Thesis for the Degree of Master of Science (20p)

The Four Vertex Theorem for Polygons

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Abstract

We[1] prove the four vertex theorem. Moreover we look at a discrete notion of curvature for polygons and prove an analogous four vertex theorem for polygons.

[1] I want to thank my supervisor Jan Stevens, whose patience and support has proven to be inversely proportionally large to my ability to plan in advance, something that has been crucial under the circumstances of which this thesis has been written. A big thanks to my friends at Göteborg University, especially Alain, Anton, Daniel, Magnus and Ottmar. I blame solely the city of Berlin for everything not going along plans and for making everything take longer time than it would without all the great friends, interesting people and fantastic music thriving here. Thank you mathematics, for changing my mind and introducing nightmares about polygons. At last my greatest thanks to my parents, for always supporting me in everything I do. I love you!

Contents

1 Introduction

Consider a smooth closed curve in the plane, without self-intersections. What can be said about invariants of such a curve? An answer to this question in the case of convex curves was given by Mukhopadhyaya[7] almost one hundered years ago. He stated that every smooth simple closed plane curve has at least four vertices, that is four local extrema on its curvature function. The curvature measures how much the curve fails to be straight at a specific point on the curve, and the case of exactly four extrema is exhibited by an oval. Since then the result has been proved in many more settings and in greater generality. It was later proved for non-convex curves, for curves in higher dimensions and so on. See [12] for a survey. Later on a sort of converse to the four vertex theorem was proved. Given a curvature function with four local extrema, there exists a plane curve having that as curvature function. It was proven in full generality by Dahlberg[2]. See [3] for an excellent survey with proofs.

Here we will use a rather geometric approach to the more general theorem about the non convex curves and its relation to circles of support. From this perspective we will investigate discrete analogs of plane curves, polygons. We define the appropriate notions of curvature for polygons to be able to state and prove in full generality, the rather recent four vertex theorem for polygons in the plane. We will follow the approach of Dahlberg[1] rather closely.

2 Smooth curves

We will study closed *smooth curves* in Euclidean space: equivalence classes of smooth maps $\gamma : S^1 \to \mathbb{R}^n$ under the equivalence relation that if $\phi : S^1 \to S^1$ is a diffeomorphism, then we say that curves γ and $\phi \circ \gamma$ are equivalent up to reparametrization. A *generic curve* in $\mathbb{R}^n$ is a curve whose $n-1$ derivatives always are linearly independent at all points. In the following we assume all curves to be generic, since it is natural in the sense that the generic curves form a dense open set in the Whitney C^∞-topology. See [5] for a proof. We define the *arc length* of a curve as

$$s(t) = \int_{t_0}^{t} |\gamma'(t)| \, dt \ .$$

When $|\gamma'(t)| \equiv 1$ we say that γ is *arc length parametrized* since we then get that $\int_{t_0}^{t} dt = t$. This is often the most convenient parametrization to use and we will assume hereafter that a curve is parametrized by arc length. Sometimes we write $\gamma(t) = \big(\gamma_1(t), \ldots, \gamma_n(t)\big)$. By *simple* curves we mean proper embeddings into the ambient space, i.e curves that nowhere self-intersect. Hereafter by a closed curve, we mean a generic simple smooth closed curve unless otherwise noted. We will now go through some notions needed to thoroughly understand the four vertex theorem. We choose to do it more generically than really needed for the simple planar case, but this will enlighten the transition to the discrete case. Since we only look at generic curves and the first $n-1$ derivatives are linearly independent at each point, we will always have the first osculating $(n-1)$-planes defined at each point:

The *osculating k-plane* of a curve in $\mathbb{R}^n$ at a point is the k-dimensional affine space spanned by the k first linearly independent derivatives of the curve at that point. By the genericity condition, the dimension of the osculating k-plane is k for k up to $n-1$.

We denote the osculating k-plane by Φ_k and since dim $\Phi_k = k$ we note that $\Phi_1 \subset \Phi_2 \subset \ldots \subset \Phi_{n-1}$ and that the osculating planes form a *flag*, which is called the *osculating flag of the curve* at that point. To get the *complete flag* we add the whole space $\Phi_n = \mathbb{R}^n$ which is spanned by the n derivatives in case they are linearly independent.

When $k = n-1$, we simply call the osculating k-plane the *osculating hyperplane*.

An osculating hyperplane naturally divides the ambient space into two parts. If the curve is contained in one of the parts solely, the osculating hyperplane is called a *osculating hyperplane of support*.

For each point of a curve we look at the complete flag of the curve at that point and we define vectors $e_i(t), i = 1, \ldots, n$, by letting

$$e_i(t) \in \Phi_i(t) \ ,$$

$$e_i(t) \perp \Phi_{i-1}(t) \ ,$$

$$|e_i(t)| = 1 \ ,$$

and an orientation for $i = 1, \ldots, n-1$:

$$\langle e_i(t), \gamma^{(i)}(t) \rangle > 0 \ .$$

At last we choose the sign of e_n so that the frame is positively oriented, w.r.t. the standard basis of $\mathbb{R}^n$.

In particular we get $e_1(t) = \gamma'(t)$ and the vectors form an orthonormal frame, the *Frenet frame* of the curve at the point $p = \gamma(t)$. The rate of change of the Frenet frame under a motion along the curve is described by the *Frenet formulae*:

$$\frac{de_i}{dt} = -k_{i-1}(t)e_{i-1} + k_i(t)e_{i+1},\ i = 1, \ldots, n\ ,$$

where we assume that $k_0 = k_n = 0,\ e_{n+1} = 0$.
The functions $k_i(t)$ are called the *higher curvatures* of the curve. When $n = 2$, we simply set $\kappa(t) = k_1(t)$.

Theorem 2.1 (Fundamental theorem of curves)**.** *Given functions $k_1, \ldots, k_{n-1}$ where for $i = 1, \ldots, n-2$ we have that $k_i(t) > 0$ for all t, there exists a curve γ in $\mathbb{R}^n$ for which they are the higher curvatures. Moreover, if γ_1 and γ_2 have $k_1, \ldots, k_{n-1}$ as higher curvatures, then they differ only by a rigid motion.*

For a proof in the case $n = 3$, see for example [4]. The theorem basically follows from the existence and uniqueness of solutions of linear differential equations, and the fact that viewed as a linear system of equations $\left(e'_1, \ldots, e'_n\right)^t = A\left(e_1, \ldots, e_n\right)^t$, the matrix A is skew-symmetric and sparse with $a_{i,i+1} = -a_{i+1,i}$ as only entries not being zero.

What is the geometric meaning of the higher curvatures? For a convex plane curve, it can be shown that the curvature is always positive, and that it measures the rate of change of the angle of the tangent to a fixed vector in the plane. But we will give the following more geometric interpretation. Consider three consecutive points on the curve. Each three points define a circle uniquely. By moving the points infinitesimally close to each other, the three points tend to one, and the circle defined at this point is called the *osculating circle.*

Proposition 2.2. *If $k_1(t) \neq 0$, the osculating circle is unique and the center of the osculating circle is given by*

$$\gamma(t) + \frac{1}{k_1(t)}e_2(t).$$

Proof. We give a proof for the case $n = 2$. Take $t_1, t_2, t_3 \in U$ where $U \subset S^1$ is a small enough neighborhood around t. Then it is clear that $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ don't lie on a straight line since $k_1(t) \neq 0$. So the circle through $\gamma(t_1), \gamma(t_2), \gamma(t_3)$ exists, with center p and radius r and we may assume without loss of generality that $t_1 < t_2 < t_3$. Set

$$F(t) = |\gamma(t) - p|^2 - r^2\ .$$

Since F has three different roots, applying the mean value theorem we find two numbers ξ_1, ξ_2 such that $t_1 < \xi_1 < t_2 < \xi_2 < t_3$ and $F'(\xi_1) = F'(\xi_2) = 0$.

Applying the mean value theorem once again we find η, $\xi_1 < \eta < \xi_2$ such that $F''(\eta) = 0$. We therefore get

$$F'(\xi_1) = 2\ \langle \gamma'(\xi_1),\, \gamma(\xi_1) - p\rangle = 0\ , \tag{2.1}$$
$$F''(\eta) = \langle \gamma''(\eta),\, \gamma(\eta) - p\rangle + \langle \gamma'(\eta),\, \gamma'(\eta)\rangle = 0\ . \tag{2.2}$$

By (2.1) we see that $p = \gamma(\xi_1) + \rho m$ where $\rho \in \mathbb{R}$ and $m \perp \gamma'(\xi_1)$, with the orientation so that $\langle m,\, e_2(t)\rangle > 0$.
We write m as a linear combination of the basis of the Frenet frame at t:

$$m = \sum_1^2 \alpha_i e_i(t)$$

where $\alpha_i = \langle m,\, e_i(t)\rangle$. Now

$$\begin{aligned}\alpha_1^2 &= \langle m,\, e_1(t)\rangle^2 = \langle m,\, e_1(t)\rangle^2 - \langle m,\, e_1(\xi_1)\rangle^2\\ &= \langle m,\, e_1(t) - e_1(\xi_1)\rangle^2 \to 0\ ,\ \text{as}\ \ \xi_1 \to t\ .\end{aligned}$$

The fact that $|m| = 1$ now gives that $\alpha_2 = 1$. Substituting $p = \gamma(\xi_1) + \rho m$ into (2.2) we get

$$\langle \gamma''(\eta),\, \gamma(\eta) - \gamma(\xi_1)\rangle - \rho m + \langle \gamma'(\eta),\, \gamma'(\eta)\rangle = 0$$

and so

$$\rho = \frac{\langle \gamma''(\eta),\, \gamma(\eta) - \gamma(\xi_1)\rangle + \langle \gamma'(\eta),\, \gamma'(\eta)\rangle}{\langle \gamma''(\eta),\, m\rangle}\ .$$

Since ξ_1, ξ_2 and η tends to t as t_1, t_2, t_3 tends to t, we get

$$\begin{aligned}p &\to \gamma(t)\frac{\langle \gamma''(t),\, \gamma(t) - \gamma(t)\rangle + \langle \gamma'(t),\, \gamma'(t)\rangle}{\langle \gamma''(t),\, e_2\rangle} e_2(t)\\ &= \gamma(t) + \frac{1}{k_1(t)} e_2(t)\end{aligned}$$

and the result follows. □

We will now see how the notions of osculating hyperplanes and osculating circles unite in the concept of contact.

Definition 2.1 (Order of contact)**.** Let $F : \mathbb{R}^n \to \mathbb{R}$ define a $(n-1)$-hypersurface M in $\mathbb{R}^n$. We say that a curve γ and $F^{-1}(0)$ have a *k-fold* or an *order of contact k* at a point $p = \gamma(t_0)$ provided the function g defined by

$$g(t) = F(\gamma_1(t), \ldots, \gamma_n(t)) = F(\gamma(t))$$

satisfies

$$g(t_0) = g'(t_0) = \ldots = g^{(k-1)}(t_0) = 0 \ ,$$
$$g^{(k)} \neq 0 \ .$$

Dropping the last condition, we say that the order of contact is at *least* k at a point.

Intuitively this says how similar the two manifolds are at the point and gives us a way to measure how flat a curve is by looking at the contact of the curve with the osculating hyperplane, or later on, how much it curves by looking at spheres of a certain order of contact with the curve.

Note that for the osculating hyperplane, F at a given point $\gamma(t_0)$ is given by

$$F(x) = \langle x - \gamma(t_0),\, e_n(t_0)\rangle \ ,$$

the plane that goes through $\gamma(t_0)$ and is orthogonal to the vector $e_n(t_0)$. From this we see that a curve always has contact of at least order n with its osculating hyperplane.

As a special case, a plane curve always has contact of at least order 2 with its tangent, the osculating 1-plane.

From Proposition 2.2 we conclude that the curvature corresponds to the inverse of the radius of the osculating circle at each point of the curve. Hence the osculating circle for a point of a plane curve is given by

$$F(x) = \left| x - (\gamma(t_0) + \frac{1}{\kappa(t_0)} e_2(t_0)) \right|^2 - \frac{1}{\kappa(t_0)}^2 \ . \tag{2.3}$$

Setting

$$p = \gamma(t_0) + \frac{1}{\kappa(t_0)} e_2(t_0)$$

we have for F as in (2.3) that

$$g'(t) = 2\,\langle \gamma'(t),\, \gamma(t) - p\rangle \tag{2.4}$$
$$g''(t) = \langle \gamma''(t),\, \gamma(t) - p\rangle + 1 \tag{2.5}$$

Hence from the proof of Proposition 2.2, in particular (2.1) and (2.2), it is clear that the order of contact at a general point of a curve with its osculating circle is at least 3. From this observation one can generalize and define the *osculating sphere* of a point of a curve to be the $(n-1)$-sphere with order of at least $n+1$ with the curve. Similar to the concept of support plane above, we say that an osculating sphere is an *osculating sphere of support* if the curve is contained solely in the closure of one of the two components of $\mathbb{R}^n$ determined by that sphere.

Definition 2.2. A point of a curve γ in $\mathbb{R}^n$ for which the osculating sphere has order of contact greater or equal to $n+2$ is called a *vertex*.

In conclusion we see that the study of curvature is equivalent to studying the radius of the osculating circle. We now see that extraordinarily flat points of a curve corresponds to higher order of contact with its osculating hyperplane, and we call these points *flattenings* of the curve. In the same sense extraordinarily curved points of a curve corresponds to points of higher order of contact with its osculating sphere, the *vertices* of the curve. The vertices will be the main object of study in the next sections.

Note that for plane curves, vertices corresponds to points where $\kappa' = 0$, i.e local extrema of the curvature function. To see this, we look at (2.5) to see that

$$g'''(t) = \langle \gamma'''(t), \gamma(t) - \big(\gamma(t_0) + \frac{1}{\kappa(t_0)} e_2(t_0)\big)\rangle$$

and use the Frenet formulae to see that

$$\begin{aligned} 0 = g'''(t_0) &= \langle \kappa'(t_0)e_2(t_0), -\frac{1}{\kappa(t_0)} e_2(t_0)\rangle + \langle \kappa(t)e_2'(t_0), -\frac{1}{\kappa(t_0)} e_2(t_0)\rangle \\ &= \langle \kappa'(t_0)e_2(t_0), -\frac{1}{\kappa(t_0)} e_2(t_0)\rangle + \langle \kappa(t_0)^2 e_1(t_0), -\frac{1}{\kappa(t_0)} e_2(t_0)\rangle \\ &= -\frac{\kappa'(t_0)}{\kappa(t_0)} \,. \end{aligned}$$

Hence for a curve to have order of contact greater or equal to 4, a necessary and sufficient condition is that $\kappa'(t) = 0$. Hence vertices of a curve corresponds to local extrema of the curvature function.
We are now able to state one of the main theorems of this thesis.

3 Smooth four vertex theorem

In this section γ will always be a plane curve, vertices mean local extrema of the curvature function or equivalently critical values of the radius of the osculating circle.

Theorem 3.1 (The four vertex theorem)**.** *Let γ be a closed simple plane curve, not a circle. Then γ has at least two local minima and two local maxima on its curvature function κ.*

In particular this means that γ has at least four vertices. Note that if γ is a circle, every point of γ is a vertex.

Proof. We follow the ideas of [8]. Consider the unique circumcircle C of γ, and denote its radius by r. We do the following observations:

- $\gamma \cap C$ contains at least two points and if only two, they would have to lie on a distance of $2r$ from eachother.
- There can be no arc D of C bigger than a half circle such that $D \cap \gamma = \emptyset$.
- In a small enough neighborhood of the curve around a point p in $C \cap \gamma$ we must have that $\kappa \geq \frac{1}{r}$ and that neighborhood of γ lies inside C.

Note that it is enough to find four consecutive points of γ, say $p_1, \ldots, p_4$ where $\kappa(p_1), \kappa(p_3) \geq \frac{1}{r}$ and $\kappa(p_2), \kappa(p_4) < \frac{1}{r}$ for some r since the κ then is forced to have two local minima and two local maxima.

From now on, assume that $C \cap \gamma$ has at least n components and $n \geq 2$. Let $p_1, \ldots, p_n$ be points ordered cyclically along $C \cap \gamma$, one in each component, and thus naturally dividing γ into curves $\gamma_1, \ldots, \gamma_n$ bounded pairwise by these points. From the observations above we have n points on γ where

$$\kappa \geq \frac{1}{r} \ .$$

We now claim that there are also n points where

$$\kappa < \frac{1}{r} \ .$$

Pick a component γ_i. Without loss of generality we may take $i = 1$. From the observations above we may assume that the end points of the component, denoted p_1 and p_2 lie in an open semi-circle of C. By an Euclidean motion we can put γ so that the center of C is at the origin and p_1 and p_2 may be connected by a vertical line. Since the curve between p_1 and p_2 is not contained in C it must contain a point q that is not contained in the corresponding arc of C, yet lies on the right of the vertical line connecting p_1 and p_2. The three points p_1, p_2 and q determine a circle C' with radius r'. Moreover C' must have greater radius than C since q lies inside C. But by moving C' succesively to the left we hit a point q' on the curve between p_1 and p_2 such that there is no other point intersecting C' if moved further to the left. Note that the curve between p_1 and p_2 and C' have the same orientation since γ is simple, and therefore have the same direction at q'. But then the curve lies locally outside C' and by the above consideration, we must have

$$\kappa(q') \leq \frac{1}{r'} < \frac{1}{r} \ .$$

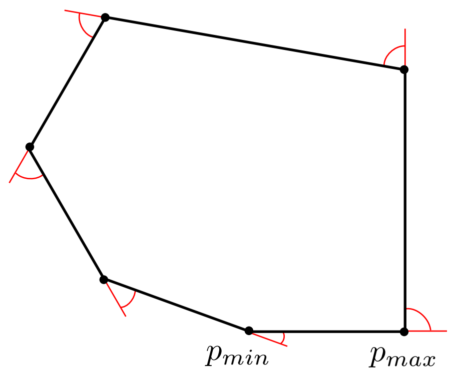

Figure 1: Polygon with only two local extrema of its exterior turning angle

So we have found two vertices for every component of $C \cap \gamma$. Left to consider is only the case of one component, i.e. $n = 1$. we then again have p_1, p_2 as endpoints of this component. But then the result is obvious since the arc must be at least a half-circle, and we may find a q' in the same way we found q' above, where $\kappa < \frac{1}{r}$. Then we may find two points in between q' and p_1 and q' and p_2 respectively where $\kappa > \frac{1}{r}$. At last in between these two points there has to be a point of which the curvature function has a local minimum. □

4 Discrete curves

Let's turn our attention to discrete curves in the plane: polygons. As we will see, there is a notion of curvature for polygons, giving rise to a completely analogous collection of four vertex theorems for them, as we have for the smooth curves.

We will no longer talk about vertices in the sense from above, since this word is reserved for the points of which the polygon is made out of. So denote a plane simple polygon by γ and is vertex set by $V(\gamma)$.

Since the curvature of a curve in the plane can be interpreted as the rate of change of the angle of the tangent, one might be tempted to define the curvature of a polygon as the *exterior* angle of two consecutive edges. However, it is easy to construct polygons that only have two extremum of its curvature function with this definition, see Figure(1). In the light of the correspondence between curvature and osculating circles, we instead define a

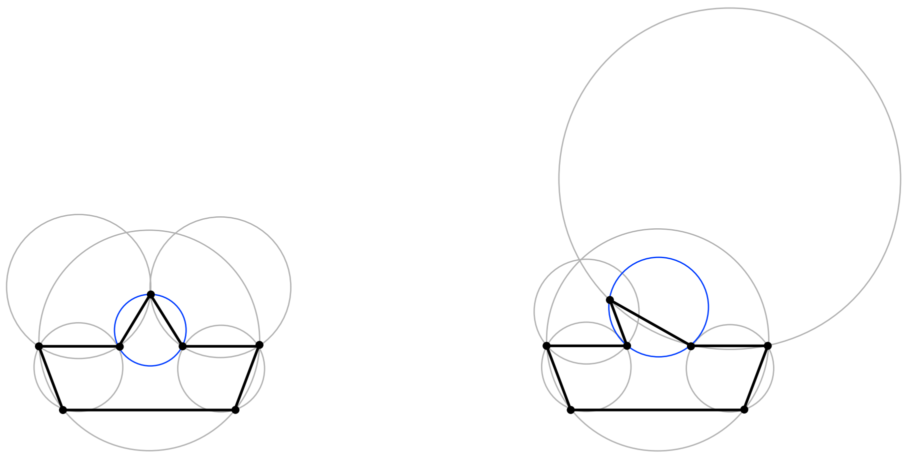

Figure 2: Locally regular and not locally regular

curvature function as the inverse of the radius of the circle of three consecutive points of a polygon.

Definition 4.1. If p_{i-1}, p_i, p_{i+1} are three consecutive non collinear points of a polygon, we call the circle passing through p_{i-1}, p_i, p_{i+1} the *circumcircle*. Sometimes we write circumcircle *through* p_i for short.

Definition 4.2. If the circle passing through three consecutive vertices $p_{i-1}, p_i, p_{i+1} \in V(\gamma)$ has all of $V(\gamma) \setminus \{p_{i-1}, p_i, p_{i+1}\}$ in one of the two components determined by the circle we call it a *support circle*. It is called *minimal*- or *maximal* circle of support if the vertices are in the outer- or inner component respectively.

Definition 4.3. If $p_i \in V(\gamma)$, let α_i denote the *internal* angle between the edges $p_{i-1}p_i$ and $p_i p_{i+1}$. This is well defined since γ is simple. For each $p_i \in V(\gamma)$, let r_i denote the radius of the circumcircle through the vertices p_{i-1}, p_i, p_{i+1}. We then define the *discrete curvature* $k(p_i) = \frac{1}{r_i}$ if $\alpha_i < \pi$ and $k(p_i) = -\frac{1}{r_i}$ if $\alpha_i \geq \pi$. Moreover if p_{i-1}, p_i, p_{i+1} lies on a line we set $k(p_i) = 0$.

If E is a subset of $V(\gamma)$ we write $k(E) < c$ if $k(p) < c$ for all $p \in E$ and similarly we write $k(E) > c$ if $k(p) > c$ for all $p \in E$. At last $k(E) = c$ means $k(p) = c$ for all $p \in E$.

For brevity we introduce the following notation.

Definition 4.4. A subset $I \subset V(\gamma)$ of consecutive vertices of a polygon γ is called an *interval*. If $I = \{p_i, p_{i+1}, \ldots, p_j\}$ is an interval, we set $I^\circ = \{p_{i+1}, p_{i+2}, \ldots, p_{j-1}\}$ and $\overline{I} = \{p_{i-1}, p_i, \ldots, p_j, p_{j+1}\}$.

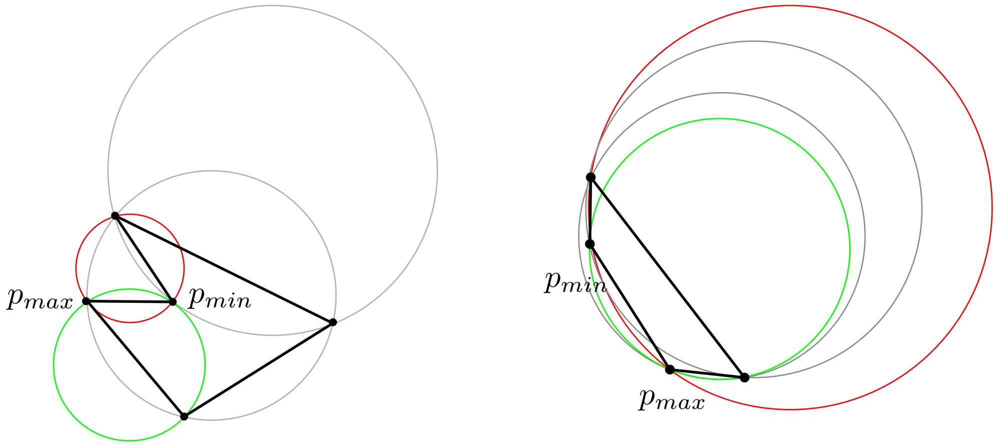

Figure 3: Not locally regular polygons with only two local extrema on their curvature functions

Definition 4.5. We say that p is a *local maximum* of k if $k(q) < k(p)$ for all $q \in \overline{I} \setminus I$ where I is the largest interval of vertices containing p such that $k(I) = k(p)$. *Local minimum* is defined analogously.

When we've come this far, it is not difficult to find polygons with only two local extrema on its curvature function, see Figure(3). We therefore need the following concept.

Definition 4.6. A polygon γ is called *locally regular* if for all vertices p_i in $V(\gamma)$ not on a line, the circumcenter of p_{i-1}, p_i, p_{i+1} is inside the closed angular sector $\angle p_{i-1}, p_i, p_{i+1}$. See Figure(2).

This name is adopted from [1]. In the literature there is a bit confusing terminology. Sedykh uses an additional condition, which is equivalent to the polygon being a subgraph of the Delaunay triangulation of $V(\gamma)$[2]. In [11] this condition is called *normality* and the local regularity of Dahlberg is called *goodness*. Together these two conditions form a class of curves called *nonsingular*. In [10] however, the Delaunay condition is called *regularity* whereas Dahlberg's condition alone is called *nonsingularity*. To avoid confusion, we have adopted Dahlberg's terminology, since the Delaunay property is not needed in the general form of the Discrete Four Vertex Theorem anyway.

In a way both these definitions are transparent, in that by adding vertices to the polygon on the line segments, and thus not adding any real curvature,

[2]A triangulation of a point set is called Delaunay, if the circumcircle of each triangle is the boundary of an empty disk, i.e it contains no other vertices

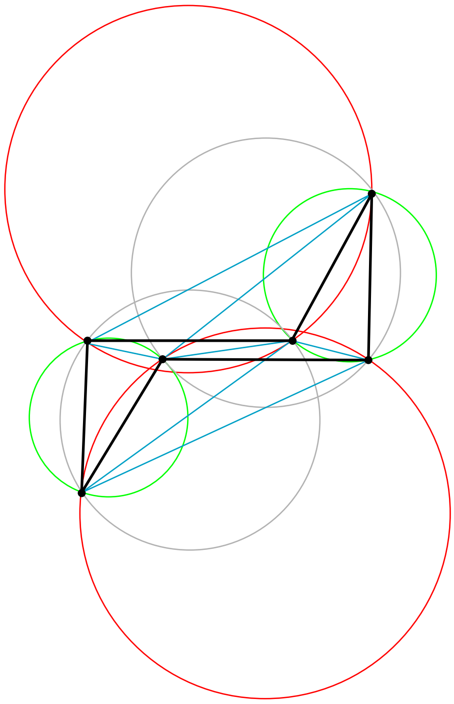

Figure 4: Polygon not being a subgraph of the Delaunay triangulation of its point set, but having two local minima and two local maxima on its discrete curvature function.

one can make an arbitrary polygon satisfy local regularity and the Delaunay condition. Moreover, since it is clear that approximating a smooth curve by a polygon with sufficiently many points, so that the edge lengths tends to zero, the discrete curvature tends to the smooth curvature. Therefore a sufficiently good approximation will also satisfy the local regularity and the Delaunay condition. However, there are situations when one approximates a smooth curve with curvature dependent edge lengths, that is, placing points of the polygon where the curve has large curvature. We call this a *curvature-approximated* polygon. Given an upper bound n it is possible to construct a smooth curve γ such that a polygon γ' that is a curvature approximation of γ with $|V(\gamma')| \leq n$ always fail to have the Delaunay property. Again see Figure(4). However, by curvature-approximating a smooth curve with sufficiently many points, the polygon will be locally regular since each triple of points on each circumcircle will stay on one halfcircle of the circumcircle, when traversed in order, in order to keep edge length to curvature constant.

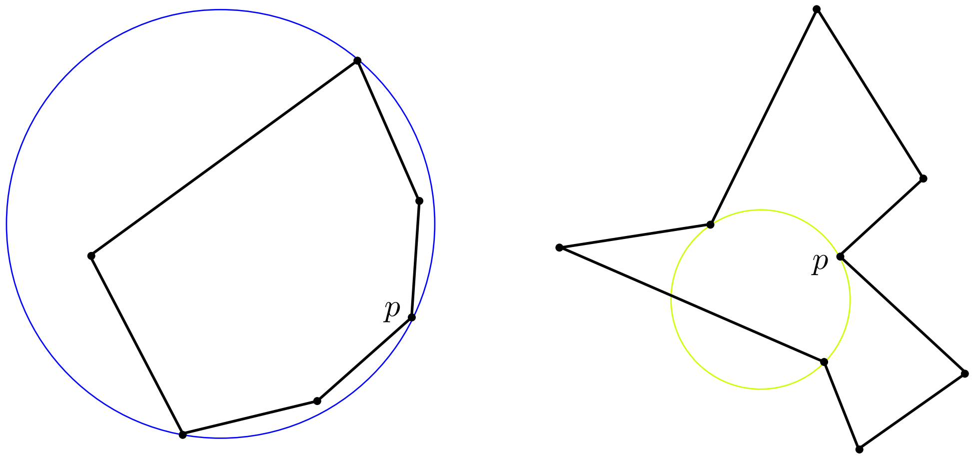

Figure 5: An element of $\mathcal{F}(p)$ and an element of $\mathcal{G}(p)$

5 Discrete four vertex theorem

A polygon is *convex* if the region bounded by it is a convex set. It is called *strictly convex* if it is convex and has no three consecutive vertices on a line.

In the smooth setting, the following holds: Every strictly convex curve (i.e without flattenings) has two minimal- and two maximal support circles. For a proof see [6].

Similar to the smooth case, every strictly convex polygon has two maximal and two minimal support circles. This was first proved in [9] although we will follow the approach from [1] here. Using this we will be able to prove a four vertex theorem for polygons in the plane.

We now introduce families of disks, whose boundary are almost minimal- and maximal support circles. They have almost the given support circle property, only lacking that the three points that define them are necessarily consecutive.

For $p \in V(\gamma)$, let $\mathcal{F}(p)$ denote the family of closed disks W such that $p \in \partial W$, $V(\gamma) \subset W$ and ∂W contains at least three points in $V(\gamma)$. Similarly let $\mathcal{G}(p)$ denote the family of all open disks such that $p \in \partial W$, $V(\gamma) \cap W = \emptyset$ and ∂W contains at least three points in $V(\gamma)$. See Figure(5).

Lemma 5.1. *Let γ be a strictly convex polygon. Then $\mathcal{F}(p)$ and $\mathcal{G}(p)$ are both non-empty for all $p \in V(\gamma)$.*

Proof. Pick $p \in V(\gamma)$ and let $\mathcal{A}(p) = \mathcal{F}(p)$ or $\mathcal{A}(p) = \mathcal{G}(p)$. If $\mathcal{A}(p) = \mathcal{F}(p)$, let W be a closed disk with $p \in \partial W$ such that $V(\gamma) \subset W$. In the case $\mathcal{A}(p) = \mathcal{G}(p)$, let W be an open disk with $p \in \partial W$ and $V(\gamma) \cap W = \emptyset$. Note

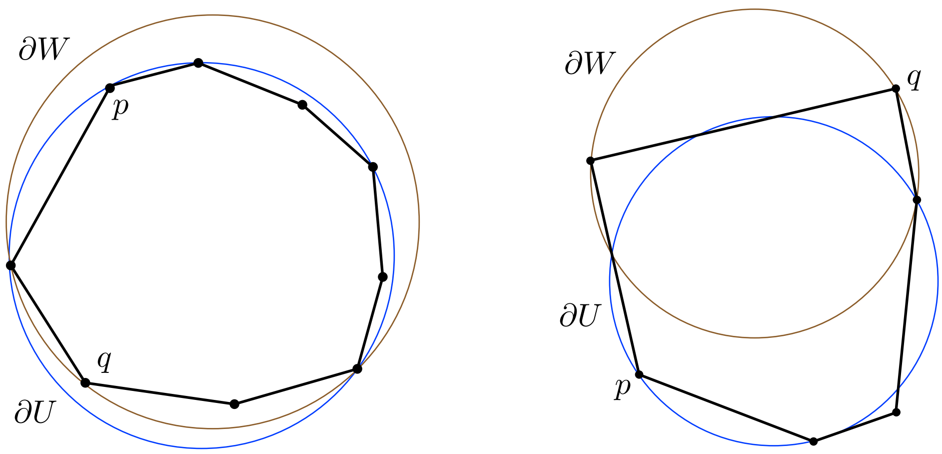

Figure 6: An illustration of Lemma 5.2 for $\mathcal{F}(p)$ and $\mathcal{G}(p)$.

that the strict convexity property is required in the first case, but not the latter. Now pick a point $q_1 \in \partial W \setminus p$. Consider the pencil of circles through p and q_1. It contains a circle ∂W_1 with the desired support circle property and the property that $(\partial W_1 \setminus \{p\}) \cap V(\gamma) \neq \emptyset$. So take $q_2 \in (\partial W_1 \setminus \{p\}) \cap V(\gamma)$. Again considering the pencil of circles through p and q_2 we find a closed disk W_2 that belongs to $\mathcal{A}(p)$. □

Lemma 5.2. *Let γ be a strictly convex polygon, let $p \in V(\gamma)$ and let $\mathcal{A}(p) = \mathcal{F}(p)$ or $\mathcal{A}(p) = \mathcal{G}(p)$. Pick $U \in \mathcal{A}(p)$ and assume $I \neq \emptyset$ is a maximal interval contained in $V(\gamma) \setminus \partial U$. If $q \in I$ and $W \in \mathcal{A}(q)$ then $\partial W \cap V(\gamma) \subset \overline{I}$. See Figure(6).*

Proof. There is nothing to prove if $\overline{I} = V(\gamma)$. So assume that $\overline{I}$ is a proper subset of $V(\gamma)$. Let $p_1, p_2 \in \overline{I} \setminus I$ be the endpoints of $\overline{I}$ and let L be the line through p_1 and p_2. Then L divides the plane into two open half planes H and H^*, with H chosen such that $q \in H$. There are two cases:

1. $\mathcal{A}(p) = \mathcal{F}(p)$: We have $\{p_1, p_2\} \subset W$ and by convexity of γ, $I = H \cap V(\gamma)$. Hence $\partial W \cap U \subset \overline{H}$. In particular we get
$$\partial W \cap V(\gamma) \subset (\partial W \cap U) \cap V(\gamma) \subset \overline{I}\,.$$
2. $\mathcal{A}(p) = \mathcal{G}(p)$: We first note that neither U nor W cannot be contained in one-another since $q \in \partial W$ but $q \notin \partial U$. Moreover we have $\{p_1, p_2\} \cap W = \emptyset$ and $p_1, p_2 \in \partial U \cap V(\gamma)$. Assume now that the conclusion doesn't hold and that there is a $q^* \in (\partial W \cap V(\gamma)) \setminus \overline{I}$. Then $q^* \in H^*$

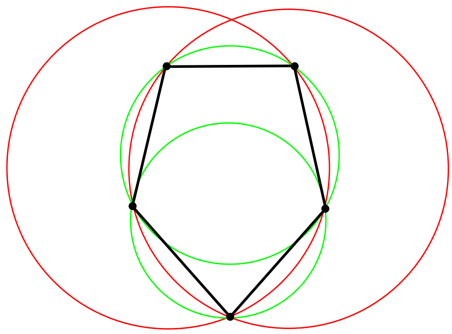

Figure 7: Every strictly convex polygon has two distinct minimal and maximal support circles

since $\overline{I} = V(\gamma) \cap \overline{H}$. By strict convexity we have that the open line segment p_1p_2 contains the closed line segment $\overline{qq^*}$ intersected with L. Therefore $\overline{W} \cap L$ is contained in the closed line segment $\overline{p_1p_2}$. And so $\partial W \cap \partial U \subset \overline{H}$ giving $\overline{W} \cap H^* \subset U$. Since U is open and $U \cap V(\gamma) = \emptyset$ we get a contradiction. □

Proposition 5.3. *Every strictly convex polygon, whose vertices do not lie on a circle, has two distinct minimal and two distinct maximal support circles. See Figure(7).*

Proof. Recall Definition 4.2 and again put $\mathcal{A}(p) = \mathfrak{F}(p)$ or $\mathcal{A}(p) = \mathfrak{G}(p)$. Let $p \in V(\gamma)$, and look at $I = V(\gamma) \cap \partial U$ for $U \in \mathcal{A}(p)$. If I is an interval, we have already found one support circle. Looking at $V(\gamma) \setminus \partial U$, we can partition it into a set of maximal nonempty intervals $I_1, \ldots, I_n$. Note that $n \geq 2$ if I is not an interval, and the only possibility that $n = 1$ is if I is an interval and we already have one support circle. By repeated use of the lemma above, there are points $q_k \in I_k$ such that if $W_k \in \mathcal{A}(q_k)$ then $V(\gamma) \cap \partial W_k \subset \overline{I_k}$ and $V(\gamma) \cap \partial W_k$ is an interval. Now look at $V(\gamma) \cap \partial W_k$. Since $V(\gamma) \cap \partial W_k$ is connected there is a $q_k^* \in \partial W_k$ such that the circumcircle of q_k^* is the boundary of W_k. Moreover since $q_j \notin \partial W_k$ if $j \neq k$, we have found n extra distinct support circles and hence, we are finished. □

For an edge e of γ, let h_e be the closed half plane determined by the line of e and the interior of γ. That is we take h_e such that if taken a sufficiently

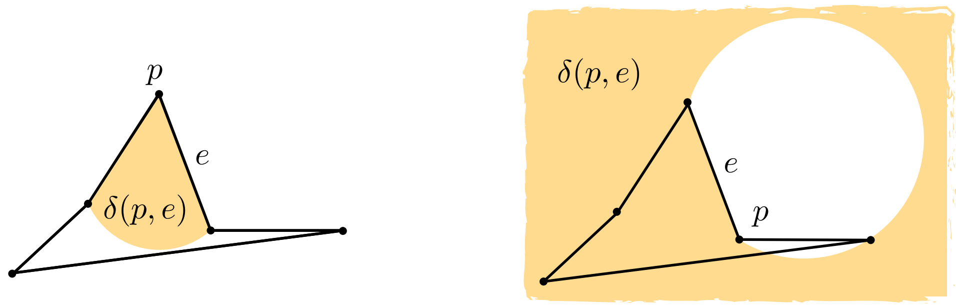

Figure 8: $\delta(p,e)$ for positive and negative curvature

small ball W around the midpoint of e, $W \cap h_e = W \cap \omega(\gamma)$, where $\omega(\gamma)$ is the interior of γ. Let ω denote the closed circumdisk enclosed by the circumcircle of p and set

$$\delta(p,e) = \begin{cases} \omega \cap h_e & k(p) > 0 \\ h_e & k(p) = 0 \\ \overline{\omega^c} \cup (\omega \cap h_e) & k(p) < 0 \, . \end{cases}$$

See Figure(8).

For the next lemma and proposition we consider convex polygons.

Lemma 5.4. *Let e be an edge of a simple locally regular polygon γ, with endpoints p and q. Then the following holds*

1. $k(p) \geq 0 \Rightarrow \delta(p,e) \subset h_e$
2. $k(p) \leq 0 \Rightarrow h_e \subset \delta(p,e)$
3. $k(p) \leq k(q) \Rightarrow \delta(q,e) \subset \delta(p,e)$.

Proof. The first properties follow from the definition. Moreover, if $k(p)k(q) \leq 0$ the third property follow from the first two. Therefore it is enough to consider the case $k(p)k(q) > 0$. Then, since γ is locally regular, the circumcircles must have their centers on the same side of e. There are two cases:

1. $0 < k(p) \leq k(q)$: Then the circumcircle for p is larger than the circumcircle for q and the lemma follows.
2. $0 > k(q) \geq k(p)$: Then the circumcircle for p is smaller than the circumcircle for q and the lemma follows.

□

Proposition 5.5. *Every strictly convex, locally regular polygon γ, not having all its vertices on a circle, has two local minima and maxima on its curvature function.*

Proof. Let $q_{min}, q_{max} \in V(\gamma)$ be such that

$$k(q_{min}) = \min\{k(p) \mid p \in V(\gamma)\} ,$$

$$k(q_{max}) = \max\{k(p) \mid p \in V(\gamma)\} .$$

Assume that the conclusion fails. Then k must be monotone on the two maximal intervals in $V(\gamma) \setminus \{q_{min}, q_{max}\}$. So for any $p \in V(\gamma)$ be different from q_{max} we can find points $p_1, \ldots, p_n$ such that $p_1 = q_{max}$, $p_n = p$, $k(p_{i+1}) \leq k(p_i)$. Denote the edge $p_i p_{i+1}$ by e_i for each $i = 1, \ldots, n-1$. Then from the above lemma it follows that

$$\delta(p_i, e_i) \subset \delta(p_{i+1}, e_i) .$$

Remember that because of strict convexity we have that the closed interior of γ is a subset of h_{e_i} for all $i = 1, \ldots, n-1$. Hence we get that restricted to the closed interior of γ, the closed circumdisk of q_{max} is a subset of the closed circumdisk of p for all $p \in V(\gamma)$. For a minimal support circle C this means that the circumcircle of q_{max} and C coincide, since a minimal support circle contains no vertices in its open interior. This contradicts Proposition 5.3 that says that γ has at least two distinct minimal support circles, which are distinct. □

After this next lemma, we are able to prove the four vertex theorem for polygons.

Lemma 5.6. *Let p_{i-1}, p_i, p_{i+1} be three vertices of a locally regular polygon, and let r be the radius of the circumcircle of the three vertices. Then any disk Δ containing p_{i-1}, p_i, p_{i+1} with $p_i \in \partial\Delta$ must have radius $R \geq r$, with equality if and only if $\partial\Delta$ is the circumcircle through p_{i-1}, p_i, p_{i+1}.*

Proof. Let Ω be the circumdisk of p_{i-1}, p_i, p_{i+1} and let r denote its radius. Consider Δ, a disk other than the circumdisk with $p_i \in \partial\Delta$. By possible shrinking Δ we can assume that $p_{i-1}, p_i \in \partial\Delta$. Let H be the closed halfplane determined by the line p_{i-1}, p_i containing p_{i-1}, p_i, p_{i+1}. We center a disk δ in the middle of the edge between p_{i-1} and p_i, with radius $\frac{|p_{i-1}p_i|}{2}$. Then $\delta \cap H \subset \Omega \cap H$ since the polygon is locally regular. Let $\alpha = \angle p_{i-1}, p_i, p_{i+1}$. Now if $0 < \alpha \leq \frac{\pi}{2}$ the center of Ω is inside the triangle $\triangle p_{i-1}, p_i, p_{i+1}$ and Ω is clearly the smallest enclosing disk. So assume that $\alpha > \frac{\pi}{2}$. If the center of Δ is called z, then $z \in H$, because otherwise $\Delta \cap H \subset \delta \cap H$ which is impossible since $p_{i+1} \in \Delta$ and $p_{i+1} \notin \delta$ since $\alpha > \frac{\pi}{2}$. But now, if the radius of Δ is less than r, we have $\Omega \cap \Delta \cap \mathrm{Int}(H) = \emptyset$ which is impossible since $z \in H$. $\square$

Theorem 5.7 (The Four-vertex theorem for plane polygons)**.** *Let γ be a simple plane polygon, assume γ to be locally regular and that the vertices not lie on a circle. Then the discrete curvature k of γ has at least two local minima and two local maxima.*

Proof of Theorem 5.7. Let $q_{min}, q_{max} \in V(\gamma)$ be such that

$$k(q_{min}) = \min\{k(p) \mid p \in V(\gamma)\} ,$$

$$k(q_{max}) = \max\{k(p) \bigm| p \in V(\gamma)\} .$$

If the conclusion doesn't hold, γ must be non-convex due to the Proposition 5.5. Moreover $k(q_{min}) \leq 0$, since a non-convex curve has negative curvature. Furthermore k must be monotone on the subintervals of $V(\gamma)$ that have q_{min} and q_{max} as endpoints, because otherwise we would have one second minimum and maximum in these intervals. Now just as in the smooth case, we consider the smallest possible circumcircle C of γ and we let r be its radius. We also let Ω be the open disc enclosed by this circumcircle, i.e. we have $\partial\Omega = C$. Now put

$$E = V(\gamma) \cap C .$$

By Lemma 5.6 we know that

$$k(E) \geq \frac{1}{r}$$

and if $p_i \in E$ but any of p_{i-1} or p_{i+1} is not in E, then

$$k(p_i) > \frac{1}{r} .$$

What does E look like? We claim that it is not connected. Assume that E is an interval and $|E| = N$. If $N = 2$ then $E = \{p_a, p_b\}$ and $|p_a p_b| = 2r$ since p_a and p_b must be on a diameter of Ω. But on the other hand $k(p_a) > \frac{1}{r}$ which is a contradiction. Therefore $N > 2$. But in that case $I^\circ \neq \emptyset$ and $k(I^\circ) = \frac{1}{r}$ whereas $k(E \setminus I^\circ) > \frac{1}{r}$. Since $q_{min} \notin E$ and $k(q_{min}) \leq 0$, k must have two local minima: one inside E and one outside, which is goes against our assumption. So E is not connected, and therefore there is at least two maximal subintervals in $V(\gamma) \setminus E$. If

$$F = \{p \in V(\gamma) \bigm| k(p) \leq 0\} ,$$

F must be connected. But since $E \cap F = \emptyset$ there must be a maximal subinterval $J \subset V(\gamma) \setminus E$ such that $F \cap J = \emptyset$ and therefore $k(J) > 0$. So if

$$J^* = \overline{J} \setminus J ,$$

then $|J^*| = 2$, $J^* \subset E$ and $k(J^*) \geq \frac{1}{r}$. Let Γ be the (non-closed) sub-polygon of γ such that $V(\Gamma) = V(\overline{J})$. Then Γ separates Ω into two domains U and V, where we let U be the one containing the interior of γ. Let Γ^* be the (smooth) subarc of $\partial U \cap C$ that has the two elements of J^* as its endpoints. Then $\partial U = \Gamma \cup \Gamma^*$ and $k(J) > 0$ implies that U is convex. By selecting consecutive points $\{A_i\}_1^n$, $A_i \in \partial U \setminus \Gamma$ we can create another polygon γ^* with vertex set $V^* = J^* \cup \{A_i\}_1^n$ and discrete curvature function k^*. Note that $k^*(V^*) > 0$ and moreover, since $k^*(J) = k(J)$ we have that $k^*(J) > \frac{1}{r}$. Since ∂U cannot be less than a semicircle, by choosing A_1 and A_n close enough to J^*, we can assure that γ^* is locally regular. In particular $k^*(J^*) > \frac{1}{r}$ since we can apply Lemma 5.6, giving

$$k^*(\overline{J}) > \frac{1}{r} .$$

Then finally, let S be the set of vertices whose circumcircles are maximum support circles for γ^*, and thus also enclosing γ. Now for $p \in V(\gamma^*) \setminus \overline{J}$, the circumcircle of p coincides with C. Moreover we have $k^*(S) \leq \frac{1}{r}$. Together with the fact above that $k^*(\overline{J}) > \frac{1}{r}$ we see that

$$S = V(\gamma^*) \setminus \overline{J} ,$$

thus making C the unique maximal support circle of γ^*. This contradicts Proposition 5.3 which in particular stays that γ^* has two distinct maximal support circles. □

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