connected component a region of the tropical hypersurface. The regions generate an ordinary polyhedral decomposition of $\mathbb{R}^d$ whose codimension-1-skeleton is precisely the tropical hypersurface. In the example above the regions are the four closed intervals $(-\infty, -2], [-2, 1], [1, 2], [2, \infty)$.

It is worthwhile to explicitly dualize the construction of the polyhedron $\mathscr{D}(F)$. The idea is to associate with a tropical polynomial the point configuration given by its support and to interpret the coefficients as a height function.

The convex hull of $\text{supp}(F)$ is the Newton polytope of $F$, denoted as $\mathcal{N}(F)$. That is, if $F = \bigoplus_{u \in S} a_u \odot x_1^{a_{u_1}} \odot x_2^{a_{u_2}} \odot \cdots \odot x_d^{a_{u_d}}$ then

$$\mathcal{N}(F) = \text{conv}\{u \in S | a_u \neq \infty\} \subset \mathbb{R}^d.$$ 

The set

$$\tilde{\mathcal{N}}(F) = \text{conv}\left\{(u, r) \in \mathbb{Z}^d \times \mathbb{R} | u \in S, r \geq a_u\right\} \subset \mathbb{R}^{d+1}$$

is called the extended Newton polyhedron $\tilde{\mathcal{N}}(F)$ of $F$. Formally, the extended Newton polyhedron is the convex hull of infinitely many points. So one has to make sure its name is justified. This means that we have to show that $\tilde{\mathcal{N}}(F)$ can be described by finitely many linear inequalities. In view of the Weyl-Minkowski-Theorem A.1 this amounts to verify that the extended Newton polyhedron can be written as the sum of a polytope and a finitely generated cone. The latter is accomplished by the following.

**Observation 1.6.** The extended Newton polyhedron is the Minkowski sum of a polytope and a single ray. More precisely, we have

$$\tilde{\mathcal{N}}(F) = \text{conv}\{(u, a_u) | u \in \text{supp}(F)\} + \text{pos}\{e_{d+1}\}.$$

**Example 1.7.** We continue the Example [1.5]. The support of the tropical polynomial $F$ is the four point set $\{0, 1, 2, 3\}$ on the real line, and so the Newton polytope $\mathcal{N}(F)$ is the interval $[0, 3]$. In this case the extended Newton polyhedron has the four vertices $(0, 4), (1, 2), (2, 1),$ and $(3, 3)$ in $\mathbb{R}^2$. Figure 1.2 shows the extended Newton polyhedron $\tilde{\mathcal{N}}(F)$.

Before we will continue with the analysis of the Newton polytopes and the extended Newton polyhedra we wish to look at the situation in some greater generality.

### 1.2 Regular Subdivisions

Consider a finite set of points $A$ in $\mathbb{R}^d$. A (polyhedral) subdivision of $A$ is a finite polytopal complex whose vertices lie in the set $A$ and that covers the convex hull.
conv $A$. If all cells of the subdivision are simplices it is called a \textit{triangulation}. A trivial example to keep in mind is the following: Each finite set of points is trivially subdivided by its convex hull. The example below, however, is more interesting.

\textbf{Example 1.8.} Let $A = \{(u,v) \in \mathbb{Z}^2 : u \geq 0, v \geq 0, u + v \leq 4\}$ be the set of lattice points in the triangle $\text{conv}\{(0,0),(4,0),(0,4)\}$. Figure 1.3 shows a subdivision of $A$ with ten maximal cells, nine triangles and one quadrangle. The four points $(0,3), (1,2), (1,3)$ and $(3,1)$ do not occur as vertices of any cell.

Now we look at an arbitrary \textit{height function} $\omega : A \to \mathbb{R}$. This gives rise to the unbounded polyhedron

$$U(A, \omega) = \text{conv}\{(u, \omega(u)) | u \in A\} + \text{pos}\{e_{d+1}\} \quad (1.2)$$

in $\mathbb{R}^{d+1}$. A face of $U(A, \omega)$ is called a \textit{lower face} if it has an outward pointing normal vector $h$ satisfying $\langle h, e_{d+1} \rangle < 0$. That is to say, the normal vector $h$ is pointing downward. In general, a face of an unbounded polyhedron may be unbounded or bounded.

\textbf{Observation 1.9.} The lower faces of $U(A, \omega)$ are precisely those which are bounded.

The lower faces form a subcomplex in the boundary of $U(A, \omega)$, and projecting down yields the \textit{regular subdivision} $\Sigma(A, \omega)$ of $A$ induced by $\omega$. If the points in $A$ are in convex position, i.e., $A$ is the set of vertices of a convex polytope, then the vertices of any regular subdivision are precisely the points in $A$. If $A \subset \mathbb{R}^d$ is arbitrary and $\omega$ maps each point $x$ to its Euclidean norm $\|x\|$ then the regular subdivision of $A$ induced by $\omega$ is known as the \textit{Delaunay subdivision} of $A$. This is dual to the Voronoi diagram of $A$. In this sense regular subdivisions generalize the Delaunay subdivision.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{support_and_extended_newton_polyhedron.png}
\caption{Support and extended Newton polyhedron of a univariate tropical polynomial}
\end{figure}
of a point configuration. Therefore, they also go by the name “weighted Delaunay subdivision”.

![Fig. 1.3 Subdivision of 15 points in $\mathbb{R}^2$](image)

**Example 1.10.** The subdivision in Figure 1.3 is regular. A height function $\omega$ is given by

$\begin{align*}
(0,0) &\to 8 
(1,0) &\to 4 
(0,1) &\to 2 
(2,0) &\to 1 
(1,1) &\to 0 \\
(0,2) &\to 1 
(3,0) &\to 2 
(2,1) &\to 0 
(1,2) &\to 0 
(0,3) &\to 4 
(1.3)
\end{align*}$

The vertices of the quadrangular cell $\text{conv}\{(1,1), (2,1), (2,2), (0,4)\}$ are lifted to height zero. The point $(1,3)$ also lies in that quadrangle, but it is lifted higher than all the vertices. The interior point $(1,2)$, however, is also lifted to height zero. In fact, this is the only point in the set $A$ which is lifted to a point which lies in the relative interior of a face of the polyhedron $U(A, \omega)$. Therefore, it receives a special mark in Figure 1.3.

There are two things to keep in mind. First, most point configurations admit subdivisions which are not regular; see Problem 1.22 below. Second, the set of height functions which define one fixed regular subdivision is a relatively open polyhedral cone. In particular, such a height function is never unique. For more details see Section A.4 in the appendix.

The attentive reader will have noticed that the description of the extended Newton polyhedron of a tropical polynomial $F$ in Observation 1.6 agrees with the definition in (1.2), where the set $A$ is the support of $F$ and the height function is given by the coefficients. That is to say, studying the extended Newton polyhedra of tropical polynomials amounts to the same as studying height functions and regular subdivisions of configurations of finitely many lattice points. The regular subdivision of $\text{supp} F$ induced by the coefficients of $F$ is denoted as $\mathcal{S}(F)$. In view of the following result $\mathcal{S}(F)$ is called the *dual subdivision* of the tropical hypersurface $\mathcal{I}(F)$.

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Theorem 1.11. Let $F$ be a $d$-variate tropical polynomial.

1. There is an inclusion reverses bijection between the faces of $\mathcal{T}(F)$ and the cells of the dual subdivision $\mathcal{H}(F)$ of $\text{supp} F$. In this way $k$-dimensional cells of $\mathcal{T}(F)$ are mapped to $(d - k)$-dimensional cells of $\mathcal{H}(F)$.

2. The latter cells are obtained from orthogonally projecting the bounded $(d - k)$-dimensional faces of $\tilde{\mathcal{N}}(F)$ by omitting the last coordinate.

3. The vertices of $\tilde{\mathcal{N}}(F)$ or, equivalently, the $0$-dimensional cells of $\mathcal{T}(F)$ of $\mathcal{N}(F)$ bijectively correspond to the facets of $\mathcal{H}(F)$ and thus also to the regions of $\mathcal{T}(F)$.

Proof. Let $F(x) = \bigoplus_{u \in S} a_u \odot x^u$ with $S = \text{supp}(F)$. Each facet of the polyhedron $\mathcal{H}(F)$ corresponds to some term of $F$. The point $(w, F(w)) \in \mathbb{R}^{d+1}$ is contained in the facet defined by $a_u \odot x^u$ if and only if the minimum $F(w)$ equals $a_u + w_1 \cdot u_1 + \cdots + w_d \cdot u_d$. Letting $(w, 1) := (w_1, w_2, \ldots, w_d, 1) \in \mathbb{R}^{d+1}$ this is equivalent to saying that the linear form $(w, 1)$ attains its minimum on the polyhedron $\tilde{\mathcal{N}}(F)$ at the point $(u, a_u)$.

This means that $-(w, 1)$ is an element of the (outer) normal cone of the point $(u, a_u)$ in the boundary of $\mathcal{N}(F)$. Conversely, for each vector $-(w, 1)$ in the normal cone of $(u, a_u)$ the point $w$ is contained in the facet of $\mathcal{H}(F)$ defined by $a_u \odot x^u$. Moreover, the vectors of the form $(v, 1)$ and their positive multiples are precisely those vectors in $\mathbb{R}^{d+1}$ which form an angle of less than $\pi/2$ with the direction $e_{d+1}$ of the lifting.

A face of $\mathcal{N}(F)$ which admits $-(v, 1)$ as an outer normal vector is a lower face. By Observation 1.9 the lower faces are precisely the bounded ones. This yields a bijection between the faces of $\mathcal{H}(F)$ and the lower normals cones of $\mathcal{N}(F)$. As a poset the normal fan of $\mathcal{N}(F)$ is anti-isomorphic to the face lattice of $\tilde{\mathcal{N}}(F)$. A $k$-dimensional lower face of the extended Newton polyhedron $\tilde{\mathcal{N}}(F)$ now corresponds to a face of $\mathcal{H}(F)$ of dimension $d - k$. This bijection reverses the inclusion, and this proves the first claim.

The linear projection from $\mathbb{R}^{d+1}$ to $\mathbb{R}^d$ which omits the last coordinate maps the poset of lower faces of $\mathcal{N}(F)$ to the polytopal complex $\mathcal{H}(F)$ subdividing the point set $\text{supp}(F)$. The $0$-dimensional cells of $\mathcal{T}(F)$ form a subset of $\text{supp}(F)$, and their convex hull coincides with $\mathcal{N}(F)$. This shows the rest. 

Example 1.12. The height function in (1.3), which induces the regular subdivision in Figure 1.3, gives rise to the tropical polynomial

$$F(x,y) = \min \left(8, 4 + x, 2 + y, 1 + 2x, x + y, 1 + 2y, 2 + 3x, 2x + y, x + 2y, 4 + 3y, 8 + 4x, 4 + 3x + y, 2x + 2y, 2 + x + 3y, 4y \right).$$

The tropical hypersurface $\mathcal{T}(F)$ is the tropical plane curve shown in Figure 1.4. Each region is marked with the unique tropical monomial which attains its minimum at all points of that region. For instance, $F(0, 0) = 0$, and that minimum is attained at $x + y, 2x + y, x + 2y$ and $4y$. These four terms correspond to the vertices of the
1.3 Minimum versus Maximum

Choosing min as our tropical addition is by no means canonical. Let us explore what happens if we exchange min by max. Notice that the equality \( \min(-x, -y) = -\max(x, y) \) establishes an isomorphism of semirings between \( (\mathbb{R}, \min, +) \) and

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Fig. 1.4 Tropical plane curve of degree four and its regions

The point \((2, 2)\), which is marked yellow in Figure 1.3, corresponds to the term \(2x + 2y\) which also attains zero at \((0,0)\). Neither \(2x + 2y\) nor, e.g., \(2 + x + 3y\) correspond to any region of the tropical plane curve \(\mathcal{T}(F)\). However, in contrast to \(2x + 2y\) the minimum \(F(x,y)\) is never attained at \(2 + x + 3y\).

The not necessarily regular subdivisions of a fixed point configuration \(A\) are partially ordered in a natural way. Let \(\Sigma\) and \(\Sigma'\) both be subdivisions of \(A\). If each cell of \(\Sigma\) is a subpolytope of some cell of \(\Sigma'\) then \(\Sigma\) is said to refine \(\Sigma'\). Conversely, \(\Sigma'\) coarsens \(\Sigma\). In the refinement partial ordering the trivial subdivision \(\text{conv}A\) is the unique coarsest subdivision, albeit not a proper one. The finest subdivisions are the triangulations which use all points in \(A\).
(\mathbb{R}, \max, +). To the \(d\)-variate min-tropical polynomial
\[
F(x) = \min \{a_u + \langle u, x \rangle \mid u \in S\}
\]
we can associate the (likewise \(d\)-variate) max-tropical polynomial
\[
F^*(x) = \max \{-a_u + \langle u, x \rangle \mid u \in S\}.
\]
where we replace min by max and take the negatives of the coefficients. Evaluating \(F^*\) at the point \(-p \in \mathbb{R}^d\) now gives
\[
F^*(-p) = \max \{-a_u + \langle u, -p \rangle \mid u \in S\} = \max \{-a_u + \langle u, p \rangle \mid u \in S\} = -F(p).
\]
From this computation it follows that \(F\) vanishes at \(p\) if and only if \(F^*\) vanishes at \(-p\). Therefore, we have the following duality relation
\[
\mathcal{T}^{\max}(F^*) = -\mathcal{T}^{\min}(F).
\]
of tropical hypersurfaces. In particular, the image of a min-tropical hypersurface under the reflection at the origin is a max-tropical hypersurface and vice versa. See Figures 1.5a and 1.5b below for a min-tropical line in \(\mathbb{R}^3/\mathbb{R}^1\) and its mirror image; notice that the dual \(a^*\) of the linear tropical polynomial \(a\) equals \(-a\), which is why the apex of the max-tropical hyperplane \(\mathcal{T}^{\max}(a^*)\) is \(a\).

Remark 1.13. Whenever we replace min by max we also need to redefine the notion of a regular subdivision, if we want to make use of results like Theorem 1.11. In fact, it suffices to replace (1.2) by \(\conv \{ (u, \omega(u)) \mid u \in A \} - \pos \{ e_{d+1} \}\) and to focus on the upper facets instead of the lower ones.

We conclude that it is just a matter of taste if one prefers min or max. However, later we will encounter the situation where it is convenient to consider both additive structures at the same time. For now we stick to min for our tropical addition.

1.4 The Tropical Projective Torus

In classical geometry it is often convenient to study projective instead of affine varieties. This means to deal with homogeneous polynomials rather than arbitrary ones. A \(d\)-variate tropical polynomial \(F\) is homogeneous of degree \(\delta\) if for all \(p \in \mathbb{R}^d\) and \(\lambda \in \mathbb{R}\) we have
\[
F(\lambda \odot p) = F(\lambda \cdot 1 + p) = \lambda^{\odot \delta} \odot F(p) = \delta \cdot \lambda + F(p),
\]
where \(1\) denotes the all-ones-vector of length \(d\). Suppose that \(F\) is homogeneous of degree \(\delta\). Then \(F\) vanishes at \(p \in \mathbb{R}^d\) if and only if the minimum \(F(p)\) is attained at
least twice if and only if the minimum \( F(\lambda \odot p) = \delta \cdot \lambda + F(p) \) is attained at least twice for all \( \lambda \in \mathbb{R} \). Hence it makes sense to consider the tropical hypersurfaces of homogeneous tropical polynomials in the quotient of \( \mathbb{R}^d \) obtained by factoring out the tropical scalar multiplication.

**Definition 1.14.** The quotient \( \mathbb{R}^d / \mathbb{R}^1 \) is called the **tropical projective** \((d-1)\)-torus.

The direct product \((\mathbb{C}^\times)^d\) of \(d\) copies of the multiplicative group of the complex numbers is called an **algebraic** \(d\)-torus. The name ‘torus’ comes about since for \(d = 2\), as a topological space, this is homotopy equivalent to the usual torus \( S^1 \times S^1 \), which is the oriented surface of genus one. In the tropical setting \( \mathbb{C} \) is replaced by \( \mathbb{R} \), and \( 0 \) is replaced by \( \infty \), the neutral element with respect to \( \oplus \). The word ‘projective’ refers to taking the quotient by one copy of \( \mathbb{R} \). Hence the name for \( \mathbb{R}^d / \mathbb{R}^1 \). The quotient topology lets \( \mathbb{R}^d / \mathbb{R}^1 \) inherit a topology from the natural topology on \( \mathbb{R}^d \).

By virtue of the bijective map

\[
(x_1, x_2, \ldots, x_d) + \mathbb{R}^1 = (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}^1 \\
\mapsto (x_2 - x_1, \ldots, x_d - x_1)
\]

the tropical projective torus \( \mathbb{R}^d / \mathbb{R}^1 \) is homeomorphic to \( \mathbb{R}^{d-1} \). Often, we will identify \( \mathbb{R}^d / \mathbb{R}^1 \) with \( \mathbb{R}^{d-1} \), and if we do so then always with respect to the map (1.6).

For a more thorough discussion of the topological situation see Section 5.1 below.

Specializing Corollary 1.3 to the homogeneous case yields:

**Corollary 1.15.** The tropical hypersurface \( \mathcal{T}(F) \) of a homogeneous \(d\)-variate tropical polynomial \( F \) of degree \( \delta \geq 1 \) is either empty or a pure and connected \((d-2)\)-dimensional polyhedral complex in \( \mathbb{R}^d / \mathbb{R}^1 = \mathbb{R}^{d-1} \).

Throughout the following we will treat coordinate vectors of points in \( \mathbb{R}^d / \mathbb{R}^1 \) as homogeneous coordinates, i.e., we identify \( (x_1, x_2, \ldots, x_d) \) with \( \lambda \odot (x_1, x_2, \ldots, x_d) \).

**Example 1.16.** For the homogeneous linear tropical polynomial \( (a_1 \odot x_1) \odot (a_2 \odot x_2) \odot (a_3 \odot x_3) \), with \( a_i \in \mathbb{R} \), the tropical variety equals

\[
-(a_1, a_2, a_3) + (\mathbb{R}_{\geq 0}(1, 0, 0) \cup \mathbb{R}_{\geq 0}(0, 1, 0) \cup \mathbb{R}_{\geq 0}(0, 0, 1)) + \mathbb{R}^1
\]

which, as a subset of \( \mathbb{R}^3 / \mathbb{R}^1 \), is the same as

\[
(0, a_1 - a_2, a_1 - a_3) + (\mathbb{R}_{\geq 0}(0, -1, -1) \cup \mathbb{R}_{\geq 0}(0, 1, 0) \cup \mathbb{R}_{\geq 0}(0, 0, 1)).
\]

By projecting onto the last two coordinates as in (1.6) we obtain the image shown in Figure 1.5a.

A tropical hyperplane is the tropical hypersurface of a homogeneous linear tropical polynomial. Rescaling the tropical linear form \( a \in \mathbb{R}^d \) by adding a constant vector \( \lambda \mathbf{1} \) gives the same tropical hyperplane, i.e.,

\[
\mathcal{T}(a) = \mathcal{T}((\lambda \mathbf{1}) + a) = \mathcal{T}(\lambda \odot a).
\]
This means that the set of tropical hyperplanes in $\mathbb{R}^d/\mathbb{R}1$ bijectively corresponds with the points in the tropical projective torus $\mathbb{R}^d/\mathbb{R}1$. This corresponds to the classical situation where linear hyperplanes in $\mathbb{C}^d$ (or, equivalently, projective hyperplanes in $\text{PG}_{d-1} \mathbb{C}$) are parameterized by the points in the projective space $\text{PG}_{d-1} \mathbb{C}$.

The special point $-a$, where the minimum is attained in all coordinates simultaneously, is called the apex of the tropical hyperplane $T(a)$.

Let us examine the tropical plane curves of degree two. A general homogeneous tropical polynomial of degree two in three indeterminates equals

$$
(a_{200} \odot x_1^{\odot 2}) \oplus (a_{110} \odot x_1 \odot x_2) \oplus (a_{101} \odot x_1 \odot x_3) \\
\quad \oplus (a_{020} \odot x_2^{\odot 2}) \oplus (a_{011} \odot x_2 \odot x_3) \oplus (a_{002} \odot x_3^{\odot 2})
$$

(1.7)

for six parameters $a_{ijk} \in \mathbb{T}$. Here we restrict our attention to the case where all exponents are non-negative, i.e., we are looking at tropical analogs of ordinary rather than Laurent polynomials. The tropical hyperplane defined by (1.7) is a tropical plane conic. An example with the parameter sequence

$$(a_{200}, a_{110}, a_{101}, a_{020}, a_{011}, a_{002}) = (6, 5, 5, 6, 5, 7)$$

(1.8)

is shown in Figure 1.6.

Notice that the Newton polytope of the tropical polynomial in (1.7) is contained in the dilated triangle $2 \Delta_2$, where $\Delta_2$ is the regular triangle spanned by the standard basis vectors $e_1, e_2, e_3$ in $\mathbb{R}^3$.

A tropical polynomial in $d$ indeterminates which is homogeneous of degree $\delta \geq 1$ is said to have full support if its support equals $\delta \Delta_{d-1} \cap \mathbb{Z}^d$, the set of all lattice points in the dilated simplex $\delta \Delta_{d-1} \cap \mathbb{Z}^d$. In particular, in this case the vertices $\delta e_k$ of $\delta \Delta_{d-1}$ are present in the support, and hence the Newton polytope equals $\delta \Delta_{d-1}$.

The following is a direct consequence of Theorem 1.11.

**Corollary 1.17.** Let $F$ be a homogeneous $d$-variate tropical polynomial of degree $\delta \geq 1$ with full support. Then the tropical hypersurface $T(F)$ is dual to the 1-coskeleton of the regular subdivision of the dilated simplex $\delta \Delta_{d-1}$ induced by the coefficients of $F$. 
Clearly, in a tropical (non-Laurent) polynomial $F$ of fixed degree only finitely many exponents occur, and this restricts the choice for the facet normals of the dome $\mathcal{D}(F)$. This way the degree of $F$ imposes restrictions on the combinatorics of the tropical hypersurface $\mathcal{T}(F)$.

### 1.5 Constant Coefficients

It is worthwhile to look into the special case of a tropical polynomial $F$ whose coefficients are all the same. In this case the privileged subdivision $\mathcal{S}(F)$ is trivial, as all the lattice points in the Newton polytope $\mathcal{N}(F)$ which correspond to monomials are raised to the same height. In particular, the dome $\mathcal{D}(F)$ is a translation of the Newton polytope, and the extended Newton polyhedron $\widetilde{\mathcal{N}}(F)$ is the Minkowski sum of $\mathcal{N}(F)$ and the upward pointing ray. Clearly, in all of this, the exact common value of the coefficients does not matter. We can specialize Corollary 1.4 to constant coefficients to yield the following.

**Corollary 1.18.** Let $F$ be a min-tropical polynomial with constant coefficients. Then the tropical hypersurface $\mathcal{T}(F)$ coincides with the codimension-1-skeleton of the inner normal fan of its Newton polytope $\mathcal{N}(F)$.

If we replace min by max the tropical hypersurface consists of outer normal cones.

**Example 1.19.** A planar example is

$$F(x,y) = \min\{x, y, 2x, 2y, x+2y, 2x+y\}, \quad (1.9)$$

where all coefficients equal zero. This is a bivariate tropical polynomial of degree three, which is not homogeneous, and so $F$ defines a tropical plane cubic in $\mathbb{R}^2$; see Figure 1.7.
For instance, the edge which joins the vertices \((1,0)\) and \((0,2)\) of the Newton polygon \(\mathcal{N}(F)\) has slope \(-2\). In the plane edges and facets are the same, and the vector \((1,2)\) is an inner normal vector on that facet. Therefore the ray \(\mathbb{R}_{\geq 0}(1, 2)\) is contained in the tropical plane curve \(\mathcal{T}(F)\).

![Fig. 1.7 A tropical cubic in \(\mathbb{R}^2\) with constant coefficients and its Newton polygon.](image)

**Problems**

**Problem 1.20.** Let \(F : \mathbb{R}^d \to \mathbb{R}\) be a tropical polynomial. Show that for any vector \(y \in \mathbb{R}^d\) the map 
\[ F_y : \mathbb{R}^d \to \mathbb{R}, \; x \mapsto F(x + y) \]
is a tropical polynomial, too.

**Problem 1.21.** Draw the tropical hypersurface defined by the tropical polynomial
\[
(4 \odot x_1 \odot^3) \oplus (1 \odot x_1 \odot x_2 \odot x_3) \oplus (4 \odot x_2 \odot^3) \\
\oplus (1 \odot x_2 \odot^2 \odot x_3) \oplus (1 \odot x_2 \odot x_3 \odot^2) \oplus (6 \odot x_3 \odot^3).
\]
How does the privileged subdivision of the Newton polytope look like?

**Problem 1.22.** Give an example of a point configuration and a subdivision which is not regular.

A *tropical conic* is a tropical hypersurface in \(\mathbb{R}^3/\mathbb{R}1\) of a homogeneous tropical polynomial of degree two.

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1.5 Constant Coefficients

**Problem 1.23.** What are the combinatorially distinct types of tropical conics in \( \mathbb{R}^3 / \mathbb{R}^1 \)? By the way, what is a good definition for "combinatorially distinct" in this context?

The natural way of defining the multiplication of matrices makes sense tropically as well. We use ‘⊙’ as a symbol for tropical matrix multiplication.

**Problem 1.24.** Let \( A \in \mathbb{R}^{m \times n} \) be a matrix. For which vectors \( b \in \mathbb{R}^m \) does the tropical linear system of equations \( A \circ x = b \) have a solution?

**Remarks**

Since the exponential function \( \exp : \mathbb{R} \to \mathbb{R}_{>0} \) is a monotonic bijection and since we further have \( \exp(x + y) = \exp(x) \cdot \exp(y) \) the tropical semiring is also isomorphic to \( (\mathbb{R}_{>0}, \min, \cdot) \) and \((\mathbb{R}_{>0}, \max, \cdot)\). In the literature the triplet \((T, \oplus, \odot)\) is often called "tropical semifield" in order to stress that the tropical multiplication \( \odot \) does have inverses, i.e., \((T, \oplus, \odot)\) satisfies all axioms of a field save the existence of inverses with respect to the tropical addition \( \oplus \).

For the foundations of the theory of polyhedral subdivisions as well as its applications we refer to [67] and [39]. Delaunay subdivisions and Voronoi diagrams are discussed, e.g., in [87] Chapters 6 and 7. Regular subdivisions are also called "coherent".

For an introduction to tropical curves and applications see [80]. There is a tropical version of the Riemann–Roch Theorem due to Gathmann and Kerber [62]. Their proof rests on a general Riemann–Roch Theorem for graphs by Baker and Norine [14].