Parabolic Problems for the Anderson Model

II. Structure of High Peaks and Lifshitz Tails*

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Summary. This is a continuation of our previous work [9] on the investigation of intermittency for the parabolic equation $(\partial/\partial t)u = \mathcal{H}u$ on $\mathbb{R}_+ \times \mathbb{Z}^d$ associated with the Anderson Hamiltonian $\mathcal{H} = \kappa\Delta + \xi(\cdot)$ for homogeneous random potentials $\xi(\cdot)$. For the Cauchy problem with nonnegative homogeneous initial condition we study the second order asymptotics of the statistical moments $\langle u(t,0)^p \rangle$ and the almost sure growth of $u(t,0)$ as $t \to \infty$. We also deal with the Lifshitz tails of the spectral distribution function ('integrated density of states') of $\mathcal{H}$. Here we mainly treat the important case of i.i.d. potentials and discuss the crucial role of double exponential tails of $\xi(0)$ for the formation of high intermittent peaks of the solution $u(t,\cdot)$ with asymptotically finite size. The challenging motivation for this paper was to achieve a better understanding of the geometric structure of such high exceedances which in one or another sense provide the main contribution to the solution. This is essential for different asymptotic problems related to the parabolic Anderson model. The behavior of the moments and the Lifshitz tails is also studied for correlated potentials.

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Contents

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Introduction

This paper is a natural continuation of our article [9]. The subject is the same, asymptotic analysis as $t \to \infty$ of the parabolic Anderson problem with homogeneous random potential $\xi(\cdot)$:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \kappa \Delta u + \xi(x) u, \qquad (t, x) \in \mathbb{R}_+ \times \mathbb{Z}^d, \\ u(0, x) &\equiv 1. \end{aligned}$$

In [9] we used rather 'soft' qualitative arguments to prove intermittency for the solution u under minimal conditions on the potential $\xi(\cdot)$. For a general discussion of intermittency and further references see the lectures [11].

Roughly speaking, intermittency means that, in contrast with homogenization, the spatial structure of $u(t, \cdot)$ is highly irregular for large t. In one or another sense the essential part of the solution is believed to consist of islands of high peaks which are located far from each other. If this picture is true, then the sizes of these islands as well as the heights and shapes of the corresponding peaks will be crucial for different asymptotic questions related to our Anderson problem. A detailed understanding of the geometric structure of intermittent solutions will therefore be extremely useful.

The challenging stimulus for the present paper was the question about the asymptotic size of the mentioned islands of high exceedances and its implications for different asymptotic problems. But, instead of directly investigating the spatial structure of $u(t, \cdot)$, we will study the second order asymptotics of the moments $\langle u(t, x)^p \rangle$, $p = 1, 2, \ldots$, and the almost sure growth of $u(t, x)$ as $t \to \infty$ for fixed x. We will also consider the Lifshitz tails of the spectral distribution function ('integrated density of states') associated with the Anderson Hamiltonian

$$\mathcal{H} = \kappa \Delta + \xi(\cdot).$$

In the larger part of the paper we will restrict ourselves to the important case (especially popular in the theory of random operators) when the potential $\xi(\cdot)$ consists of independent, identically distributed random variables unbounded from above. In this case the solution $u(t,\cdot)$ is known to develop an intermittent behavior as $t \to \infty$, see [9]. Implicitly, our results and their proofs will allow a rather detailed insight into the geometry of the peaks.

In the i.i.d. case, a crucial role is played by double exponential tails with parameter ϱ, $0 < \varrho < \infty$:

$$\mathrm{Prob}(\xi(0) > r) = \exp\{-e^{r/\varrho}\}, \qquad r \to \infty.$$

Such tail behavior leads to islands of asymptotically finite size. In the case of 'heavier' tails (corresponding to $\varrho = \infty$ and including Gaussian potentials) the islands consist of isolated single lattice sites. On the other hand, for 'almost bounded' potentials (with faster decaying tails corresponding to $\varrho = 0$) the optimal peaks form very large flat islands. Qualitatively, the last situation is similar to the picture presented by A.-S. Sznitman in a series of papers on Brownian motion in a Poissonian environment, see e.g. [13] and [14].

In *Section 1* we will prove that

$$\langle u(t,0)^p \rangle = \exp\Big\{ H(pt) - 2d\kappa\chi(\frac{\varrho}{\kappa})pt + o(t) \Big\} \tag{0.1}$$

as $t \to \infty$, where H is the cumulant generating function of $\xi(0)$ and

$$e^{H(pt)} := \Big\langle e^{pt\xi(0)} \Big\rangle$$

is supposed to be finite for $t \geq 0$. The last condition guarantees the existence of all statistical moments of the (homogeneous and ergodic) solution $u(t,\cdot)$. The shapes of the high exceedances of the solution determine the function χ which may be expressed in terms of a variational problem. We will see that $\chi(0) = 0$, $0 < \chi(\varrho) < 1$ for $0 < \varrho < \infty$, and $\chi(\infty) = 1$. In the case when $\varrho = \infty$, the factor $\exp\{-2d\kappa pt\}$ in (0.1) may be easily explained by use of the Feynman-Kac formula

$$u(t,0) = \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\}, \tag{0.2}$$

where $x(t)$ is simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$. Namely, this factor will appear if the random walk is forced to stay at 0 until time t. Indeed,

$$\begin{aligned} u(t,0) &\geq \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s))\, ds \right\} \mathbb{1}(x(s) = 0 \text{ for } s \in [0,t]) \\ &= e^{t\xi(0)}\, \mathbb{P}_0(x(s) = 0 \text{ for } s \in [0,t]) \\ &= e^{t\xi(0) - 2d\kappa t}, \end{aligned}$$

and therefore

$$\langle u(t,0)^p \rangle \geq e^{H(pt) - 2d\kappa pt}.$$

Hence, in this case the random walk prefers to stay at one and the same lattice site for almost all the time. This makes it plausible that for $\varrho = \infty$ the islands of high exceedances consist of single lattice sites and that in general $\chi(\varrho/\kappa)$ is closely related to the size of these islands. In fact, the solution to the mentioned variational problem, which is given by a nonlinear difference equation, is expected to determine the nonrandom shape of the relevant peaks. Note also that the simple universal bounds

$$e^{H(pt)-2d\kappa pt} \leq \langle u(t,0)^p \rangle \leq e^{H(pt)}$$

are valid for arbitrary homogeneous potentials $\xi(\cdot)$ with finite cumulant generating function H, see [9] for a proof of the upper bound.

In *Section 2* we will show under reasonable regularity assumptions that

$$u(t,0) = \exp\left\{ t\psi(d\log t) - 2d\kappa\chi(\frac{\varrho}{\kappa})t + o(t) \right\} \tag{0.3}$$

as $t \to \infty$ for almost all realizations of the random potential $\xi(\cdot)$. Thereby the function ψ is again fully determined by the tail behavior of the distribution of $\xi(0)$. In fact, $\psi(d\log t)$ describes the almost sure asymptotics of the maximum of the potential $\xi(\cdot)$ in a ball of radius t as $t \to \infty$. Note that for unbounded from above potentials the moments $\langle u(t,0)^p \rangle$ grow much faster then the solution $u(t,0)$ itself, which is one more manifestation of intermittency. Hence, the leading terms in the asymptotic formulas (0.1) and (0.3) are totally different. But the second order correction terms, which contain the essential information about the geometry of the relevant peaks, coincide. This means that in both cases the advantageous peaks have the same shape but different heights. This coincidence is closely related to the special properties of the double exponential distribution and will be explained below.

In *Section 3* we will prove under certain restrictions that the tail $\bar{N}(\lambda) = 1 - N(\lambda)$ of the spectral distribution function N of the Anderson Hamiltonian $\mathcal{H}$ satisfies

$$\log \bar{N}(\lambda) \sim \log \bar{F}\left(\lambda + 2d\kappa\chi(\frac{\varrho}{\kappa})\right)$$

as $\lambda \to \infty$, where $\bar{F}(\lambda) := \mathrm{Prob}(\xi(0) > \lambda)$. Here again the correction term $2d\kappa\chi(\varrho/\kappa)$ appears. For general background information on Lifshitz tails we refer to Pastur and Figotin [12].

Section 4 contains an alternative proof of (0.1) for $0 < \varrho < \infty$ which makes the role of the geometry of high peaks more transparent. After that our results for the moments and the Lifshitz tails will be generalized to a large class of correlated potentials $\xi(\cdot)$. In such generality, the proofs are more involved and rely on a large part on the ideas and results of the previous sections. We do not intend to treat the general case from the very beginning, since this would have made it more difficult for the reader to extract the basic ideas and techniques.

Conceptionally our results are closely related to the spectral analysis of the Anderson Hamiltonian $\mathcal{H}$. According to localization theory (see e.g. Aizenman and Molchanov [1]), the upper part of the spectrum of $\mathcal{H}$ in $l^2(\mathbb{Z}^d)$ consists of a complete system of exponentially localized (random) eigenfunctions e_n corresponding to (non-isolated) eigenvalues λ_n. Hence, one may try to expand the solution $u(t,\cdot)$ of our Cauchy problem in a Fourier series with respect to these eigenfunctions:

$$u(t,\cdot) = \sum_n e^{\lambda_n t}(e_n, \mathbb{1})e_n(\cdot).$$

This suggests the following general picture known in the physics literature, see e.g. Lifshitz, Gredescul, and Pastur [10]:

(i) The main contribution to the solution $u(t,\cdot)$ and its statistical moments is given by local maxima of the potential $\xi(\cdot)$ of height comparable with the asymptotic formulas (0.1) and (0.3), respectively.

(ii) The main contribution to the Feynman-Kac representation (0.2) of the solution is given by those trajectories of the random walk $x(s)$, $s \in [0,t]$, which spend the overwhelming time near these high exceedances of the potential.

(iii) The eigenvalues responsible for the asymptotics of $u(t,0)$ and $\langle u(t,0)^p\rangle$ are 'generated' by these local maxima, and the corresponding eigenfunctions are localized nearby and produce the high peaks of the solution $u(t,\cdot)$. Since the relevant maxima are separated from each other by a huge distance, these eigenfunctions may be considered as ground states corresponding to such peaks ('potential wells'). Of course, the eigenfunctions are not strictly positive, but their negative parts (caused by interaction between distinct maxima) are very small and may be evaluated by means of an appropriate cluster expansion.

Let us now explain how and why the double exponential tails enter this picture in the i.i.d. case. The fundamental property of the double exponential distribution is that

$$\text{Prob}\left(\xi(x) > h + \varphi(x),\, |x| \le R\right) = \exp\left\{-e^{h/\varrho}\sum_{|x|\le R} e^{\varphi(x)/\varrho}\right\}.$$

This means that, independent of their common height h, two local peaks of the potential of the form $h+\varphi(\cdot)$ and $h+\tilde\varphi(\cdot)$ occur with the same frequency if and only if

$$\sum_{|x|\le R} e^{\varphi(x)/\varrho} = \sum_{|x|\le R} e^{\tilde\varphi(x)/\varrho}.$$

Because of the above spectral theoretical considerations, we conclude from this that, both for $u(t,0)$ and $\langle u(t,0)^p\rangle$, the shape $\varphi(\cdot)$ of the typical peaks (normalized by $\sum_x e^{\varphi(x)/\varrho} = 1$) maximizes the principle eigenvalue $\lambda(\tilde\varphi)$ of the operator $\kappa\Delta + \tilde\varphi(\cdot)$ among all shapes $\tilde\varphi(\cdot)$ with

$$\sum_x e^{\tilde\varphi(x)/\varrho} = 1. \tag{0.4}$$

The corresponding positive eigenfunction describes the shape of the advantageous peaks of the solution $u(t,\cdot)$ near the relevant local maxima of the potential. We will see in Section 2 that under the constraint (0.4) the maximum of $\lambda(\tilde{\varphi})$ coincides with the term $-2d\kappa\chi(\varrho/\kappa)$ in (0.1) and (0.3). For technical reasons, in Section 1 we will describe χ by means of a different, but equivalent, variational problem. For $\varrho \to \infty$, the shapes $\tilde{\varphi}(\cdot)$ in (0.4) become more and more δ-like. But for $\varphi(0) = 0$ and $\varphi(x) = -\infty$, $x \neq 0$, the principle eigenvalue $\lambda(\varphi)$ equals $-2d\kappa$ in accordance with $\chi(\infty) = 1$. If $\varrho \to 0$, then $\tilde{\varphi}(\cdot)$ becomes more and more flat. But $\lambda(\varphi) = 0$ for $\varphi \equiv 0$, which makes it plausible that $\chi(0) = 0$.

1. Asymptotics of the statistical moments

1.1. Statement of the result

This section deals with the random Cauchy problem

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= \kappa\Delta u(t,x) + \xi(x)u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d, \\ u(0,x) &= u_0(x), \qquad x \in \mathbb{Z}^d, \end{aligned} \tag{1.1}$$

for the Anderson tight binding Hamiltonian

$$\mathcal{H} := \kappa\Delta + \xi(\cdot).$$

Thereby κ denotes a positive diffusion constant and Δ is the lattice Laplacian:

$$\Delta f(x) := \sum_{y:\,|y-x|=1} [f(y) - f(x)], \qquad x \in \mathbb{Z}^d.$$

The potential $\xi(\cdot)$ and the nonnegative initial datum $u_0(\cdot)$ are random fields on a joint probability space. The underlying probability measure and expectation will be denoted by $\mathrm{Prob}(\cdot)$ and $\langle\cdot\rangle$, respectively.

We will assume throughout that the field $\xi(\cdot)$ consists of independent, identically distributed random variables whose cumulant generating function H is finite on the positive half-axis:

$$H(t) := \log \left\langle e^{t\xi(0)} \right\rangle < \infty \qquad \text{for } t \geq 0.$$

The field $u_0(\cdot)$ is supposed to be nonnegative, homogeneous, and independent of $\xi(\cdot)$. We further require that

$$0 < \langle u_0(0)^p \rangle < \infty \qquad \text{for } p = 1, 2, \ldots \tag{1.2}$$

The above assumptions guarantee that a.s. the Cauchy problem (1.1) admits a unique nonnegative solution u. For each $t \geq 0$, $u(t,\cdot)$ is a spatially homogeneous random field, and

$$0 < \langle u(t,x)^p \rangle < \infty \qquad \text{for } p = 1, 2, \ldots \text{ and } (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d.$$

If, in addition, $u_0(\cdot)$ is ergodic, then $u(t,\cdot)$ is also ergodic for all t. For details see [9], Sections 2 and 3.

The objective of this section is to study the asymptotic behavior of the moments $\langle u(t,x)^p \rangle$, $p = 1,2,\dots$, as $t \to \infty$ under the following regularity assumption on the cumulant generating function H.

Assumption (H). There exists ϱ, $0 \le \varrho \le \infty$, such that

$$\lim_{t\to\infty} \frac{H(ct) - cH(t)}{t} = \varrho c \log c \tag{1.3}$$

for all $c \in (0,1)$.

In order to explain the meaning of this assumption let us consider the case when the potential is double exponentially distributed with parameter ϱ, $0 < \varrho < \infty$:

$$\mathrm{Prob}(\xi(0) > r) = \exp\{-e^{r/\varrho}\}, \qquad r \in \mathbb{R}.$$

Then $H(t) = \log \Gamma(\varrho t+1) = \varrho t \log(\varrho t) - \varrho t + o(t)$, and (1.3) is fulfilled. Therefore, roughly speaking, for $0 < \varrho < \infty$, Assumption (H) tells us that the upper tail of the distribution of $\xi(0)$ behaves like that of a double exponential distribution. In the case $\varrho = \infty$ the tail is 'heavier', i.e. we are 'beyond' the double exponential situation. Finally, $\varrho = 0$ means that the tail decays faster than in the double exponential case, and we will say that the potential $\xi(\cdot)$ is 'almost bounded.'

Remark 1.1. a) If $0 \le \varrho < \infty$, then Assumption (H) says that the function $\exp\{H(t)/t\}$ is regularly varying with exponent ϱ.

b) If $0 \le \varrho < \infty$, then the convergence in (1.3) is uniform on $[0,1]$. For $\varrho = \infty$, the convergence to $-\infty$ is uniform on each compact subset of $(0,1)$. This follows from the observation that the function on the left of (1.3) is convex in c.

By $\mathcal{P}(\mathbb{Z})$ we will denote the space of probability measures on $\mathbb{Z}$. We next introduce the *Donsker-Varadhan functional* $S\colon \mathcal{P}(\mathbb{Z}) \to \mathbb{R}_+$ and the *entropy functional* $I\colon \mathcal{P}(\mathbb{Z}) \to \mathbb{R}_+$ defined by

$$S(p) := \sum_{x\in\mathbb{Z}} \left(\sqrt{p(x+1)} - \sqrt{p(x)}\right)^2, \qquad p \in \mathcal{P}(\mathbb{Z}),$$

and

$$I(p) := -\sum_{x\in\mathbb{Z}} p(x)\log p(x), \qquad p \in \mathcal{P}(\mathbb{Z}),$$

respectively. Our result will be described in terms of the cumulant generating function H and the function

$$\chi(\varrho) := \frac{1}{2} \inf_{p\in\mathcal{P}(\mathbb{Z})} \left[S(p) + \varrho I(p)\right], \qquad 0 \le \varrho < \infty. \tag{1.4}$$

One easily checks that χ is strictly increasing and concave and $0 \le \chi < 1$. Moreover, $\chi(0) = 0$ and $\lim_{\varrho\to\infty} \chi(\varrho) = 1$. Set $\chi(\infty) := 1$.

We are now ready to state the main result of this section.

Theorem 1.2. *Let Assumption (H) be satisfied. Then*

$$\langle u(t,0)^p\rangle = \exp\left\{H(pt) - 2d\kappa\chi(\frac{\varrho}{\kappa})pt + o(t)\right\} \tag{1.5}$$

as $t \to \infty$ for $p = 1, 2, \ldots$

Remark 1.3. a) It will become obvious from the proof that the same asymptotics holds true for $\langle u(t,x_1)\ldots u(t,x_p)\rangle$, $x_1,\ldots,x_p \in \mathbb{Z}^d$, as well as for the moments of the fundamental solution $q(t,x,y)$ of our Cauchy problem. One only has to check that the large deviation principles of Lemma 1.5 below are also valid for the correspondingly modified measures.

b) As explained in the Introduction, the asymptotics (1.5) allows the following interpretation. For $0 < \varrho < \infty$, the main contribution to the statistical moments is given by high peaks of the solution $u(t,\cdot)$ which form islands of asymptotically bounded size and unboundedly increasing distance. In the case $\varrho = \infty$, these islands consist of isolated single lattice sites, whereas for $\varrho = 0$ the sizes of the islands grow unboundedly as $t \to \infty$.

c) For $0 < \varrho < \infty$, the infimum in (1.4) is attained. A probability measure on $\mathbb{Z}$ is a solution to this variational problem if and only if it is of the form const v_ϱ^2, where v_ϱ is a nonnegative solution of the nonlinear difference equation

$$\Delta v_\varrho + 2\varrho v_\varrho \log v_\varrho = 0 \qquad \text{on } \mathbb{Z} \tag{1.6}$$

with minimal l^2-norm $\|v_\varrho\|_2$. Moreover,

$$\chi(\varrho) = \varrho \log \|v_\varrho\|_2.$$

For sufficiently large ϱ, the minimal l^2-solution of (1.6) is unique modulo shifts. For small ϱ this is an open problem. As $\varrho \downarrow 0$,

$$\chi(\varrho) = \frac{\varrho}{4}\log\frac{1}{\varrho} + O(\varrho)$$

and v_ϱ has an asymptotically Gaussian shape of width $1/\sqrt{\varrho}$. The proof of these facts may be found in the forthcoming paper [8]. Note also that a similar problem occurs in Bolthausen and Schmock [2] in connection with the investigation of self-attracting random walks. Let us further remark that, for $0 < \varrho < \infty$, the term $2d\kappa\chi(\varrho/\kappa)$ in the expansion (1.5) is concave and strictly increasing as a function of the diffusion constant κ. This is obvious from (1.4).

d) A deeper analysis in the spirit of the almost sure considerations in Section 2 indicates that the typical shapes of the above mentioned high peaks of our solution are (time-dependent) multiples of $v_{\varrho/\kappa} \otimes \cdots \otimes v_{\varrho/\kappa}$. See also the discussion at the end of Section 4.1.

To prove Theorem 1.2 we first remark that the logarithmic asymptotics of the moments $\langle u(t,0)^p\rangle$ is independent of the particular choice of the initial field

$u_0(\cdot)$. Under our general assumptions on the random fields $\xi(\cdot)$ and $u_0(\cdot)$, we have

$$\langle u_0(0)\rangle^p \leq \frac{\langle u(t,0)^p\rangle}{\langle \overline{u}(t,0)^p\rangle} \leq \langle u_0(0)^p\rangle,$$

where $\overline{u}$ is the solution to our Cauchy problem with initial datum identically one, see e.g. step 2 in the proof of Theorem 3.2 in [9]. We will therefore assume from now on that u_0 is identically one.

As a next step towards the proof of Theorem 1.2, we will express the moments of $u(t,0)$ by means of local times of random walks on $\mathbb{Z}^d$. To this end, we exploit the Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \xi(x(s))\,ds\right\}, \tag{1.7}$$

where $(x(t),\mathbb{P}_x)$ denotes simple symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and $\mathbb{E}_x$ stands for expectation with respect to $\mathbb{P}_x$. Let $p \in \mathbb{N}$ be fixed until the end of the proof. Consider p independent copies $x_1(t),\ldots,x_p(t)$ of the random walk $x(t)$, and denote by $\mathbb{P}_0^p$ and $\mathbb{E}_0^p$ probability and expectation given $x_1(0) = \cdots = x_p(0) = 0$, respectively. Let

$$l_{t,i}(z) := \int_0^t 1\!\!1\,(x_i(s) = z)\;ds$$

be the local time of the i-th random walk spent at $z \in \mathbb{Z}^d$ during the time interval $[0,t]$, and introduce the total local time

$$l_t(z) := \sum_{i=1}^p l_{t,i}(z).$$

It then follows from (1.7) that

$$u(t,0)^p = \mathbb{E}_0^p \exp\left\{\sum_{z\in\mathbb{Z}^d} l_t(z)\xi(z)\right\}.$$

Averaging over the random field $\xi(\cdot)$ leads to

$$\langle u(t,0)^p\rangle = \mathbb{E}_0^p \exp\left\{\sum_z H(l_t(z))\right\}. \tag{1.8}$$

We next note that the *occupation time measures*

$$L_t(\cdot) := \frac{l_t(\cdot)}{pt}$$

satisfy the *weak* large deviation principle as $t \to \infty$ with rate function being a d-dimensional analogue of the Donsker-Varadhan functional S, cf. Donsker and

Varadhan [5]. In the next subsection we will explain how to get appropriate upper and lower bounds for the expectation on the right of (1.8) by 'compactifying' the state space of our random walks and then applying the *full* large deviation principle for the corresponding occupation time measures. After that, in Section 1.3, we will see how the variational expressions in these upper and lower bounds fit together to arrive at (1.5).

1.2. Compactification and application of large deviations

Given $R \in \mathbb{N}$, let $\mathbb{T}_R^d := \{-R, \dots, R\}^d$ denote the centered lattice cube of length $2R+1$. By introducing the periodic distance

$$d_R^\pi(x,y) := \min_{z \in (2R+1)\mathbb{Z}^d} |x-y-z|, \qquad x,y \in \mathbb{T}_R^d,$$

we may consider $\mathbb{T}_R^d$ as d-dimensional lattice torus.

Let $u^{R,\pi}$ and $u^{R,0}$ denote the solutions to the initial boundary value problem for the equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u + \xi(x) u \qquad \text{on } \mathbb{R}_+ \times \mathbb{T}_R^d$$

with periodic and zero boundary conditions, respectively, and initial datum identically one. The following lemma enables us to reduce the study of the moments to the consideration of a large finite box $\mathbb{T}_R^d$. This has the advantage that the probability laws of the associated occupation time measures live on the *compact* state space $\mathcal{P}(\mathbb{T}_R^d)$.

Lemma 1.4. *Let u be the solution to the Cauchy problem (1.1) with initial datum $u_0 \equiv 1$. Then*

$$\left\langle u^{R,0}(t,0)^p \right\rangle \le \langle u(t,0)^p \rangle \le \left\langle u^{R,\pi}(t,0)^p \right\rangle \tag{1.9}$$

for all $R \in \mathbb{N}$, $t \ge 0$, and $p = 1, 2, \dots$

The derivation of these bounds relies on probabilistic formulas for the moments and only works for i.i.d. potentials. Rather than directly exploiting the bounds (1.9), we will use later on the corresponding inequalities for their probabilistic representations. But Lemma 1.4 explains the idea on a more analytic language. Moreover, in Section 4.1 this simple lemma will be taken as starting point for a totally different proof of Theorem 1.2.

We consider the 'periodized' local times

$$l_t^R(z) := \sum_{x \in (2R+1)\mathbb{Z}^d} l_t(z+x), \qquad z \in \mathbb{T}_R^d,$$

which may be regarded as total local times of p independent random walks on $\mathbb{T}_R^d$ with generator $\kappa\Delta$ and periodic boundary conditions. Let

$$L_t^R(\cdot) := \frac{l_t^R(\cdot)}{pt}$$

be the associated occupation time measures on $\mathbb{T}_R^d$. Let further τ_R^p denote the first time when one of the random walks $x_1(t), \dots, x_p(t)$ exits $\mathbb{T}_R^d$.

We already know that the moments of the solution u to (1.1) with initial datum $u_0 \equiv 1$ admit the representation (1.8). In analogy with this, we find that

$$\left\langle u^{R,0}(t,0)^p \right\rangle = \mathbb{E}_0^p \exp\left\{ \sum_{z \in \mathbb{T}_R^d} H(l_t(z)) \right\} 1\!\!1 (\tau_R^p > t) \tag{1.10}$$

and

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = \mathbb{E}_0^p \exp\left\{ \sum_{z \in \mathbb{T}_R^d} H(l_t^R(z)) \right\}. \tag{1.11}$$

Proof of Lemma 1.4. The lower bound for $\langle u(t,0)^p \rangle$ is obvious from (1.8) and (1.10). Since the cumulant generating function H is convex and $H(0) = 0$, we have

$$\sum_{k=1}^n H(\lambda_k) \le H\left(\sum_{k=1}^n \lambda_k \right)$$

for all $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$. Hence,

$$\sum_{z \in \mathbb{Z}^d} H(l_t(z)) \le \sum_{z \in \mathbb{T}_R^d} H(l_t^R(z)).$$

Using this, we obtain the upper bound for $\langle u(t,0)^p \rangle$ from the probabilistic representations (1.8) and (1.11). $\square$

The main tools for deriving asymptotic formulas for the moments (1.10) and (1.11) are large deviations for the occupation time measures of p independent random walks on $\mathbb{T}_R^d$ with zero and periodic boundary conditions (Lemma 1.5 below). That is, we will consider large deviations for the subprobability measures

$$\mu_t^{R,0}(B) := \mathbb{P}_0^p \left(L_t(\cdot) \in B, \tau_R^p > t \right)$$

and the probability measures

$$\mu_t^{R,\pi}(B) := \mathbb{P}_0^p \left(L_t^R(\cdot) \in B \right)$$

on $\mathcal{P}(\mathbb{T}_R^d)$. In this context, we need the Donsker-Varadhan functionals $S_d^{R,0}$ and $S_d^{R,\pi}$ on $\mathcal{P}(\mathbb{T}_R^d)$ defined by

$$S_d^{R,0}(p) := \sum_{\substack{\{x,y\} \subset \mathbb{Z}^d \\ |x-y|=1}} \left(\sqrt{p(x)} - \sqrt{p(y)} \right)^2, \qquad p \in \mathcal{P}(\mathbb{T}_R^d),$$

and

$$S_d^{R,\pi}(p) := \sum_{\substack{\{x,y\}\subset \mathbb{T}_R^d \\ d_R^\pi(x,y)=1}} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2, \qquad p \in \mathcal{P}(\mathbb{T}_R^d),$$

respectively, where, by convention, in the first formula $p(x) := 0$ for $x \notin \mathbb{T}_R^d$. Note that these expressions coincide with the Dirichlet form at $\sqrt{p}$ of the operator $-\Delta$ on $l^2(\mathbb{T}_R^d)$ with either zero or periodic boundary condition.

We intend to apply the following finite dimensional large deviation results which may be derived as particular cases from Donsker and Varadhan [4] or Gärtner [7].

Lemma 1.5. *Given $R \in \mathbb{N}$, the following holds true as $t \to \infty$.*

a) The subprobability measures $\mu_t^{R,0}$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,0}$.

b) The probability measures $\mu_t^{R,\pi}$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,\pi}$.

Remark 1.6. Since the formulation of a large deviation principle for unnormalized measures may appear to be unconventional, let us remark that assertion a) of Lemma 1.5 may be rephrased as follows. The probability measures $\mu_t^{R,0}(\cdot)/\mu_t^{R,0}(\mathbb{T}_R^d)$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,0} - \min(\kappa S_d^{R,0})$ and

$$\lim_{t\to\infty} \frac{1}{pt} \log \mu_t^{R,0}(\mathbb{T}_R^d) = -\min(\kappa S_d^{R,0}).$$

We are now in a position to derive the desired asymptotic formulas for $\langle u^{R,0}(t,0)^p\rangle$ and $\langle u^{R,\pi}(t,0)^p\rangle$. Passing from the description by local times to the description by occupation time measures, we may rewrite (1.11) in the form

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = e^{H(pt)} \mathbb{E}_0^p \exp\left\{ pt \sum_{z\in\mathbb{T}_R^d} \frac{H(L_t^R(z)pt) - L_t^R(z)H(pt)}{pt} \right\}. \tag{1.12}$$

Let us first consider the case when $0 \le \varrho < \infty$. Then Remark 1.1 b) to Assumption (H) implies that the expression under the last sum becomes uniformly close to $\varrho L_t^R(z) \log L_t^R(z)$ as $t \to \infty$, and we arrive at

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = e^{H(pt)+o(t)} \mathbb{E}_0^p \exp\left\{ -pt\varrho I_d^R(L_t^R(\cdot)) \right\}, \tag{1.13}$$

where I_d^R is the entropy functional on $\mathcal{P}(\mathbb{T}_R^d)$:

$$I_d^R(p) := -\sum_{z\in\mathbb{T}_R^d} p(z) \log p(z), \qquad p \in \mathcal{P}(\mathbb{T}_R^d).$$

We may now apply the Laplace-Varadhan method for the large deviation probabilities of Lemma 1.5 b) to see that

$$\mathbb{E}_0^p \exp\left\{-pt\varrho I_d^R(L_t^R(\cdot))\right\} = \exp\left\{-pt\min\left[\kappa S_d^{R,\pi} + \varrho I_d^R\right] + o(t)\right\}. \tag{1.14}$$

Combining (1.13) with (1.14), we arrive at assertion b) of the next lemma.

Lemma 1.7. *Let Assumption (H) be satisfied. Then the following holds true as $t \to \infty$ for arbitrary $R \in \mathbb{N}$ and $p = 1, 2, \dots$*

a) If $0 \le \varrho < \infty$, then

$$\left\langle u^{R,0}(t,0)^p \right\rangle \ge \exp\left\{H(pt) - pt\min\left[\kappa S_d^{R,0} + \varrho I_d^R\right] + o(t)\right\}.$$

b) If $0 \le \varrho < \infty$, then

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle \le \exp\left\{H(pt) - pt\min\left[\kappa S_d^{R,\pi} + \varrho I_d^R\right] + o(t)\right\}.$$

c) If $\varrho = \infty$, then

$$\left\langle u^{R,0}(t,0)^p \right\rangle = \exp\left\{H(pt) - 2d\kappa pt + o(t)\right\}, \tag{1.15}$$

and the same asymptotics is valid for $\langle u^{R,\pi}(t,0)^p \rangle$.

The proof of assertion a) follows the same lines as that of b). Instead of (1.11) and Lemma 1.5 b), one has to use (1.10) and Lemma 1.5 a), respectively. To prove assertion c) assume that $\varrho = \infty$. The expression on the right of (1.15) is a trivial lower bound for $\langle u^{R,0}(t,0)^p \rangle$ which is obtained from (1.10) by forcing all random walks $x_1(t), \dots, x_p(t)$ to stay at 0 during the whole time interval $[0,t]$. In view of Lemma 1.4, it now only remains to show that the expression on the right of (1.15) may also serve as an upper bound for $\langle u^{R,\pi}(t,0)^p \rangle$. From (1.12) and Remark 1.1 b) we conclude that

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle \le e^{H(pt)} \left[\mathbb{P}_0^p\left(L_t^R(z) \notin (\varepsilon, 1-\varepsilon) \text{ for all } z \in \mathbb{T}_R^d\right) + o(e^{-\gamma t})\right]$$

for any $\varepsilon \in (0,1)$ and arbitrarily large γ. Here we have also used that the expression under the sum on the right of (1.12) is always nonpositive. But the large deviation principle for $L_t^R(\cdot)$ (Lemma 1.5 b)) tells us that the probability on the right behaves like

$$\exp\left\{-pt\min\left\{\kappa S_d^R(p) \colon p(z) \notin (\varepsilon, 1-\varepsilon) \text{ for all } z \in \mathbb{T}_R^d\right\} + o(t)\right\}.$$

Since the minimum in the exponent tends to $2d\kappa$ as $\varepsilon \to 0$, this yields the correct upper bound.

A combination of Lemma 1.4 with Lemma 1.7 c) proves Theorem 1.2 in the case when $\varrho = \infty$. To complete the proof for $0 \le \varrho < \infty$ one has to show that the minima in the exponents on the right of the assertions a) and b) of Lemma 1.7 converge to the same limit as $R \to \infty$ and that this limit equals $2d\kappa\chi(\varrho/\kappa)$. This final step will be carried out in the next subsection in Lemma 1.10.

1.3. Properties of associated variational problems

We first consider the d-dimensional Donsker-Varadhan functional S_d and the d-dimensional entropy functional I_d defined by

$$S_d(p) := \sum_{\substack{\{x,y\}\subset\mathbb{Z}^d \\ |x-y|=1}} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2, \qquad p \in \mathcal{P}(\mathbb{Z}^d),$$

and

$$I_d(p) := -\sum_{x\in\mathbb{Z}^d} p(x)\log p(x), \qquad p \in \mathcal{P}(\mathbb{Z}^d),$$

respectively. Note that S_1 and I_1 coincide, respectively, with the functionals S and I introduced in Section 1.1.

We claim that our d-dimensional variational problems split into the sum of d one-dimensional problems.

Lemma 1.8. *For* $0 \le \varrho < \infty$,

$$\inf\left[S_d + \varrho I_d\right] = d\inf\left[S + \varrho I\right].$$

If $0 < \varrho < \infty$, *then the infimum on the left is attained at* $p \in \mathcal{P}(\mathbb{Z}^d)$ *if and only if* p *is a product measure,*

$$p = \bigotimes_{i=1}^{d} p_i,$$

and the infimum on the right is attained at all $p_i \in \mathcal{P}(\mathbb{Z})$, $i = 1, \dots, d$.

Proof. Given $d \ge 1$ and ϱ with $0 \le \varrho < \infty$, abbreviate

$$F_d := S_d + \varrho I_d.$$

We will show that

$$\inf F_{d+1} = \inf F_d + \inf F_1. \tag{1.16}$$

First observe that

$$S_{d+1}(p_d \otimes p_1) = S_d(p_d) + S_1(p_1) \tag{1.17}$$

and

$$I_{d+1}(p_d \otimes p_1) = I_d(p_d) + I_1(p_1) \tag{1.18}$$

for all $p_d \in \mathcal{P}(\mathbb{Z}^d)$ and $p_1 \in \mathcal{P}(\mathbb{Z})$. This implies that the expression on the left of (1.16) does not exceed that on the right. To obtain the opposite inequality fix $p \in \mathcal{P}(\mathbb{Z}^{d+1})$ arbitrarily. Denote by p_d and p_1 the marginals of the first d and the last component of p, respectively, and consider the conditional laws

$$p_d(x|y) := \frac{p(x,y)}{p_1(y)} \quad \text{and} \quad p_1(y|x) := \frac{p(x,y)}{p_d(x)}, \qquad (x,y) \in \mathbb{Z}^d \times \mathbb{Z}.$$

Then

$$S_{d+1}(p) = \sum_{y \in \mathbb{Z}} p_1(y) S_d(p_d(\cdot\,|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) S_1(p_1(\cdot\,|x))$$

and

$$\begin{aligned} I_{d+1}(p) &= \sum_{y \in \mathbb{Z}} p_1(y) I_d(p_d(\cdot\,|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) I_1(p_1(\cdot\,|x)) \\ &\quad + \sum_{x \in \mathbb{Z}^d} \left[\sum_{y \in \mathbb{Z}} \Big(p_1(y) p_d(x|y) \log p_d(x|y) \Big) - p_d(x) \log p_d(x) \right]. \end{aligned}$$

Since the function $x \log x$, $x \geq 0$, is strictly convex, an application of Jensen's inequality shows that the expression in the square brackets is nonnegative and vanishes identically if and only if $p = p_d \otimes p_1$. Hence,

$$F_{d+1}(p) \geq \sum_{y \in \mathbb{Z}} p_1(y) F_d(p_d(\cdot\,|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) F_1(p_1(\cdot\,|x)). \tag{1.19}$$

This yields the desired lower bound. Moreover, if $0 < \varrho < \infty$, then in (1.19) equality holds only if $p = p_d \otimes p_1$. Together with (1.17) and (1.18), this shows that the infimum on the left of (1.16) is attained at p if and only if p has the form $p_d \otimes p_1$ and the infima on the right are attained at p_d and p_1, respectively. □

Remark 1.9. It is obvious from the above proof that assertions analogous to Lemma 1.8 are valid for the functionals $S_d^{R,\pi} + \varrho I_d^R$ and $S_d^{R,0} + \varrho I_d^R$ considered in Section 1.2.

Recall that the function χ has been defined in (1.4). The next lemma fills the outstanding gap in the proof of Theorem 1.2.

Lemma 1.10. *For $0 \leq \varrho < \infty$ and each $R \in \mathbb{N}$,*

$$\min \left[S_d^{R,\pi} + \varrho I_d^R \right] \leq \inf \left[S_d + \varrho I_d \right] \leq \min \left[S_d^{R,0} + \varrho I_d^R \right]. \tag{1.20}$$

Moreover,

$$\begin{aligned} \lim_{R \to \infty} \min \left[S_d^{R,\pi} + \varrho I_d^R \right] &= \lim_{R \to \infty} \min \left[S_d^{R,0} + \varrho I_d^R \right] \\ &= \inf \left[S_d + \varrho I_d \right] = 2d\chi(\varrho). \end{aligned} \tag{1.21}$$

Proof. Because of Lemma 1.8 and Remark 1.9, it will be enough to consider the case $d = 1$. For convenience, we will suppress the dimension index in our notation.

The right inequality in (1.20) is obvious. To derive the left inequality, we fix $p \in \mathcal{P}(\mathbb{Z})$ arbitrarily and consider the 'periodized' measure

$$p_R(z) := \sum_{x \in (2R+1)\mathbb{Z}} p(z+x), \qquad z \in \mathbb{T}_R.$$

It will then be enough to check that

$$S^{R,\pi}(p_R) \le S(p) \tag{1.22}$$

and

$$I^R(p_R) \le I(p). \tag{1.23}$$

As a consequence of the Cauchy-Schwarz inequality, we have

$$\left(\sqrt{p_R(y)} - \sqrt{p_R(z)}\right)^2 \le \sum_{x \in (2R+1)\mathbb{Z}} \left(\sqrt{p(y+x)} - \sqrt{p(z+x)}\right)^2$$

for all $y, z \in \mathbb{T}_R$. This yields (1.22). Inequality (1.23) follows from the fact that the function $\varphi(x) := -x \log x$, $x \ge 0$, is concave and $\varphi(0) = 0$ and therefore

$$\varphi(p_R(z)) \le \sum_{x \in (2R+1)\mathbb{Z}} \varphi(p(z+x))$$

for all $z \in \mathbb{T}_R$.

To prove (1.21), let $p \in \mathbb{T}_R$ be a measure at which the minimum of $S^{R,\pi} + \varrho I^R$ is attained. Because of shift invariance, we may assume without loss of generality that

$$p(-R) + p(R) \le \frac{2}{2R+1}.$$

Then

$$S^{R,0}(p) - S^{R,\pi}(p) = 2\sqrt{p(-R)}\sqrt{p(R)} \le \frac{2}{2R+1}.$$

This implies that

$$\min\left[S^{R,0} + \varrho I^R\right] - \min\left[S^{R,\pi} + \varrho I^R\right] \le \frac{2}{2R+1}.$$

Together with (1.20), this proves the convergence relations in (1.21). For $d = 1$ the last equality on the right of (1.21)) is the definition of χ. $\square$

2. Almost sure asymptotics

2.1. Statement of the result

In this section we will study the almost sure behavior of the solution $u(t,x)$ to our basic Cauchy problem (1.1) as $t \to \infty$ for fixed $x \in \mathbb{Z}^d$.

We will assume throughout that the potential $\xi(\cdot)$ consists of independent, identically distributed random variables with *continuous* distribution function F satisfying $F(r) < 1$ for all r (i.e. $\xi(\cdot)$ is unbounded from above a.s.). The initial datum $u_0(\cdot)$ is assumed to be nonnegative, a.s. not to vanish identically, and to satisfy the growth condition

$$\limsup_{|x|\to\infty} \frac{\log \log_+ u_0(x)}{\log |x|} < 1 \qquad \text{a.s.} \tag{2.1}$$

where $\log_+ x := \log(x \vee e)$.

Let us introduce the non-decreasing function

$$\varphi(r) := \log \frac{1}{1 - F(r)}, \qquad r \in \mathbb{R},$$

and its left-continuous inverse

$$\psi(s) := \min\{r : \varphi(r) \geq s\}, \qquad s > 0.$$

Note that ψ is strictly increasing and $\varphi(\psi(s)) = s$ for all $s > 0$. The function ψ has been determined in such a way that the distribution of the field $\xi(\cdot)$ coincides with that of $\psi(\eta(\cdot))$, where $\eta(\cdot)$ is a field of independent, exponentially distributed random variables with mean 1. Hence, we may and will assume without loss of generality that $\xi(\cdot) = \psi(\eta(\cdot))$. This will allow us to study the high peaks of $\xi(\cdot)$ by investigating those of the 'standard' field $\eta(\cdot)$.

We next formulate our crucial restriction on the tail behavior of the distribution function F.

Assumption (F). There exists ϱ, $0 \leq \varrho \leq \infty$, such that

$$\lim_{s\to\infty} [\psi(cs) - \psi(s)] = \varrho \log c \tag{2.2}$$

for all $c \in (0,1)$. If $\varrho = \infty$, then we demand in addition that

$$\lim_{s\to\infty} [\psi(s + \log s) - \psi(s)] = 0. \tag{2.3}$$

Roughly speaking, if $0 < \varrho < \infty$, then assumption (2.2) requires that the upper tail of F behaves like that of a double exponential distribution with parameter ϱ. The case $\varrho = 0$ is that of an 'almost bounded' potential. If $\varrho = \infty$, then we are 'beyond' the double exponential tails, and (2.3) mainly restricts the tails to be not as 'heavy' as for exponentially distributed variables. In particular, problem (1.1) admits a unique nonnegative solution u, and this solution is given by the Feynman-Kac formula

$$u(t,0) = \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(s))\,ds\right\} u_0(x(t)), \tag{2.4}$$

cf. [9], Sections 2 and 3. This sounds very similar to what was assumed in the previous section. In fact, we will prove later (Lemma 2.3 below) that Assumption (F) is slightly stronger than Assumption (H). The latter was imposed on the cumulant generating function in Section 1.1.

Remark 2.1. The following assertions are easily verified.

a) For $0 \le \varrho < \infty$, (2.2) says that e^{ψ} is regularly varying with exponent ϱ.

b) Condition (2.3) is equivalent to

$$\lim_{s\to\infty} [\psi(s + c\log s) - \psi(s)] = 0 \qquad \text{for all } c \in \mathbb{R}.$$

If $0 \le \varrho < \infty$, then (2.3) follows from (2.2). As a consequence of (2.3), $\psi(s) = o(s/\log s)$.

c) Assumption (F) implies that

$$\lim_{r\to\infty} \frac{\varphi(r+\beta)}{\varphi(r)} = e^{\beta/\varrho} \qquad \text{for all } \beta \in \mathbb{R}$$

(with the obvious definition of $e^{\beta/\varrho}$ for $\varrho = 0$ and $\varrho = \infty$) and

$$\varphi(r) + \log\varphi(r) \le \varphi(r+\beta)$$

for each $\beta > 0$ and all sufficiently large r. If $F(r)$ is strictly increasing for large r, then the converse is also true.

d) If $0 \le \varrho < \infty$, then

$$\psi(\varphi(r)) = r + o(1) \qquad \text{as } r \to \infty.$$

The almost sure asymptotics of $u(t,x)$ as $t \to \infty$ will now be characterized in terms of the function ψ and the function χ which was introduced in Section 1.1 by means of the Donsker-Varadhan functional S and the entropy functional I.

Theorem 2.2. *Let Assumption (F) be satisfied. If $d = 1$, suppose in addition that $\langle \log(1 + \xi(0)^-)\rangle < \infty$. Then almost surely*

$$u(t,0) = \exp\left\{\psi(d\log t)t - 2d\kappa\chi(\frac{\varrho}{\kappa})t + o(t)\right\} \qquad \text{as } t \to \infty. \tag{2.5}$$

The leading term in this asymptotic expansion is related to the maximum of the potential $\xi(\cdot)$ along those paths of the random walk $x(t)$ which give the main contribution to the Feynman-Kac formula (2.4). As we will see,

$$\max_{|x|\le t}\xi(x)=\psi(d\log t)+o(1)\qquad \text{a.s.}$$

For 'heavy' tails violating assumption (2.3), this non-random asymptotics breaks down and the second order term in (2.5) is expected to be superimposed by random fluctuations.

Although the leading terms in the expansion of $\langle u(t,0)\rangle$ and $u(t,0)$ are totally different, the second order terms coincide. As explained in the Introduction, this comes from the fact that, in the double exponential case, the typical shapes of high exceedances of $\xi(\cdot)$ which contribute to $\langle u(t,0)\rangle$ and $u(t,0)$, respectively, are the same independent of their different height. Our proof also indicates the following interpretation. Assume that $0<\varrho<\infty$ and that the minimal l^2-solution v_ϱ of equation (1.6) is unique modulo shifts. Let the initial total mass $\sum u_0(x)$ be finite a.s. Then, as $t\to\infty$, the main contribution to the total mass of $u(t,\cdot)$ will be given by widely spaced high peaks the local shapes of which consist of (time-dependent) multiples of $v_{\varrho/\kappa}\otimes\cdots\otimes v_{\varrho/\kappa}$. These peaks of $u(t,\cdot)$ correspond to high exceedances of the potential $\xi(\cdot)$ of the form

$$\psi(d\log t)-2d\kappa\chi(\varrho/\kappa)+2\varrho\log\left(v_{\varrho/\kappa}\otimes\cdots\otimes v_{\varrho/\kappa}\right). \tag{2.6}$$

The proof of Theorem 2.2 will be broken down into several steps. In the Sections 2.2–2.4 we will collect all the ingredients necessary for the proof which will then be fit together in Section 2.5.

We close this subsection by revealing the precise relationship between Assumption (F) and Assumption (H) from Section 1.1. As before, assume that the distribution function F is continuous and $F(r)<1$ for all r. Let H denote the associated cumulant generating function.

Lemma 2.3. *Assume that $H(t)<\infty$ for all $t>0$ or, equivalently, $\psi(s)=o(s)$ as $s\to\infty$.*

a) If $0\le\varrho<\infty$, then the following two conditions are equivalent:

$$\lim_{s\to\infty}\left[\psi(cs)-\psi(s)\right]=\varrho\log c\qquad \textit{for all } c\in(0,1) \tag{2.7}$$

and

$$\lim_{t\to\infty}\left[\frac{H(ct)}{ct}-\frac{H(t)}{t}\right]=\varrho\log c\qquad \textit{for all } c\in(0,1). \tag{2.8}$$

Moreover, either of them implies that

$$\frac{H(t)}{t}=\psi(t)+\varrho\log\varrho-\varrho+o(1)\qquad \textit{as } t\to\infty. \tag{2.9}$$

b) Suppose that $\varrho=\infty$. Then assumption (2.7) implies (2.8). If $\psi(s)$ is continuously differentiable and $\psi'(s)$ is strictly decreasing for large s, then (2.8) also implies (2.7).

Without additional regularity assumptions on ψ like that in assertion b), (2.7) does not follow from (2.8) when $\varrho = \infty$. To construct a counterexample consider the discrete distribution function F with mass at e^n and 'Gaussian-like' tail $1 - F(e^n) = \exp\{-e^{2n}\}$, $n \in \mathbb{N}$.

Proof of Lemma 2.3. 1^0 Recall that $\xi(0) = \psi(\eta(0))$, where $\eta(0)$ is exponentially distributed with mean 1. Using this, we find that

$$e^{H(t)} = t \int_0^\infty \exp\left\{t\left[\psi(ct) - c\right]\right\} dc, \qquad t > 0.$$

An application of the Laplace method yields

$$\sup_{c>0}\left[\psi(ct) - c\right] - \frac{1}{t} \le \frac{H(t)}{t} \le \sup_{c>0}\left[\psi(ct) - \theta c\right] + \frac{1}{t}\log\frac{1}{1-\theta} \tag{2.10}$$

for $\theta \in (0,1)$ and all $t > 0$. Indeed,

$$e^{H(t)} \ge t \int_\beta^{\beta+\frac{1}{t}} \exp\left\{\left[\psi(ct) - c\right]\right\} dc \ge \exp\left\{t\left[\psi(\beta t) - \beta - \frac{1}{t}\right]\right\}$$

for all $\beta > 0$. This gives the lower bound for $H(t)/t$. On the other hand,

$$\begin{aligned} e^{H(t)} &= t \int_0^\infty \exp\left\{t\left[\psi(ct) - \theta c\right]\right\} e^{-(1-\theta)tc}\, dc \\ &\le \frac{1}{1-\theta} \exp\left\{t \sup_{c>0}\left[\psi(ct) - \theta c\right]\right\} \end{aligned}$$

for $\theta \in (0,1)$. This is the upper bound.

We are now going to prove assertion a). It will be sufficient to verify that both (2.7) and (2.8) imply (2.9).

2^0 We first show that (2.7) implies (2.9) for $0 \le \varrho < \infty$. Since ψ is increasing, it follows from (2.7) that

$$\psi(ct) = \psi(t) + \left[\varrho + o_u(1)\right]\log c + \tilde{o}_u(1),$$

where $o_u(1)$ and $\tilde{o}_u(1)$ tend to zero as $t \to \infty$ uniformly in $c > 0$. Substituting this in (2.10) and noting that $\sup_{c>0}[\varrho\log c - c] = \varrho\log\varrho - \varrho$, we obtain (2.9).

3^0 We next show that (2.8) implies (2.9) for $0 < \varrho < \infty$. To this end, we introduce the shifted distribution functions

$$F_t(r) := F\left(\frac{H(t)}{t} + r\right), \qquad r \in \mathbb{R},\ t > 0.$$

Then

$$\int e^{\beta t r}\, F_t(dr) = \exp\left\{H(\beta t) - \beta H(t)\right\},$$

and it follows from assumption (2.8) that

$$\lim_{t\to\infty}\frac{1}{t}\log\int e^{\beta tr}\,F_t(dr)=\varrho\beta\log\beta=:G(\beta)\qquad\text{for }\beta>0.$$

Since the function G is continuously differentiable on $(0,\infty)$ and has infinite negative slope at 0, we conclude from this by a standard Cramèr argument that the probability measures associated with F_t on the (left-compactified) space $[-\infty,\infty)$ satisfy the full large deviation principle with scale t and rate function J being the Legendre transform of G:

$$J(\alpha)=\varrho\exp\left\{\frac{\alpha}{\varrho}-1\right\},\qquad\alpha\in[-\infty,\infty),$$

cf. e.g. Freidlin and Wentzell [6], Chap. 5. In particular,

$$1-F\left(\frac{H(t)}{t}-\varrho\log\varrho+\varrho\right)=\exp\left\{-(1+o(1))t\right\},$$

i.e.

$$\varphi\left(\frac{H(t)}{t}-\varrho\log\varrho+\varrho\right)=(1+o(1))t.$$

Using (2.8) once more, one easily derives from this assertion (2.9).

4^0 We now assume that (2.8) is fulfilled for $\varrho=0$. To avoid heavy notation let us further assume that ψ is continuous and, hence, the supremum of $\psi(ct)-c$ is attained at some point $c_t>0$ for sufficiently large t. The modifications for noncontinuous ψ will be obvious. Using (2.10), we find that, for $\beta>1$ and $\theta\in(0,1)$,

$$\begin{aligned}\frac{H(\beta t)}{\beta t}-\frac{H(\theta t)}{\theta t}&\geq\max_{c>0}\left[\psi(ct)-\frac{c}{\beta}\right]-\max_{c>0}\left[\psi(ct)-c\right]+o(1)\\&\geq\left(1-\frac{1}{\beta}\right)c_t+o(1).\end{aligned}$$

Since, by assumption, the expression on the left tends to zero, we obtain $c_t\to 0$ as $t\to\infty$. Using (2.10) once more, we conclude that

$$\begin{aligned}\frac{H(\theta t)}{\theta t}&\leq\max_{c>0}\left[\psi(ct)-c\right]+o(1)=\psi(c_t t)-c_t+o(1)\\&\leq\psi(\theta t)+o(1)\end{aligned}$$

for all $\theta\in(0,1)$, i.e.

$$\frac{H(t)}{t}\leq\psi(t)+o(1).\tag{2.11}$$

On the other hand, for arbitrary $\delta>0$,

$$\frac{H(t)}{t}\geq\psi(\delta t)-\delta+o(1),$$

i.e.

$$\psi(t) \leq \frac{H(t/\delta)}{t/\delta} + \delta + o(1) = \frac{H(t)}{t} + \delta + o(1),$$

and we arrive at the inequality opposite to (2.11). In this way we have shown that (2.9) holds for $\varrho = 0$.

5^0 We now turn to the proof of assertion b). Let us first show that (2.7) implies (2.8) if $\varrho = \infty$. To this end fix γ and θ with $0 < \gamma < \theta < 1$ arbitrarily. Then (2.10) implies that

$$\frac{H(t)}{t} - \frac{H(\gamma\theta t)}{\gamma\theta t} \geq \sup_{c>0} \left[\psi(ct) - c\right] - \sup_{c>0} \left[\psi(\gamma ct) - c\right] + o(1).$$

It will therefore be enough to show that the expression on the right tends to infinity as $t \to \infty$. Since (2.7) holds for $\varrho = \infty$, both suprema on the right may be taken over $c \geq 1$ only for large t. Hence, we may continue as follows:

$$\begin{aligned} &= \sup_{c\geq 1} \left[\psi(ct) - c\right] - \sup_{c\geq 1} \left[\psi(\gamma ct) - c\right] + o(1) \\ &\geq \inf_{s\geq t} \left[\psi(s) - \psi(\gamma s)\right]. \end{aligned}$$

By assumption, the last expression tends to infinity as $t \to \infty$, and we are done.

6^0 It remains to consider the case when (2.8) is fulfilled for $\varrho = \infty$, $\psi(s)$ is continuously differentiable, and $\psi'(s)$ is strictly decreasing for large s. We will again exploit inequality (2.10). Since $H(t)/t \to \infty$, the supremum of $\psi(ct) - c$ is attained at some point c_t (for large t) such that $c_t t \to \infty$. But

$$\psi'(c_t t) = \frac{1}{t},$$

and therefore our assumptions ensure that $c_t t$ varies continuously in t. We then obtain for $0 < \gamma < \theta < 1$ the estimate

$$\begin{aligned} \frac{H(\theta t)}{\theta t} - \frac{H(\gamma t)}{\gamma t} &\leq \sup_{c>0} \left[\psi(ct) - c\right] - \sup_{c>0} \left[\psi(\gamma ct) - c\right] + o(1) \\ &\leq \psi(c_t t) - \psi(\gamma c_t t) + o(1). \end{aligned}$$

By assumption, the expression on the left tends to infinity as $t \to \infty$. Therefore

$$\psi(c_t t) - \psi(\gamma c_t t) \to \infty \qquad \text{as } t \to \infty$$

for all $\gamma \in (0, 1)$. Hence, (2.7) is valid for $\varrho = \infty$. $\square$

2.2. Percolation bounds

In dimension $d \geq 2$, there is no need to impose any restrictions on the lower tail of the distribution function F. As a consequence of a percolation effect, in the Feynman-Kac representation (2.4) the random walk is able to bypass clusters of extremely negative peaks of the potential $\xi(\cdot)$. This subsection contains the related notation and auxiliary results.

Given $x \in \mathbb{Z}^d$ and $r \geq 0$, let $B_r(x) := \{y \in \mathbb{Z}^d\colon |y - x| \leq r\}$ denote the closed ball in $\mathbb{Z}^d$ with center x and radius r. Here and in the sequel $|\cdot|$ stands for the lattice norm on $\mathbb{Z}^d$. We will abbreviate $B_r(0)$ by B_r.

Given a natural number R, we will say that $x, y \in \mathbb{Z}^d$ are *R-neighbors* if $|x - y| \leq R$. A subset W of $\mathbb{Z}^d$ will be called *R-connected* if any two sites x, y of W may be joined by a path $x = z_0 \to z_1 \to \cdots \to z_n = y$ of R-neighbors in W. The minimum of the lengths $\sum |z_k - z_{k-1}|$ of all such paths will be denoted by $d_W^R(x, y)$. Hence, $d_W^R(x, y)$ measures the distance of x and y inside the R-connected set W. Each subset of $\mathbb{Z}^d$ splits into *R-connected components.*

For each $R \in \mathbb{N}$, consider the random variables

$$\xi_R(z) := \min_{x \in B_R(z)} \xi(x), \qquad z \in (2R+1)\mathbb{Z}^d.$$

Note that $\xi_R(\cdot)$ is a field of i.i.d. random variables on the sublattice $(2R+1)\mathbb{Z}^d$. Define the level sets

$$A_\alpha^+(R) := \left\{z \in (2R+1)\mathbb{Z}^d\colon \xi_R(z) > \alpha\right\}, \qquad \alpha \in \mathbb{R}.$$

Lemma 2.4. *Suppose that $d \geq 2$. Then, for each $R \in \mathbb{N}$, one finds a level $\alpha = \alpha_R$ such that the following holds true.*

a) A.s. there exists a unique infinite $(2R{+}1)$-connected subset $W^+ = W^+(R)$ of $A_\alpha^+(R)$, and $\mathrm{Prob}(0 \in W^+) > 0$.

b) There exists $\vartheta_R > 1$ such that a.s.

$$\limsup_{|y| \to \infty, y \in W^+} \frac{d_{W^+}^{2R+1}(x, y)}{|x - y|} \leq \vartheta_R$$

for all $x \in W^+$.

Proof. This repeats the proof given in [9], Section 2.4, with the random field $\xi(\cdot)$ on $\mathbb{Z}^d$ replaced by $\xi_R(\cdot)$ on the sublattice $(2R+1)\mathbb{Z}^d$. $\square$

We will assume from now on that, for each $R \in \mathbb{N}$, a level $\alpha = \alpha_R$ has been chosen as in Lemma 2.4 and the level set $A_\alpha^+(R)$ and the infinite percolation cluster $W^+(R)$ are defined accordingly.

As before, let $(x(t), \mathbb{P}_x)$ denote random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$. By τ_x and $\tau(r)$ we denote the first hitting times of the site $x \in \mathbb{Z}^d$ and the complement of the ball B_r, respectively.

Lemma 2.5. *a) For arbitrary $r > 0$ and $t > 0$, we have*

$$\mathbb{P}_0(\tau(r) \leq t) \leq 2^{d+1} \exp\left\{-r \log \frac{r}{d\kappa t} + r\right\}.$$

b) Suppose that $d \geq 2$. *Fix* $R \in \mathbb{N}$ *arbitrarily. Then there exists* $\vartheta_R > 1$ *such that for each* $t > 0$ *a.s.*

$$\mathbb{E}_0 \exp\left\{\int_0^{\tau_x} \xi(x(s))\, ds\right\} \mathbb{1}(\tau_x \leq t) \geq \exp\{-\vartheta_R |x| \log |x|\} \tag{2.12}$$

and

$$\mathbb{E}_x \exp\left\{\int_0^{\tau_0} \xi(x(s))\, ds\right\} \mathbb{1}(\tau_0 \leq t) \geq \exp\{-\vartheta_R |x| \log |x|\} \tag{2.13}$$

for all sufficiently large $x \in \bigcup_{z \in W^+(R)} B_R(z)$. *In dimension* $d = 1$, *corresponding estimates are valid a.s. for* all *sufficiently large* $|x|$ *provided that* $\langle \log(1 + \xi(0)^-)\rangle < \infty$.

This is a slight modification of Lemma 4.3 in [9]. The proof of part b) relies on the percolation bound in Lemma 2.4 b).

2.3. High exceedances of the random potential

In this subsection we consider random fields $\xi(\cdot)$ which satisfy Assumption (F) for some $\varrho \in [0, \infty]$. We will show that almost surely as $t \to \infty$ the high exceedances of the field $\xi(\cdot)$ in the ball B_t are of order $\psi(d \log t)$ and, for $\varrho \in (0, \infty]$, form islands of bounded size which are separated from each other by an arbitrarily large distance. After that we will prove that almost surely the set of local peaks of the shifted potential $\xi(\cdot) - \psi(d \log t)$ in B_t is asymptotically described by the class of profiles $h(\cdot)$ for which

$$\sum_x e^{h(x)/\varrho} \leq 1.$$

Our results will first be formulated for the field $\eta(\cdot)$ of independent, exponentially distributed random variables with mean 1. As a corollary, we will then obtain the corresponding statements for the transformed field $\xi(\cdot) = \psi(\eta(\cdot))$.

Let us begin with the almost sure behavior of the maxima of the field $\eta(\cdot)$.

Lemma 2.6. *We have*

$$\limsup_{t \to \infty} \frac{\left|\max\limits_{x \in B_t} \eta(x) - \log |B_t|\right|}{\log \log |B_t|} \leq 1 \qquad a.s.$$

The proof of this classical result will be given for the sake of completeness only.

Proof of Lemma 2.6. Fix $\theta > 1$ and an increasing sequence (t_n) of positive numbers so that $|B_{t_n}| \sim \theta^n$ as $n \to \infty$. Since

$$\log |B_{t_{n+1}}| - \log |B_{t_n}| = o(\log \log |B_{t_n}|)$$

and

$$\log\log|B_{t_{n+1}}| \sim \log\log|B_{t_n}|,$$

it will be enough to prove the statement for the sequence (t_n) instead of t. For each $c > 1$, we get

$$\begin{aligned}
&\operatorname{Prob}\left(\max_{x\in B_{t_n}} \eta(x) > \log|B_{t_n}| + c\log\log|B_{t_n}|\right) \\
&\qquad \le |B_{t_n}|\operatorname{Prob}\left(\eta(0) > \log|B_{t_n}| + c\log\log|B_{t_n}|\right) \\
&\qquad = \frac{1}{(\log|B_{t_n}|)^c} \sim \frac{1}{(n\log\theta)^c}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Prob}\left(\max_{x\in B_{t_n}} \eta(x) < \log|B_{t_n}| - c\log\log|B_{t_n}|\right) \\
&\qquad = \left(1 - \frac{(\log|B_{t_n}|)^c}{|B_{t_n}|}\right)^{|B_{t_n}|} \le \exp\left\{-(\log|B_{t_n}|)^c\right\} \\
&\qquad = \exp\left\{-(n\log\theta)^c(1+o(1))\right\}.
\end{aligned}$$

Hence, the probabilities on the left of both inequalities are summable over n for $c > 1$, and our assertion follows by an application of the Borel-Cantelli lemma. □

Taking into account Remark 2.1 b), we obtain the corresponding result for the transformed field $\xi(\cdot)$.

Corollary 2.7. *Let Assumption (F) be satisfied. Then almost surely*

$$\max_{x\in B_t} \xi(x) = \psi(\log|B_t|) + o(1) \qquad \text{as } t\to\infty.$$

Remark 2.8. Since $|B_t|$ behaves like $(2t)^d$, we may replace in Lemma 2.6, and therefore also in Corollary 2.7, $\log|B_t|$ by $d\log t$. Corollary 2.7 therefore explains the appearance of the term $\psi(d\log t)$ in our considerations.

Given $\gamma > 0$ and $t > 0$, consider the point process of high exceedances

$$\tilde{E}_t^\gamma := \left\{x \in B_t \colon \eta(x) > e^{-\gamma}\log|B_t|\right\}.$$

We next want to show that almost surely for large t the set $\tilde{E}_t^\gamma$ consists of islands the size of which does not exceed e^γ. After that this result will be reformulated in terms of the high exceedances of $\xi(\cdot)$.

Lemma 2.9. *For each $\gamma > 0$ and each natural number R, the following is true almost surely. There exists a random time $t_0 = t_0(\gamma, R, \eta(\cdot)) > 0$ such that for $t > t_0$ each R-connected component of $\tilde{E}_t^\gamma$ consists of at most e^γ elements.*

Proof. Fix $\gamma > 0$, $R \in \mathbb{N}$, and $\theta > 1$ arbitrarily. Consider an increasing sequence (t_n) such that $|B_{t_n}| \sim \theta^n$ as $n \to \infty$. Then

$$e^{-\gamma} \log |B_{t_{n+1}}| = e^{-\gamma+o(1)} \log |B_{t_n}|, \qquad n \to \infty.$$

Because of this it will suffice to prove our lemma for the sequence (t_n) instead of t.

Fix a natural number $m > e^\gamma$ arbitrarily and denote by $A_t^{\gamma,m}$ the event that $\tilde{E}_t^\gamma$ contains an R-connected subset of m elements. By the Borel-Cantelli lemma it will be enough to check that

$$\sum_n \operatorname{Prob}\left(A_{t_n}^{\gamma,m}\right) < \infty. \tag{2.14}$$

There are at most $C_{m,R}|B_t|$ R-connected subsets of B_t consisting of m elements, where $C_{m,R}$ is a positive constant which depends on m and R only. For each of these sets the probability to be contained in $\tilde{E}_t^\gamma$ equals

$$\exp\left\{-me^{-\gamma} \log |B_t|\right\} = |B_t|^{-me^{-\gamma}}.$$

Therefore,

$$\operatorname{Prob}\left(A_{t_n}^{\gamma,m}\right) \le C_{m,R}|B_{t_n}|^{1-me^{-\gamma}} \sim C_{m,R}\theta^{-(me^{-\gamma}-1)n}.$$

Since $me^{-\gamma} > 1$, we arrive at (2.14). □

Given $\gamma > 0$ and $t > 0$, consider now the point process

$$E_t^\gamma := \left\{ x \in B_t \colon \xi(x) > \max_{B_t} \xi - \gamma \right\}.$$

Corollary 2.10. *Let Assumption (F) be satisfied for some $\varrho \in (0,\infty]$. Then for each $\gamma > 0$ and each natural number R, the following is true almost surely. There exists a random time $t_0 = t_0(\gamma, R, \xi(\cdot)) > 0$ such that for $t > t_0$ each R-connected component of E_t^γ consists of at most $e^{\gamma/\varrho}$ elements.*

In other words, for $0 < \varrho \le \infty$, the high exceedances of the potential $\xi(\cdot)$ form islands of asymptotically bounded size which are located far from each other. For $\varrho = \infty$, these islands shrink to single lattice sites as $t \to \infty$.

Proof of Corollary 2.10. Suppose first that $0 < \varrho < \infty$. Then in Lemma 2.9 the point process $\tilde{E}_t^{\gamma/\varrho}$ coincides in law with

$$\left\{ x \in B_t \colon \xi(x) > \psi\left(e^{-\gamma/\varrho} \log |B_t|\right) \right\}$$

and, by Assumption (F),

$$\psi\left(e^{-\gamma/\varrho} \log |B_t|\right) = \psi\left(\log |B_t|\right) - \gamma + o(1).$$

Combining this with Corollary 2.7, we arrive at the desired result. The case $\varrho = \infty$ may be treated similarly. □

We now turn to the investigation of the typical shapes of high peaks of the field $\eta(\cdot)$ (resp. $\xi(\cdot)$) in a large ball around 0.

Lemma 2.11. *For each $R \in \mathbb{N}$ and almost all realizations of the random field $\eta(\cdot)$,*

$$\limsup_{t\to\infty} \max_{x\in B_t} \frac{\sum_{y\in B_R(x)} \eta(y)}{\log|B_t|} \le 1. \tag{2.15}$$

Proof. Fix $\theta > 1$ arbitrarily and select an increasing sequence (t_n) so that $|B_{t_n}| \sim \theta^n$ as $n \to \infty$. It will be enough to prove (2.15) for the sequence (t_n) instead of t. Fix further $\gamma > 1$ arbitrarily. Then, applying Chebyshev's exponential inequality, we obtain

$$\begin{aligned}
&\operatorname{Prob}\left(\max_{x\in B_{t_n}} \sum_{y\in B_R(x)} \eta(y) > \gamma^2 \log|B_{t_n}|\right)\\
&\quad\le |B_{t_n}| \operatorname{Prob}\left(\gamma^{-1} \sum_{y\in B_R} \eta(y) > \gamma \log|B_{t_n}|\right)\\
&\quad\le |B_{t_n}| \exp\{-\gamma \log|B_{t_n}|\} \left\langle \exp\left\{\gamma^{-1} \sum_{y\in B_R} \eta(y)\right\}\right\rangle\\
&\quad= \left(\frac{\gamma}{\gamma-1}\right)^{|B_R|} |B_{t_n}|^{1-\gamma} \sim \left(\frac{\gamma}{\gamma-1}\right)^{|B_R|} \theta^{-(\gamma-1)n}.
\end{aligned}$$

Hence, the above probabilities are summable over n, and our assertion again follows from the Borel-Cantelli lemma. □

Corollary 2.12. *Let Assumption (F) be satisfied for some $\varrho \in (0,\infty)$. Then for each $R \in \mathbb{N}$ and almost all realizations of the random field $\xi(\cdot)$,*

$$\limsup_{t\to\infty} \max_{x\in B_t} \sum_{y\in B_R(x)} \exp\{[\xi(y) - \psi(\log|B_t|)]/\varrho\} \le 1.$$

Remark 2.13. The corresponding assertion for $\varrho = \infty$ is obvious from Corollary 2.7 and Corollary 2.10. In this case, given $\gamma < 0 < \delta$ and $R \in \mathbb{N}$, the following holds true a.s. for sufficiently large t. In each ball $B_R(x)$, $x \in B_t$, the shifted potential $\xi(\cdot) - \psi(\log|B_t|)$ exceeds γ at not more than one lattice site and does not exceed δ at all.

Proof of Corollary 2.12. Since $\eta(\cdot) = \varphi(\xi(\cdot))$, the assertion of Lemma 2.11 may be rewritten in the form

$$\limsup_{t\to\infty} \max_{x\in B_t} \sum_{y\in B_R(x)} \frac{\varphi(\xi(y))}{\log|B_t|} \le 1 \qquad \text{a.s.} \tag{2.16}$$

Assume that $0 < \varrho < \infty$. It then follows from Remark 2.1 c) that

$$\lim_{t\to\infty} \frac{\varphi\left(\psi(\log|B_t|) + \beta\right)}{\log|B_t|} = e^{\beta/\varrho} \qquad \text{uniformly in } \beta \le \beta_0 \tag{2.17}$$

for each β_0. Because of Corollary 2.7, a.s. the field $\xi(\cdot) - \psi(\log|B_t|)$ is bounded from above on B_{t+R} uniformly for large t. Taking this into account, we conclude from (2.17) that a.s.

$$\frac{\varphi(\xi(y))}{\log|B_t|} = \exp\left\{\left[\xi(y) - \psi(\log|B_t|)\right]/\varrho\right\} + o(1)$$

uniformly in $y \in B_{t+R}$ as $t \to \infty$. Substituting this in (2.16), we arrive at the desired result. $\square$

We are now going to derive bounds on the profiles of high peaks opposite to that given in Lemma 2.11 and Corollary 2.12. To this end we will need to consider percolation clusters. Recall that, for each $R \in \mathbb{N}$, we fixed a level $\alpha = \alpha_R$ as in Lemma 2.4 and denoted by $A^+(R) = A^+_\alpha(R)$ and $W^+(R)$ the associated level set on the sublattice $(2R+1)\mathbb{Z}^d$ and its infinite $(2R+1)$-connected component, respectively. We will assume without loss of generality that the random field $\xi(\cdot)$ admits a representation of the form

$$\xi(x) = (1-\zeta(x))\xi_-(x) + \zeta(x)\xi_+(x), \qquad x \in \mathbb{Z}^d,$$

where the random variables $\zeta(x)$, $\xi_-(x)$, $\xi_+(x)$ are mutually independent, $\zeta(x)$ attains the values 0 and 1 with probability $\mathrm{Prob}(\xi(x) \le \alpha)$ and $\mathrm{Prob}(\xi(x) > \alpha)$, respectively, $\xi_-(x) \le \alpha < \xi_+(x)$, and the distributions of $\xi_-(x)$ and $\xi_+(x)$ coincide with the conditional laws of $\xi(x)$ given $\xi(x) \le \alpha$ and $\xi(x) > \alpha$, respectively. Note that $\zeta(x) = 1$ if and only if $\xi(x)$ exceeds the level α. Accordingly, the field $\eta(\cdot) = \varphi(\xi(\cdot))$ admits the decomposition

$$\eta(x) = (1-\zeta(x))\eta_-(x) + \zeta(x)\eta_+(x), \qquad x \in \mathbb{Z}^d, \tag{2.18}$$

where $\eta_\pm(x) := \varphi(\xi_\pm(x))$. In particular, we have $\eta_-(x) \le \varphi(\alpha) \le \eta_+(x)$ and $\mathrm{Prob}(\eta_+(x) > s) = \exp\{\varphi(\alpha) - s\}$ for $s > \varphi(\alpha)$.

Lemma 2.14. *a) Suppose that $d \ge 2$. Given a natural number R and a function $h\colon B_R \to \mathbb{R}_+$ with*

$$\sum_{x\in B_R} h(x) < 1, \tag{2.19}$$

the following holds true a.s. There exists a positive (random) time t_0 such that for all $t > t_0$ one finds a (random) site $z_0 \in W^+(R)$ such that $B_R(z_0) \subseteq B_t$ and

$$\eta(z_0 + \cdot\,) > h(\cdot)\log|B_t| \qquad \text{on } B_R. \tag{2.20}$$

b) With $W^+(R)$ replaced by $A^+(R)$, the above assertion is also true in dimension $d = 1$.

Proof. a) Fix $R \in \mathbb{N}$ and $h\colon B_R \to \mathbb{R}_+$ satisfying (2.19) arbitrarily. Suppose without loss of generality that h is strictly positive. Since $\log|B_{n+1}| \sim \log|B_n|$, we may restrict ourselves to natural values of t. For $t \geq R$, define

$$W_t^+(R) := W^+(R) \cap B_{t-R}.$$

The balls $B_R(z)$, $z \in W_t^+(R)$, are pairwise disjoint and contained in B_t. Recall that $\mathrm{Prob}(0 \in W^+(R)) > 0$ (Lemma 2.4 a)). Hence, we conclude from Birkhoff's ergodic theorem that there exists a positive constant C_R such that a.s.

$$|W_t^+(R)| \geq C_R |B_t| \qquad \text{for sufficiently large } t. \tag{2.21}$$

Consider the events

$$E_{t,z} := \{\eta(z + \cdot\,) > h(\cdot) \log|B_t| \text{ on } B_R\},$$

$t \in \mathbb{N}$, $z \in (2R+1)\mathbb{Z}^d$. Using the decomposition (2.18) and taking into account that $\zeta(x) = 1$ for all $x \in B_R(z)$ if $z \in W_t^+(R)$, the events $E_{t,z}$ coincide with

$$E_{t,z}^+ := \{\eta_+(z + \cdot\,) > h(\cdot) \log|B_t| \text{ on } B_R\}$$

for $z \in W_t^+(R)$. Therefore an application of the Borel-Cantelli lemma with respect to the conditional law given $\zeta(\cdot)$ reduces the proof of assertion a) to the verification of

$$\sum_{t=R}^{\infty} \mathrm{Prob}\left(\bigcap\nolimits_{z \in W_t^+(R)} (E_{t,z}^+)^c \,\middle|\, \zeta(\cdot)\right) < \infty \qquad \text{a.s.} \tag{2.22}$$

Since the random cluster $W_t^+(R)$ depends on $\zeta(\cdot)$ only and the events $E_{t,z}^+$ are mutually independent and independent of $\zeta(\cdot)$, we obtain a.s.

$$\begin{aligned}
\mathrm{Prob}\left(\bigcap\nolimits_{z \in W_t^+(R)} (E_{t,z}^+)^c \,\middle|\, \zeta(\cdot)\right) &= \big(1 - \mathrm{Prob}(E_{t,0}^+)\big)^{|W_t^+(R)|} \\
&\leq \exp\big\{-|W_t^+(R)|\,\mathrm{Prob}(E_{t,0}^+)\big\} \\
&= \exp\Big\{-|W_t^+(R)| e^{|B_R|\varphi(\alpha)} |B_t|^{-\sum_{x \in B_R} h(x)}\Big\} \\
&\leq \exp\Big\{-\tilde{C}_R |B_t|^{1-\sum_{x \in B_R} h(x)}\Big\}
\end{aligned}$$

for sufficiently large t, where $\tilde{C}_R$ denotes a positive constant. On the bottom line we have used the bound (2.21). Because of assumption (2.19), this proves (2.22).

b) With several simplifications, the proof of part b) goes along the same lines as that of part a). $\square$

Corollary 2.15. *Suppose that $d \geq 2$. Let Assumption (F) be satisfied for some $\varrho \in [0, \infty)$. Fix $R \in \mathbb{N}$ arbitrarily. Then the following is valid for almost all realizations of the random field $\xi(\cdot)$.*

a) If $\varrho = 0$, then for each $\delta > 0$ there exists a positive (random) time t_0 such that for every $t > t_0$ one finds a (random) site $z_0 \in W^+(R)$ such that $B_R(z_0) \subseteq B_t$ and

$$\xi(z_0 + \cdot\,) > \psi(\log|B_t|) - \delta \qquad \text{on } B_R.$$

b) If $0 < \varrho < \infty$, then for each function $h\colon B_R \to \mathbb{R}$ with

$$\sum_{x \in B_R} e^{h(x)/\varrho} < 1 \tag{2.23}$$

there exists a positive (random) time t_0 such that for every $t > t_0$ one finds a (random) site $z_0 \in W^+(R)$ such that $B_R(z_0) \subseteq B_t$ and

$$\xi(z_0 + \cdot\,) > \psi(\log|B_t|) + h(\cdot) \qquad \text{on } B_R.$$

c) With $W^+(R)$ replaced by $A^+(R)$, the above assertions are also true in dimension $d = 1$.

For $\varrho = 0$, assertion a) tells us that the size of the islands of high exceedances of the potential $\xi(\cdot)$ grows unboundedly as $t \to \infty$.

Proof of Corollary 2.15. Since $\xi(\cdot) = \psi(\eta)$, assertion (2.20) implies that

$$\xi(z_0 + \cdot\,) > \psi\left(h(\cdot)\log|B_t|\right) \qquad \text{on } B_R. \tag{2.24}$$

But, if $\varrho = 0$, then

$$\psi\left(h(\cdot)\log|B_t|\right) = \psi\left(\log|B_t|\right) + o(1)$$

independent of the specific choice of $h\colon B_R \to \mathbb{R}_+$ provided that h is strictly positive. This yields assertion a).

To prove b) we remark that assumption (2.23) is the same as (2.19) with $h(\cdot)$ replaced by $e^{h(\cdot)/\varrho}$. Hence, instead of (2.24) we obtain

$$\xi(z_0 + \cdot\,) > \psi\left(e^{h(\cdot)/\varrho}\log|B_t|\right) \qquad \text{on } B_R.$$

But, since $0 < \varrho < \infty$, Assumption (F) yields

$$\psi\left(e^{h(\cdot)/\varrho}\log|B_t|\right) = \psi\left(\log|B_t|\right) + h(\cdot) + o(1) \qquad \text{on } B_R,$$

and we are done.

To prove c), one has to use assertion b) of Lemma 2.14 instead of a). □

2.4. Related spectral problems

Given $R > 0$ and $h\colon B_R \to \mathbb{R}$, let us denote by $\lambda_R(h(\cdot))$ the principal eigenvalue of the operator $\kappa\Delta + h(\cdot)$ in $l^2(B_R)$ with Dirichlet boundary condition. In particular, $\lambda_t(\xi(\cdot))$ is the principal eigenvalue of the Anderson Hamiltonian

$$\mathcal{H} = \kappa\Delta + \xi(\cdot)$$

in $l^2(B_t)$ with zero boundary condition. The aim of this subsection is to prove the following theorem on the almost sure asymptotics of $\lambda_t(\xi(\cdot))$.

Theorem 2.16. *Let Assumption (F) be satisfied for some $\varrho \in [0,\infty]$. Then almost surely*

$$\lambda_t(\xi(\cdot)) = \psi(\log|B_t|) - 2d\kappa\chi(\frac{\varrho}{\kappa}) + o(1) \qquad \textit{as } t \to \infty. \tag{2.25}$$

Before carrying out the details, let us briefly explain the origin of formula (2.25) in the case when $0 < \varrho < \infty$. We have seen in Section 2.3 that almost surely the high peaks of $\xi(\cdot)$ in B_t are of the form

$$\psi(\log|B_t|) + h(\cdot),$$

where $h\colon B_R \to \mathbb{R}$ runs through the class of functions satisfying

$$\sum_{x\in B_R} e^{h(x)/\varrho} < 1 \tag{2.26}$$

and R is arbitrarily large. Since the islands of these peaks are located far from each other, the upper part of the spectrum of $\mathcal{H}$ in $l^2(B_t)$ is expected to split into the union of the spectra on the single islands. Hence, as $t \to \infty$, $\lambda_t(\xi(\cdot))$ will be close to the upper boundary of

$$\lambda_R\left(\psi(\log|B_t|) + h(\cdot)\right) = \psi(\log|B_t|) + \lambda_R(h(\cdot))$$

taken over all profiles $h\colon B_R \to \mathbb{R}$ satisfying (2.26) for arbitrarily large R. The next lemma shows that this variational expression equals $\psi(\log|B_t|) - 2d\kappa\chi(\varrho/\kappa)$. It therefore makes plausible formula (2.25).

Let S_R^0 denote the Donsker-Varadhan functional on $\mathcal{P}(B_R)$ with Dirichlet boundary condition, and let I_R be the corresponding entropy functional. These functionals are defined in the same way as the functionals $S_d^{R,0}$ and I_d^R considered in the Sections 1.2 and 1.3 with the only difference that they are now given on $\mathcal{P}(B_R)$ instead of $\mathcal{P}(\mathbb{T}_R^d)$.

Lemma 2.17. *If $0 < \varrho < \infty$, then*

$$\sup_{\sum_{x\in B_R} e^{h(x)/\varrho}<1} \lambda_R(h(\cdot)) = -\min_{\mathcal{P}(B_R)} \left[\kappa S_R^0 + \varrho I_R\right]$$

for each $R \in \mathbb{N}$. Moreover,

$$\lim_{R\to\infty} \min_{\mathcal{P}(B_R)} \left[\kappa S_R^0 + \varrho I_R\right] = 2d\kappa\chi(\frac{\varrho}{\kappa}).$$

Proof. Let us first note that

$$\sup_{\sum_{x\in B_R} e^{h(x)/\varrho}<1} \lambda_R(h(\cdot)) = \sup_{\sum_{x\in B_R} e^{h(x)/\varrho}=1} \lambda_R(h(\cdot)). \tag{2.27}$$

This follows e.g. from the observation that $\lambda_R(h(\cdot)+c) = \lambda_R(h(\cdot)) + c$ for each constant c. According to the variational principle for the largest eigenvalue,

$$\lambda_R(h(\cdot)) = \sup_{\|v\|_2=1} \sum_{x\in B_R} (\kappa\Delta v(x) + h(x)v(x))\, v(x),$$

where, by convention, $v(x) = 0$ for $x \notin B_R$. Since it is enough to take the supremum over positive v, we may use the substitution $v^2 =: p$ to rewrite it in the form

$$\lambda_R(h(\cdot)) = \sup_{p\in\mathcal{P}(B_R)} \left[\sum_{x\in B_R} h(x)p(x) - \kappa S_R^0(p)\right].$$

In other words, λ_R is the Legendre transform of κS_R^0. Using this, we find that the supremum on the right of (2.27) equals

$$\sup_{p\in\mathcal{P}(B_R)} \left\{ \sup_{\sum_{x\in B_R} e^{h(x)/\varrho}=1} \left[\sum_{x\in B_R} h(x)p(x) - \varrho \log \sum_{x\in B_R} e^{h(x)/\varrho}\right] - \kappa S_R^0(p)\right\}.$$

Now observe that the expression in the square brackets does not change by adding a constant to h. Therefore the inner supremum may be taken over all $h\colon B_R \to \mathbb{R}$, and a straightforward computation shows that it coincides with $-\varrho I_R(p)$. In this way we arrived at the first assertion of our lemma. Since each ball B_R may be embedded in between two tori $T_{R'}^d$ and $T_{R''}^d$, the second assertion is a straightforward consequence of Lemma 1.10. $\square$

It may be seen from the above proof that the maximum of $\lambda_R(h(\cdot))$ over all h with

$$\sum_{x\in B_R} e^{h(x)/\varrho} = 1 \tag{2.28}$$

is attained at h if and only if the square of the normalized positive eigenfunction of $\kappa\Delta + h(\cdot)$ in $l^2(B_R)$ minimizes the functional $\kappa S_R^0 + \varrho I_R$, i.e. if

$$h = -\frac{\kappa\Delta\sqrt{p}}{\sqrt{p}} + \text{const},$$

where $p \in \mathcal{P}(\mathbb{T}_R^d)$ is a minimizer of $\kappa S_R^0 + \varrho I_R$ and the constant adjusts h to fulfill (2.28). Now let $R \to \infty$ and take into account Remark 1.3 c) and Lemma 1.10. Then one finds that the relevant shapes of the potential should be of the form

$$h = 2\varrho \log\left(v_{\varrho/\kappa} \otimes \cdots \otimes v_{\varrho/\kappa}\right) - 2d\kappa\chi(\varrho/\kappa).$$

This is in accordance with our claims after Theorem 2.2.

We are now going to prove that the principal eigenvalue $\lambda_t(\xi(\cdot))$ indeed may be approached by the maximum of the principle eigenvalues on the islands of high peaks provided that these islands are located far from each other. Since primarily this does not have to do anything with randomness, we will formulate the result in a nonrandom setting, although the proof will heavily rely on probabilistic arguments.

Let B be a finite connected subset of $\mathbb{Z}^d$. Fix $\kappa > 0$ and a potential $V\colon B \to \mathbb{R}$ arbitrarily. We want to estimate the principle eigenvalue $\lambda^{\mathcal{G}}$ of the Hamiltonian

$$\mathcal{G} = \kappa\Delta + V \qquad \text{in } l^2(B)$$

with Dirichlet boundary condition by comparing it with the maximum of the principle eigenvalues of $\mathcal{G}$ on the 'islands of high peaks' of V. To this end, let G_i, $i = 1, \ldots, m$, denote connected subsets of B such that $\operatorname{dist}(G_i, G_j) > 1$ for $i \neq j$, where $\operatorname{dist}(\cdot,\cdot)$ denotes the lattice distance between subsets of $\mathbb{Z}^d$. For $i = 1, \ldots, m$, let g_i be a (not necessarily connected) non-empty subset of G_i. Think of the g_i's as the sites of high exceedances of V on some islands G_i in a surrounding ocean B. Let λ_i denote the principle eigenvalue of $\mathcal{G}$ in $l^2(G_i)$ with Dirichlet boundary condition, and set

$$\lambda_{\max} := \max_i \lambda_i, \quad g := \bigcup_i g_i, \quad G := \bigcup_i G_i.$$

Lemma 2.18. *(Cluster expansion) Suppose that*

$$\max_{B\setminus g} V \leq \lambda_{\max}. \tag{2.29}$$

Then the principle eigenvalue $\lambda^{\mathcal{G}}$ of $\mathcal{G}$ in $l^2(B)$ satisfies

$$\lambda_{\max} < \lambda^{\mathcal{G}} < \gamma$$

for all $\gamma > \lambda_{\max}$ for which

$$\frac{\gamma - \lambda_{\max}}{2d\kappa}\left[\left(1 + \frac{\gamma - \lambda_{\max}}{2d\kappa}\right)^{\operatorname{dist}(B\setminus G, g)} - 1\right] > \max_i |G_i|. \tag{2.30}$$

Proof. Since the principle eigenvalue $\lambda^{\mathcal{G}}$ depends on the potential V monotonically, the lower bound $\lambda^{\mathcal{G}} > \lambda_{\max}$ is obvious from replacing V by $-\infty$ outside of G. To prove the upper bound, we will apply some sort of cluster expansion of the resolvent $\mathcal{R}_\gamma$ associated with $\mathcal{G}$. We will show that, under (2.29) and (2.30), γ belongs to the resolvent set of $\mathcal{G}$. Using the probabilistic representation of the resolvent and taking into account that B is finite, it will be enough to check that

$$\mathcal{R}_\gamma 1\!\!1(x) = \mathbb{E}_x \int_0^\eta dt\, \exp\left\{\int_0^t ds\, [V(x(s)) - \gamma]\right\} < \infty \tag{2.31}$$

for all $x \in B$. Here, as before, $(x(t), \mathbb{P}_x)$ denotes symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$, and η is the first exit time from B:

$$\eta := \inf\{t \geq 0 \colon x(t) \notin B\}.$$

We next introduce stopping times $0 \le \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \dots$ of successive visits of the sets g and G^c by our random walk:

$$\begin{aligned} \sigma_0 &:= \inf\{t \ge 0 \colon x(t) \in g\}, \\ \tau_i &:= \inf\{t \ge \sigma_i \colon x(t) \notin G\}, \\ \sigma_{i+1} &:= \inf\{t \ge \tau_i \colon x(t) \in g\}, \qquad i = 0, 1, 2, \dots \end{aligned}$$

We will use these stopping time cycles to estimate the resolvent from above by a geometric series. First, we may rewrite (2.31) in the form

$$\begin{aligned} \mathcal{R}_\gamma \mathbb{1}(x) = \mathbb{E}_x \int_0^{\sigma_0 \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\} \\ + \sum_{i=0}^{\infty} \mathbb{E}_x \int_{\sigma_i \wedge \eta}^{\sigma_{i+1} \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\}. \end{aligned} \tag{2.32}$$

Since $x(s) \in B \setminus g$ for $0 < s < \sigma_0 \wedge \eta$ and because of (2.29), we have

$$\begin{aligned} \mathbb{E}_x \int_0^{\sigma_0 \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\} \\ \le \int_0^\infty dt \, e^{(\lambda_{\max} - \gamma)t} = \frac{1}{\gamma - \lambda_{\max}} < \infty \end{aligned} \tag{2.33}$$

for all $x \in B$. Using the strong Markov property together with (2.29) and $\lambda_{\max} < \gamma$, we find for $i = 0, 1, 2, \dots$ that

$$\begin{aligned} &\mathbb{E}_x \int_{\sigma_i \wedge \eta}^{\sigma_{i+1} \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\} \\ &\quad = \mathbb{E}_x \exp\left\{ \int_0^{\sigma_0} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_0 < \eta) \\ &\qquad \times \mathbb{E}_{x(\sigma_0)} \exp\left\{ \int_0^{\sigma_i} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_i < \eta) \\ &\qquad \times \mathbb{E}_{x(\sigma_i)} \int_0^{\sigma_1 \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\} \\ &\quad \le \left(\max_{y \in g} \mathbb{E}_y \exp\left\{ \int_0^{\sigma_1} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_1 < \eta) \right)^i \\ &\qquad \times \max_{y \in g} \mathbb{E}_y \int_0^{\sigma_1 \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\}. \end{aligned} \tag{2.34}$$

Combining (2.32) with (2.33) and (2.34), we see that it will be enough to show that, under the assumptions (2.29) and (2.30),

$$\mathbb{E}_x \exp\left\{ \int_0^{\sigma_1} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_1 < \eta) < 1 \tag{2.35}$$

and

$$\mathbb{E}_x \int_0^{\sigma_1 \wedge \eta} dt \, \exp\left\{ \int_0^t ds \, [V(x(s)) - \gamma] \right\} < \infty \tag{2.36}$$

for all $x \in g$.

Let us first prove assertion (2.35). Applying the strong Markov property, we obtain for $x \in g$:

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{ \int_0^{\sigma_1} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_1 < \eta) \\
&\quad = \mathbb{E}_x \exp\left\{ \int_0^{\tau_0} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\tau_0 < \eta) \\
&\qquad \times \mathbb{E}_{x(\tau_0)} \exp\left\{ \int_0^{\sigma_0} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_0 < \eta).
\end{aligned} \tag{2.37}$$

To derive an appropriate bound for the last expectation, note that $x(\tau_0) \in B \setminus G$ and $x(s) \in B \setminus g$ for $0 \le s < \sigma_0$. But, by assumption (2.29), $V - \gamma \le \lambda_{\max} - \gamma < 0$ outside of g. Moreover, σ_0 may be estimated from below by the sum of $\mathrm{dist}(B \setminus G, g)$ independent exponentially distributed random variables with mean $(2d\kappa)^{-1}$. Hence, $\mathbb{P}_x$-a.s.

$$\begin{aligned}
&\mathbb{E}_{x(\tau_0)} \exp\left\{ \int_0^{\sigma_0} ds \, [V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_0 < \eta) \\
&\qquad \le \mathbb{E}_{x(\tau_0)} e^{-(\gamma - \lambda_{\max})\sigma_0} \le \left(\frac{2d\kappa}{2d\kappa + \gamma - \lambda_{\max}} \right)^{\mathrm{dist}(B \setminus G, g)}.
\end{aligned} \tag{2.38}$$

Note that, $\mathbb{P}_x$-a.s. for $x \in g_i$, τ_0 coincides with the first exit time from G_i. Thus, for $i \in \{1, \dots, m\}$ and $x \in g_i$,

$$u(x) := \mathbb{E}_x \exp\left\{ \int_0^{\tau_0} ds \, [V(x(s)) - \gamma] \right\}, \tag{2.39}$$

coincides with the solution to the boundary value problem

$$\begin{aligned}
(\kappa\Delta + V - \gamma)\, u &= 0 \qquad \text{in } G_i, \\
u &= 1 \qquad \text{on } G_i^c.
\end{aligned}$$

With the substitution $u =: 1 + v$, this turns into

$$\begin{aligned}
(\kappa\Delta + V - \gamma)\, v &= \gamma - V \quad \text{in } G_i, \\
v &= 0 \qquad\quad \text{on } G_i^c.
\end{aligned}$$

For $\gamma > \lambda_i$, the solution exists and is given by

$$v = \mathcal{R}_\gamma^{(i)}(V - \gamma),$$

where $\mathcal{R}_\gamma^{(i)}$ denotes the resolvent of $\mathcal{G}$ in $l^2(G_i)$ with Dirichlet boundary condition. Since $V \le \lambda_i + 2d\kappa \le \gamma + 2d\kappa$ on G_i, and because of the positivity of the resolvent, we obtain

$$v(x) \le 2d\kappa \mathcal{R}_\gamma^{(i)} 1\!\!1(x) \le 2d\kappa \left(\mathcal{R}_\gamma^{(i)} 1\!\!1, 1\!\!1\right)_{G_i}, \qquad x \in G_i,$$

where $(\cdot,\cdot)_{G_i}$ is the inner product in $l^2(G_i)$. Using the spectral representation of the resolvent (i.e. its Fourier expansion with respect to the orthonormal basis of eigenfunctions of $\mathcal{G}$ in $l^2(G_i)$), we find that

$$\left(\mathcal{R}_\gamma^{(i)} 1\!\!1, 1\!\!1\right)_{G_i} \le \frac{|G_i|}{\gamma - \lambda_i}.$$

This means that

$$u(x) \le 1 + \frac{2d\kappa}{\gamma - \lambda_i} |G_i|, \qquad x \in G_i. \tag{2.40}$$

Combining (2.37) with (2.38), (2.39), and (2.40), we arrive at

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{\int_0^{\sigma_1} ds\,[V(x(s)) - \gamma]\right\} 1\!\!1(\sigma_1 < \eta) \\
&\qquad \le \left(1 + \frac{2d\kappa}{\gamma - \lambda_{\max}} \max_i |G_i|\right) \left(1 + \frac{\gamma - \lambda_{\max}}{2d\kappa}\right)^{-\operatorname{dist}(B\setminus G, g)}
\end{aligned}$$

for $x \in g$. But the expression on the right is less than 1 if and only if (2.30) is fulfilled. This proves (2.35).

It remains to verify (2.36). Given $i \in \{1, \ldots, m\}$ and $x \in g_i$, an application of the strong Markov property and (2.33) yields

$$\begin{aligned}
&\mathbb{E}_x \int_0^{\sigma_1 \wedge \eta} dt\, \exp\left\{\int_0^t ds\,[V(x(s)) - \gamma]\right\} \\
&\quad = \mathbb{E}_x \left(\int_0^{\tau_0} + \int_{\tau_0 \wedge \eta}^{\sigma_1 \wedge \eta}\right) dt\, \exp\left\{\int_0^t ds\,[V(x(s)) - \gamma]\right\} \\
&\quad = \mathcal{R}_\gamma^{(i)} 1\!\!1(x) + \mathbb{E}_x \exp\left\{\int_0^{\tau_0} ds\,[V(x(s)) - \gamma]\right\} 1\!\!1(\tau_0 < \eta) \\
&\qquad \times \mathbb{E}_{x(\tau_0)} \int_0^{\sigma_0 \wedge \eta} dt\, \exp\left\{\int_0^t ds\,[V(x(s)) - \gamma]\right\} \\
&\quad \le \mathcal{R}_\gamma^{(i)} 1\!\!1(x) + \frac{1}{\gamma - \lambda_{\max}} \mathbb{E}_x \exp\left\{\int_0^{\tau_0} ds\,[V(x(s)) - \gamma]\right\}.
\end{aligned}$$

Since $\gamma > \lambda_i$, $\mathcal{R}_\gamma^{(i)} 1\!\!1(x)$ is finite. The finiteness of the expectation on the right of the last estimate was shown before, see (2.39) and (2.40). Hence, we arrived at (2.36). This completes the proof of our lemma. $\square$

We have now collected all the auxiliary material for the proof of our theorem.

Proof of Theorem 2.16. a) *Lower bound.* Let us first assume that $0 < \varrho < \infty$. Fix $R \in \mathbb{N}$ and $h\colon B_R \to \mathbb{R}$ with

$$\sum_{x \in B_R} e^{h(x)/\varrho} < 1$$

arbitrarily. Corollary 2.15 b) (resp. c) in dimension one) tells us that a.s. for sufficiently large t there exists a site z_0 such that $B_R(z_0) \subset B_t$ and

$$\xi(z_0 + \cdot\,) > \psi(\log|B_t|) + h(\cdot) \qquad \text{on } B_R.$$

This implies that

$$\lambda_t(\xi(\cdot)) \geq \psi(\log|B_t|) + \lambda_R(h(\cdot)).$$

From this we conclude that

$$\liminf_{t\to\infty} [\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \geq \sup_{\sum_{x\in B_R} e^{h(x)/\varrho}<1} \lambda_R(h(\cdot)) \qquad \text{a.s.}$$

Together with Lemma 2.17, this yields the lower bound.

In the case $\varrho = 0$, using Corollary 2.15 a) (resp. c)), we obtain

$$\liminf_{t\to\infty} [\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \geq \lambda_R(0).$$

But $\lambda_R(0) \to 0$ as $R \to \infty$, and we are done.

For $\varrho = \infty$, the lower bound follows from Corollary 2.7 and the fact that

$$\lambda_t(\xi(\cdot)) \geq \max_{B_t} \xi - 2d\kappa.$$

The latter is obvious from the observation that, for each $x \in B_t$, $\lambda_t(\xi(\cdot))$ may be estimated from below by the principal eigenvalue of $\mathcal{H}$ on the set $\{x\}$ with zero boundary condition which equals $\xi(x) - 2d\kappa$.

b) *Upper bound.* We first treat the case $0 < \varrho < \infty$. Fix $\delta > 0$ arbitrarily and choose $R \in \mathbb{N}$ so large that

$$\frac{\delta}{2d\kappa}\left[\left(1 + \frac{\delta}{2d\kappa}\right)^R - 1\right] > e^{2d\kappa/\varrho}|B_R|. \tag{2.41}$$

We know from Corollary 2.10 that a.s. for sufficiently large t the level set

$$E_t := \left\{x \in B_t \colon \xi(x) > \max_{B_t} \xi - 2d\kappa\right\}$$

splits into $(2R{+}1)$-connected clusters of size not exceeding $e^{2d\kappa/\varrho}$. Given $x \in E_t$, denote by g_x the $(2R+1)$-connected component of E_t which contains x, and let

$$G_x := \bigcup_{y \in g_x} B_R(y) \cap B_t$$

denote its R-neighborhood in B_t. By construction, the sets G_x are connected. Moreover, any two of these sets either coincide or have a distance larger than one. Let λ_x denote the principle eigenvalue of $\mathcal{H}$ in $l^2(G_x)$ with Dirichlet boundary condition. Since

$$\lambda_x \geq \max_{G_x} \xi - 2d\kappa, \qquad x \in E_t,$$

and the potential in B_t does not exceed $\max_{B_t} \xi - 2d\kappa$ outside of $E_t = \bigcup_x g_x$, we find that

$$\lambda_{\max} := \max_{x \in E_t} \lambda_x \geq \max_{B_t \setminus \bigcup_{x \in E_t} g_x} \xi.$$

Hence, we are in a situation where we may apply Lemma 2.18 to estimate $\lambda_t(\xi(\cdot))$ from above. In our case $g = \bigcup_{x \in E_t} g_x$, $G = \bigcup_{x \in E_t} G_x$,

$$\operatorname{dist}(B_t \setminus G, g) \geq R, \text{ and } \max_{x \in E_t} |G_x| \leq e^{2d\kappa/\varrho} |B_R|.$$

Because of (2.41), this means that condition (2.30) is fulfilled for $\gamma = \lambda_{\max} + \delta$. Thus, we conclude from Lemma 2.18 that a.s. for large t,

$$\lambda_t(\xi(\cdot)) \leq \max_{x \in E_t} \lambda_x + \delta. \tag{2.42}$$

Now observe that each of the sets G_x is contained in a ball of radius $R' := e^{2d\kappa/\varrho} R$. But, according to Corollary 2.12, we have

$$\max_{x \in B_t} \sum_{y \in B_{R'}(x)} \exp\{[\xi(y) - \psi(\log|B_t|)]\} < e^{\delta/\varrho} \tag{2.43}$$

a.s. for large t. Since the principle eigenvalue depends on the potential monotonically, we conclude from (2.42) and (2.43) that

$$\begin{aligned}
\lambda_t(\xi(\cdot)) - \psi(\log|B_t|) &\leq \max_{x \in B_t} \lambda_{R'}\left(\xi(x + \cdot\,) - \psi(\log|B_t|)\right) + \delta \\
&\leq \sup_{\sum_{y \in B_{R'}} e^{h(y)/\varrho} < e^{\delta/\varrho}} \lambda_{R'}(h) + \delta \\
&= \sup_{\sum_{y \in B_{R'}} e^{h(y)/\varrho} < 1} \lambda_{R'}(h) + 2\delta
\end{aligned}$$

a.s. for large t. Hence, for each $\delta > 0$ and all sufficiently large R,

$$\limsup_{t \to \infty} [\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \leq \sup_{\sum_{y \in B_{R'}} e^{h(y)/\varrho} < 1} \lambda_{R'}(h) + 2\delta \qquad \text{a.s.}$$

Combining this with Lemma 2.17, we arrive at the desired upper bound.

The proof for $\varrho = \infty$ is similar. Given $\delta > 0$, one has to choose R so large that (2.41) holds with $e^{2d\kappa/\varrho}$ replaced by 1. A.s. for sufficiently large t, the $(2R+1)$-connected components of the level set E_t consist of single lattice sites. In particular, $G_x = B_R(x)$, $x \in E_t$. Choose $\gamma < 0$ arbitrarily. According to Remark 2.13, a.s. for large t, the shifted potential $\xi(\cdot) - \psi(\log|B_t|)$ does not exceed γ on $B_R(x)\setminus\{x\}$ and does not exceed δ at site x for each $x \in E_t$. Hence, applying Lemma 2.18, we find that

$$\limsup_{t\to\infty}\,[\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \le \lambda_R(h_{\delta,\gamma}) + \delta \qquad \text{a.s.},$$

where $h_{\delta,\gamma}(0) = \delta$ and $h_{\delta,\gamma}(x) = \gamma$ for $x \in B_R \setminus \{0\}$. But $\lambda_R(h_{\delta,\gamma})$ tends to $\delta - 2d\kappa$ as $\gamma \to -\infty$. Since $\delta > 0$ may be chosen arbitrarily small, this implies the desired bound.

The proof in the case $\varrho = 0$ is a straightforward consequence of Corollary 2.7 and the observation that

$$\lambda_t(\xi(\cdot)) \le \max_{B_t} \xi.$$

□

We close this subsection with a modification of Theorem 2.16 which takes into account the percolation effect explained in Section 2.2. Recall that $A^+(R)$ and $W^+(R)$ denote, respectively, the level set in the sublattice $(2R+1)\mathbb{Z}^d$ and its infinite $(2R+1)$-connected component considered in Lemma 2.4. For $d \ge 2$, define

$$\widehat{W}_t^+(R) := \bigcup_{z\in W^+(R)\cap B_{t-R}} B_R(z).$$

This is the R-neighborhood of the part of the infinite percolation cluster $W^+(R)$ in the ball B_{t-R}. If $d = 1$, then we define $\widehat{W}_t^+(R)$ by the same formula but with $W^+(R)$ replaced by the level set $A^+(R)$. We denote by $\lambda_t^R(\xi(\cdot))$ the principle eigenvalue of our random Hamiltonian $\mathcal{H}$ in $l^2(\widehat{W}_t^+(R))$ with Dirichlet boundary condition. Note that $\lambda_t^R(\xi(\cdot)) \le \lambda_t(\xi(\cdot))$.

Corollary 2.19. *Let Assumption (F) be satisfied for some $\varrho \in [0,\infty]$. Then almost surely*

$$\liminf_{R\to\infty}\liminf_{t\to\infty}\left[\lambda_t^R(\xi(\cdot)) - \psi(\log|B_t|)\right] \ge -2d\kappa\chi(\frac{\varrho}{\kappa}).$$

This means that the principle eigenvalue $\lambda_t(\xi(\cdot))$ is essentially 'generated' by those islands of high peaks of the potential $\xi(\cdot)$ which are located in the R-neighborhood of the cluster $W^+(R)$ for large R.

Proof of Corollary 2.19. For $0 < \varrho < \infty$ and also for $\varrho = 0$, this repeats part a) of the proof of Theorem 2.16. Namely, according to Corollary 2.15, in the proof the lattice site z_0 may be assumed to belong to $W^+(R)$. If $\varrho = \infty$, then one has to replace Corollary 2.7 by the asymptotic formula

$$\max_{x\in W^+(R)\cap B_{t-R}} \xi(x) = \psi(\log|B_t|) + o(1) \qquad \text{a.s. as } t\to\infty$$

which holds in dimension $d \ge 2$ and the corresponding formula with $W^+(R)$ replaced by $A^+(R)$ in dimension $d = 1$). To understand how to treat such a restriction to the cluster $W^+(R)$, we refer to the proof of Lemma 2.14 where a similar problem had been considered. The details are left to the reader. □

2.5. Completion of the proof

We are now finally in a position to complete the proof of Theorem 2.2. Roughly speaking, we will show by an application of the Feynman-Kac formula and the spectral representation theorem that $u(t,0)$ behaves like $e^{t\lambda_t(\xi(\cdot))}$ a.s. as $t \to \infty$. This combined with our asymptotic formula for the principle eigenvalue $\lambda_t(\xi(\cdot))$ will then yield the desired asymptotics of $u(t,0)$.

To be precise, fix $\varepsilon > 0$ arbitrarily and set

$$\underline{r}(t) := \frac{t}{(\log t)^{1+\varepsilon}} \quad \text{and} \quad \overline{r}(t) := t(\log t)^{1+\varepsilon}.$$

We want to show that under the assumptions of Theorem 2.2,

$$\exp\left\{(t-3)\lambda^R_{\underline{r}(t)}(\xi(\cdot)) + o(t)\right\} \le u(t,0) \le \exp\left\{t\lambda_{\overline{r}(t)}(\xi(\cdot)) + o(t)\right\} \tag{2.44}$$

a.s. as $t \to \infty$ for each $R \in \mathbb{N}$. We may then apply the asymptotic formulas for the principle eigenvalues $\lambda_{\overline{r}(t)}(\xi(\cdot))$ and $\lambda^R_{\underline{r}(t)}(\xi(\cdot))$ obtained in Theorem 2.16 and Corollary 2.19, respectively. Substituting them in (2.44) and taking into account that

$$\begin{aligned} \psi\left(\log|B_{\underline{r}(t)}|\right) &= \psi(d\log t) + o(1), \\ \psi\left(\log|B_{\overline{r}(t)}|\right) &= \psi(d\log t) + o(1), \end{aligned}$$

and $\psi(d\log t) = o(t)$, we arrive at the desired asymptotics (2.5). The above properties of ψ are obvious from Remark 2.1 b).

It now only remains to prove (2.44) by exploiting the Feynman-Kac representation (2.4) of $u(t,0)$. To derive the *lower bound* for $u(t,0)$, fix $R \in \mathbb{N}$ arbitrarily. Recall that $\widehat{W}^+_{\underline{r}(t)}(R)$ is the R-neighborhood of the part of the infinite percolation cluster $W^+(R)$ (resp. the level set $A^+(R)$ if $d=1$) which is contained in the ball $B_{\underline{r}(t)-R}$. Let e^R_t denote the normalized positive eigenfunction corresponding to the principle eigenvalue $\lambda^R_{\underline{r}(t)}(\xi(\cdot))$ of the random Hamiltonian $\mathcal{H}$ in $l^2(\widehat{W}^+_{\underline{r}(t)}(R))$ with Dirichlet boundary condition. Let $z_0 \in \widehat{W}^+_{\underline{r}(t)}(R)$ be a random site (depending on t and R) at which e^R_t attains its maximum. Then $|z_0| \le \underline{r}(t)$ and $(e^R_t(z_0))^2 \ge |\widehat{W}^+_{\underline{r}(t)}(R)|^{-1} \ge |B_{\underline{r}(t)}|^{-1}$. Since the initial datum u_0 is supposed a.s. not to vanish identically, we find a random site $x_0 \in \mathbb{Z}^d$ such that $u_0(x_0) > 0$ a.s. Let σ^R_t denote the first exit time of the random walk $x(t)$ from $\widehat{W}^+_{\underline{r}(t)}(R)$. As before, τ_{x_0} and τ_{z_0} are the first hitting times of x_0 and z_0, respectively. Repeatedly applying the strong Markov property to the

Feynman-Kac representation of $u(t,0)$, we find that

$$\begin{aligned} u(t,0) \geq \mathbb{E}_0 \exp&\left\{\int_0^{\tau_{z_0}} \xi(x(u))\,du\right\} 1\!\!1\left(\tau_{z_0} \leq 1\right) \\ &\times \mathbb{E}_{z_0} \exp\left\{\int_0^{t-3} \xi(x(u))\,du\right\} 1\!\!1\left(\sigma_t^R > t-3, x(t-3) = z_0\right) \\ &\times \mathbb{E}_{z_0} \exp\left\{\int_0^{\tau_0} \xi(x(u))\,du\right\} 1\!\!1\left(\tau_0 \leq 1\right) \\ &\times \inf_{1\leq s\leq 3} \mathbb{E}_0 \exp\left\{\int_0^{s} \xi(x(u))\,du\right\} 1\!\!1\left(x(s) = x_0\right) u_0(x_0). \end{aligned} \tag{2.45}$$

In other words, we have forced the random walk to hit z_0 until time 1 and then to stay in $\widehat{W}^+_{\underline{r}(t)}(R)$ during a time period of length $t-3$ at the end of which it has to return to z_0. After that $x(t)$ is forced to move from z_0 to 0 during a time interval of length not exceeding 1. The remaining time has to be spent in such a way that the random walk is at x_0 at time t. The expression on the last line of (2.45) is independent of t and strictly positive a.s. Since $|z_0| \leq \underline{r}(t) = t/(\log t)^{1+\varepsilon}$, an application of Lemma 2.5 b) shows that a.s. both the first and the third expectation on the right are of order $e^{o(t)}$ as $t \to \infty$. The main asymptotics is therefore hidden in the second expectation. But this is the probabilistic representation of the fundamental solution of $\mathcal{H}$ in $l^2(\widehat{W}^+_{\underline{r}(t)}(R))$ with zero boundary condition considered at time $t-3$ with starting point and end point equal to z_0. The spectral representation of the fundamental solution shows that the considered expectation may be estimated from below by

$$e^{(t-3)\lambda^R_{\underline{r}(t)}(\xi(\cdot))}(e_t^R(z_0))^2 \geq e^{(t-3)\lambda^R_{\underline{r}(t)}(\xi(\cdot))}|B_{\underline{r}(t)}|^{-1}.$$

In this way we arrive at the lower bound in (2.44).

To derive the *upper bound*, set $R_n(t) := n\overline{r}(t)$ for $n \in \mathbb{N}$ and $t > 0$. As before, let $\tau(R_n(t))$ denote the first exit time from the ball $B_{R_n(t)}$. Then, using the Feynman-Kac formula, we obtain

$$\begin{aligned} u(t,0) \leq \mathbb{E}_0 \exp&\left\{\int_0^{t} \xi(x(u))\,du\right\} u_0(x(t)) 1\!\!1\left(\tau(\overline{r}(t)) > t\right) \\ &+ \sum_{n=1}^{\infty} \mathbb{E}_0 \exp\left\{\int_0^{t} \xi(x(u))\,du\right\} u_0(x(t)) 1\!\!1\left(\tau(R_n(t)) \leq t < \tau(R_{n+1}(t))\right). \end{aligned} \tag{2.46}$$

We will show that the first term on the right provides the correct asymptotics and the remaining sum tends to zero as $t \to \infty$ a.s. First note that

$$v(s,x) := \mathbb{E}_x \exp\left\{\int_0^{s} \xi(x(u))\,du\right\} 1\!\!1\left(\tau(\overline{r}(t)) > s\right), \qquad (s,x) \in \mathbb{R}_+ \times B_{\overline{r}(t)},$$

is the solution of the initial boundary value problem for the parabolic equation

$$\frac{\partial v}{\partial s} = \mathcal{H}v \qquad \text{on } \mathbb{R}_+ \times B_{\overline{r}(t)}$$

with initial datum $v(0,x) \equiv 1$ and Dirichlet boundary condition. Using the spectral representation of $v(t,\cdot)$ (i.e. its Fourier expansion with respect to the eigenfunctions of $\mathcal{H}$ in $l^2(B_{\overline{r}(t)})$, we find that

$$v(t,0) \le \sum_{x \in B_{\overline{r}(t)}} v(t,x) \le e^{t\lambda_{\overline{r}(t)}(\xi(\cdot))} |B_{\overline{r}(t)}|.$$

Moreover, the growth condition (2.1) provides the bound

$$\max_{B_{\overline{r}(t)}} u_0 \le e^{o(t)} \qquad \text{a.s. as } t \to \infty.$$

Consequently,

$$\mathbb{E}_0 \exp\left\{\int_0^t \xi(x(u))\,du\right\} u_0(x(t)) 1\!\!1(\tau(\overline{r}(t)) > t) \le \exp\left\{t\lambda_{\overline{r}(t)}(\xi(\cdot)) + o(t)\right\}$$

a.s. as $t \to \infty$. This is the desired upper bound. It remains to check that the sum on the right of (2.46) tends to zero a.s. We obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(u))\,du\right\} u_0(x(t)) 1\!\!1(\tau(R_n(t)) \le t < \tau(R_{n+1}(t))) \\ &\quad \le \sum_{n=1}^{\infty} \exp\left\{t \max_{B_{R_{n+1}(t)}} \xi + \max_{B_{R_{n+1}(t)}} \log_+ u_0\right\} \mathbb{P}_0\left(\tau(R_n(t)) \le t\right). \qquad (2.47) \end{aligned}$$

Using Corollary 2.7 and taking into account that $\psi(s) = o(s)$ by Remark 2.1 b), we find that a.s.

$$\max_{B_{R_{n+1}(t)}} \xi = \psi\left(\log|B_{R_{n+1}(t)}|\right) + o(1) = o(\log R_{n+1}(t))$$

as $t \to \infty$ uniformly in n. Assumption (2.1) implies that a.s.

$$\max_{B_{R_{n+1}(t)}} \log_+ u_0 = o(R_{n+1}(t))$$

as $t \to \infty$ uniformly in n. According to Lemma 2.5 a),

$$\mathbb{P}_0\left(\tau(R_n(t)) \le t\right) \le 2^{d+1} \exp\left\{-R_n(t) \log \frac{R_n(t)}{d\kappa t} + R_n(t)\right\}.$$

Using these estimates and remembering that $R_n(t) = nt(\log t)^{1+\varepsilon}$, one easily checks that the sum on the right of (2.47) tends to zero a.s. as $t \to \infty$.

The proof of Theorem 2.2 is now complete.

Remark 2.20. A thorough analysis of the above proof shows that assumption (2.1) may be replaced by the following slightly weaker growth condition: There exist strictly positive random variables C_0 and ε_0 such that a.s.

$$\log_+ u_0(x) \le C_0 \frac{|x|}{(\log|x|)^{1+\varepsilon_0}} \qquad \text{for all } x \in \mathbb{Z}^d.$$

3. Lifshitz tails of the integrated density of states

In this section we will study the logarithmic tail behavior of the spectral distribution function ('integrated density of states') associated with the Anderson Hamiltonian

$$\mathcal{H} = \kappa\Delta + \xi(\cdot).$$

As before, we will deal with potentials $\xi(\cdot)$ consisting of i.i.d. random variables with double exponential tails. This case is of particular interest, since the upper part of the spectrum is expected to correspond to the principle eigenvalues generated by sparsely distributed islands of high peaks of the potential which have non-degenerate finite size. This constellation is crucial for the tail behavior of the spectral distribution function and will enter its asymptotics via the variational problem considered in Section 1.

We will assume throughout that the potential $\xi(x)$, $x \in \mathbb{Z}^d$, consists of i.i.d. random variables with continuous distribution function F such that $F(r) < 1$ for all r and $\langle e^{t\xi(0)} \rangle < \infty$ for all $t \geq 0$. Later on we will require in addition that Assumption (F) of Section 2 is fulfilled.

Given $R \geq 1$, let $\lambda_1^{(R)} \geq \lambda_2^{(R)} \geq \cdots \geq \lambda_{|\mathbb{T}_R^d|}^{(R)}$ denote the eigenvalues of $\mathcal{H}$ in $l^2(\mathbb{T}_R^d)$ with Dirichlet boundary condition. The associated spectral distribution function $N^{(R)}(h)$ is defined as the relative number of eigenvalues not exceeding h:

$$N^{(R)}(h) := \frac{1}{|\mathbb{T}_R^d|} \sum_{i=1}^{|\mathbb{T}_R^d|} \mathbb{1}\left(\lambda_i^{(R)} \leq h\right), \qquad h \in \mathbb{R}.$$

The *spectral distribution function* N of $\mathcal{H}$ in $l^2(\mathbb{Z}^d)$ may then be defined by

$$N(h) := \lim_{R\to\infty} N^{(R)}(h), \qquad h \in \mathbb{R}.$$

It is well-known that this limit exists a.s. and is nonrandom as a consequence of the ergodicity of $\xi(\cdot)$. Moreover, its Laplace transform coincides with the 'trace' of the fundamental solution $q(t, x, y)$ of $\mathcal{H}$:

$$\int_{-\infty}^{\infty} e^{th}\, N(dh) = \langle q(t, 0, 0) \rangle \qquad \text{for } t \geq 0, \tag{3.1}$$

provided that the expression on the right is finite for all $t \geq 0$ which is certainly true under our assumptions. Let $\bar{N}^{(R)}(h) := 1 - N^{(R)}(h)$ and $\bar{N}(h) := 1 - N(h)$ denote the upper tails of the spectral distribution functions $N^{(R)}$ and N, respectively. Let further $\bar{F}(h) := 1 - F(h)$ be the upper tail of the distribution of $\xi(0)$.

Note that

$$\xi(\cdot) - 4d\kappa \leq \mathcal{H} \leq \xi(\cdot)$$

in the sense of positive definiteness in l^2. This implies that

$$\frac{1}{|\mathbb{T}_R^d|} \sum_{x\in\mathbb{T}_R^d} \mathbb{1}(\xi(x) - 4d\kappa > h) \leq \bar{N}^{(R)}(h) \leq \frac{1}{|\mathbb{T}_R^d|} \sum_{x\in\mathbb{T}_R^d} \mathbb{1}(\xi(x) > h),$$

and the strong law of large numbers yields the trivial bounds

$$\bar{F}(h + 4d\kappa) \leq \bar{N}(h) \leq \bar{F}(h), \qquad h \in \mathbb{R}. \tag{3.2}$$

We remark that these bounds are universal in the sense that they are valid for any homogeneous ergodic potential.

Recall that the function χ is defined by the variational expression (1.4). The following theorem provides a more precise description of the asymptotics of $\bar{N}(h)$ than (3.2) because it takes into account the structure of the high peaks of the potential $\xi(\cdot)$.

Theorem 3.1. *Let Assumption (F) be satisfied for some $\varrho \in [0, \infty)$.*

a) If $0 < \varrho < \infty$, then

$$\log \bar{N}(h) \sim \log \bar{F}(h + 2d\kappa\chi(\varrho/\kappa)) \qquad \text{as } h \to \infty. \tag{3.3}$$

b) If $\varrho = 0$, then

$$\bar{F}(h + \delta) \leq \bar{N}(h) \leq \bar{F}(h) \tag{3.4}$$

for arbitrary $\delta > 0$ and all sufficiently large h.

As can be seen from Remark 2.1 c), for $0 \leq \varrho < \infty$, $\log \bar{F}(h)$ and $\log \bar{F}(h + c)$ are not asymptotically equivalent for $c \neq 0$. Hence, Theorem 3.1 is indeed an improvement upon the universal bounds (3.2). On the other hand, for $\varrho = \infty$ Remark 2.1 c) shows that $\log \bar{F}(h)$ and $\log \bar{F}(h+c)$ are asymptotically equivalent for all c. In this case assertion (3.3) is equivalent to (3.2). But, since for $\varrho = \infty$ the relevant islands of high exceedances of the potential consist of isolated single peaks, one may use more direct probabilistic methods. An expansion with respect to the 'noise' in a vicinity of such peaks will yield a much more accurate asymptotics. This will be the subject of a separate paper.

In order to prove Theorem 3.1 we introduce the shifted spectral distribution functions

$$N_t(h) := N\left(\frac{H(t)}{t} + h\right), \qquad t > 0,\ h \in \mathbb{R},$$

where, as before, H is the cumulant generating function of $\xi(0)$. We will consider N_t, $t > 0$, as probability measures on the left-compactified real line $[-\infty, \infty)$. We further abbreviate $\bar{N}_t(h) := 1 - N_t(h)$. The following lemma provides the key for our proof of Theorem 3.1.

Lemma 3.2. *Let Assumption (H) be satisfied for some $\varrho \in [0, \infty)$.*

a) If $0 < \varrho < \infty$, then the probability measures N_t on $[-\infty, \infty)$ satisfy the full large deviation principle as $t \to \infty$ with scale t and rate function $J\colon [-\infty, \infty) \to \mathbb{R}_+$ given by $J(-\infty) := 0$ and

$$J(h) = \varrho \exp\left\{\frac{h + 2d\kappa\chi(\varrho/\kappa)}{\varrho} - 1\right\} \qquad \text{for } h \in \mathbb{R}. \tag{3.5}$$

b) If $\varrho = 0$, then

$$\lim_{t\to\infty} \frac{1}{t} \log \bar{N}_t(h) = 0 \qquad \text{for } h < 0.$$

Proof. We know from Remark 1.3 a) that the fundamental solution $q(t,x,y)$ of $\mathcal{H}$ satisfies

$$\langle q(t,0,0)\rangle = \exp\left\{ H(t) - 2d\kappa\chi(\frac{\varrho}{\kappa})t + o(t)\right\}$$

as $t \to \infty$. Because of (3.1), this implies that

$$\frac{1}{t}\log \int e^{\beta th}\, N_t(dh) = \frac{H(\beta t) - \beta H(t)}{t} - 2d\kappa\chi(\frac{\varrho}{\kappa})\beta + o(1)$$

for each $\beta > 0$. Together with Assumption (H), this yields

$$\lim_{t\to\infty} \frac{1}{t}\log \int e^{\beta th}\, N_t(dh) = \varrho\beta\log\beta - 2d\kappa\chi(\frac{\varrho}{\kappa})\beta =: G(\beta) \tag{3.6}$$

for $\beta > 0$.

a) Suppose now that $0 < \varrho < \infty$. Since the limiting function G is continuously differentiable on $(0,\infty)$ and $G'(\beta) \to -\infty$ for $\beta \to 0$, we conclude from (3.6) that N_t satisfies the full large deviation principle with scale t and rate function J being the Legendre transform of G:

$$J(h) = \sup_{\beta>0}\left[h\beta - G(\beta)\right], \qquad h \in \mathbb{R},$$

which coincides with (3.5). See e.g. Freidlin and Wentzell [6], Chap. 5, for standard Cramèr arguments of such type.

b) We now turn to the case $\varrho = 0$. Fix $\beta > 0$ and $\delta > 0$ arbitrarily. We conclude from (3.6) for $\varrho = 0$ that

$$\left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty}\right) e^{\beta ht}\, N_t(dh) = e^{o(t)}.$$

The first and the third integral decay exponentially fast as $t \to \infty$. For,

$$\int_{-\infty}^{-\delta} e^{\beta th}\, N_t(dh) \le e^{-\beta\delta t}$$

and

$$\begin{aligned}\int_{\delta}^{\infty} e^{\beta th}\, N_t(dh) &\le e^{-\gamma\delta t}\int_{-\infty}^{\infty} e^{(\beta+\gamma)th}\, N_t(dh)\\ &= e^{-\gamma\delta t + o(t)}\end{aligned}$$

for $\gamma > 0$. Here we have again used (3.6). Thus,

$$e^{\beta\delta t}\bar{N}_t(-\delta) \ge \int_{-\delta}^{\delta} e^{\beta th}\, N_t(dh) = e^{o(t)}.$$

Since $\beta > 0$ may be chosen arbitrarily small, this shows that

$$\lim_{t\to\infty} \frac{1}{t} \log \bar{N}_t(-\delta) = 0,$$

and we are done. $\square$

Proof of Theorem 3.1. Under our general suppositions on $\xi(\cdot)$, Assumption (H) and Assumption (F) are equivalent for $\varrho \in [0, \infty)$. This was pointed out in Lemma 2.3. There it had also be shown that

$$\psi(t) = \frac{H(t)}{t} - \varrho \log \varrho + \varrho + o(1) \qquad \text{as } t \to \infty.$$

According to Remark 2.1 d), $\psi(\varphi(h)) = h + o(1)$. Hence, we conclude that

$$h = \frac{H(\varphi(h))}{\varphi(h)} - \varrho \log \varrho + \varrho + o(1) \qquad \text{as } h \to \infty. \tag{3.7}$$

a) If $0 < \varrho < \infty$, then we may combine (3.7) with assertion a) of Lemma 3.2 to find that

$$\begin{aligned} \log \bar{N}(h - 2d\kappa\chi(\varrho/\kappa)) &= \log \bar{N}_{\varphi(h)} \left(-\varrho \log \varrho + \varrho - 2d\kappa\chi(\varrho/\kappa) + o(1)\right) \\ &\sim -\varphi(h) = \log \bar{F}(h) \end{aligned}$$

as $h \to \infty$. Here we have also used that the rate function J is non-decreasing and continuous and

$$J\left(-\varrho \log \varrho + \varrho - 2d\kappa\chi(\varrho/\kappa)\right) = 1.$$

This clearly proves (3.3).

b) Suppose now that $\varrho = 0$. In this case (3.7) tells us that

$$h = \frac{H(\varphi(h))}{\varphi(h)} + o(1) \qquad \text{as } h \to \infty.$$

From this and assertion b) of Lemma 3.2 we conclude that

$$\log \bar{N}(h - \delta) = \log \bar{N}_{\varphi(h)}(-\delta + o(1)) \geq -\varphi(h) = \log \bar{F}(h)$$

for every $\delta > 0$ and all sufficiently large h. This yields the lower bound in (3.4). The upper bound is the same as in (3.2). $\square$

4. Another view on the moments and Lifshitz tails

4.1. An alternative proof

We have seen in Section 1.1 that the moments of the solution u to our Cauchy problem with initial datum $u_0 \equiv 1$ admit the probabilistic representation

$$\langle u(t,0)^p \rangle = \left\langle \mathbb{E}_0^p \exp\left\{ \sum_{z\in\mathbb{Z}^d} l_t(z)\xi(z) \right\} \right\rangle .$$

Theorem 1.2 was proved by first 'averaging' over all realizations of the random potential $\xi(\cdot)$ and then applying large deviations for the local times $l_t(\cdot)$. In this section we will go the other way round. We will first 'average' over $l_t(\cdot)$ and then apply large deviations for the high exceedances of the potential $\xi(\cdot)$. This approach works well under Assumption (H) at least for $0 < \varrho < \infty$. It has the advantage of making transparent the role of the typical profiles of high peaks of the potential and explains how to identify them in the same spirit as it had be done for the almost sure asymptotics in Section 2.

As before, we will assume without loss of generality that $u_0 \equiv 1$ and, hence, $\langle u(t,x)^p \rangle$ does not depend on x. It is obvious from the proof of Lemma 1.4 that the bounds

$$\left\langle u^{R,0}(t,x)^p \right\rangle \leq \langle u(t,0)^p \rangle \leq \left\langle u^{R,\pi}(t,x)^p \right\rangle$$

are valid not only for $x = 0$ but for arbitrary $x \in \mathbb{T}_R^d$. This implies that

$$\frac{1}{|\mathbb{T}_R^d|^p} \left\langle \left(\sum_{x \in \mathbb{T}_R^d} u^{R,0}(t,x) \right)^p \right\rangle \leq \langle u(t,0)^p \rangle \leq \frac{1}{|\mathbb{T}_R^d|} \left\langle \left(\sum_{x \in \mathbb{T}_R^d} u^{R,\pi}(t,x) \right)^p \right\rangle . \tag{4.1}$$

This form of the bounds has the advantage that the sums on the left and the right may be estimated by means of their spectral representations with respect to the eigenvalues and eigenfunctions of $\mathcal{H}$ in $l^2(\mathbb{T}_R^d)$. For, given $h\colon \mathbb{T}_R^d \to \mathbb{R}$, let $\lambda^{R,0}(h(\cdot))$ and $\lambda^{R,\pi}(h(\cdot))$ denote the principle eigenvalues of the operator $\kappa\Delta + h(\cdot)$ in $l^2(\mathbb{T}_R^d)$ with zero and periodic boundary conditions, respectively. Then we obtain

$$\sum_{x \in \mathbb{T}_R^d} u^{R,0}(t,x) \geq e^{t\lambda^{R,0}(\xi(\cdot))}$$

and

$$\sum_{x \in \mathbb{T}_R^d} u^{R,\pi}(t,x) \leq |\mathbb{T}_R^d| \, e^{t\lambda^{R,\pi}(\xi(\cdot))}.$$

Rough considerations indicate that the relevant exceedances of the potential $\xi(\cdot)$ should be of height $H(pt)/pt$. This motivates us to introduce the shifted potential

$$\xi_t(\cdot) := \xi(\cdot) - \frac{H(t)}{t}, \qquad t > 0.$$

Then

$$\lambda^{R,0}(\xi(\cdot)) = \frac{H(pt)}{pt} + \lambda^{R,0}(\xi_{pt}(\cdot))$$

and

$$\lambda^{R,\pi}(\xi(\cdot)) = \frac{H(pt)}{pt} + \lambda^{R,\pi}(\xi_{pt}(\cdot)).$$

With these observations and the above estimates we conclude from (4.1) that

$$|\mathbb{T}_R^d|^{-p} e^{H(pt)} \left\langle e^{pt\lambda^{R,0}(\xi_{pt}(\cdot))} \right\rangle \leq \langle u(t,0)^p \rangle \leq |\mathbb{T}_R^d|^{p-1} e^{H(pt)} \left\langle e^{pt\lambda^{R,\pi}(\xi_{pt}(\cdot))} \right\rangle . \tag{4.2}$$

To obtain asymptotic formulas for the expectations on the left and the right it remains to derive a large deviation principle for $\xi_t(\cdot)$ and afterwards to apply the Laplace-Varadhan method. Since $\xi_t(0) \to -\infty$ as $t \to \infty$, we need to add $-\infty$ to the real axis. We already know from step 3^0 of the proof of Lemma 2.3 that the distribution of $\xi_t(0)$ on the left-compactified real line $[-\infty, \infty)$ satisfies the full large deviation principle as $t \to \infty$ with scale t and rate function

$$J(h) = \varrho \exp\left\{\frac{h}{\varrho} - 1\right\}, \qquad h \in [-\infty, \infty).$$

Since the random variables $\xi_t(x)$, $x \in \mathbb{T}_R^d$, are independent and identically distributed, this implies the following result.

Lemma 4.1. *Let Assumption (H) be satisfied for some $\varrho \in (0, \infty)$. Then for each $R \in \mathbb{N}$, the probability distributions of $\xi_t(\cdot)$ on $[-\infty, \infty)^{\mathbb{T}_R^d}$ satisfy the full large deviation principle as $t \to \infty$ with scale t and rate function*

$$J^{R,\varrho}(h(\cdot)) := \frac{\varrho}{e} \sum_{x \in \mathbb{T}_R^d} e^{h(x)/\varrho}, \qquad h\colon \mathbb{T}_R^d \to [-\infty, \infty).$$

Now observe that the principle eigenvalues $\lambda^{R,0}$ and $\lambda^{R,\pi}$ may be considered as continuous functionals on $[-\infty, \infty)^{\mathbb{T}_R^d}$. If a potential $h\colon \mathbb{T}_R^d \to [-\infty, \infty)$ attains the value $-\infty$, then $\lambda^{R,0}(h(\cdot))$ and $\lambda^{R,\pi}(h(\cdot))$ are the principal eigenvalues of $\kappa\Delta + h(\cdot)$ on $\mathbb{T}_R^d \setminus \{h = -\infty\}$ with Dirichlet boundary condition on the set where $h = -\infty$ and zero or periodic boundary condition on the rest of the boundary. The functionals $\lambda^{R,0}$ and $\lambda^{R,\pi}$ are not bounded from above. Therefore, in order to be sure that the Laplace-Varadhan method is applicable, we still have to check that

$$\lim_{M\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \left\langle e^{t\lambda^{R,\pi}(\xi_t(\cdot))} \mathbb{1}\left(\lambda^{R,\pi}(\xi_t(\cdot)) > M\right)\right\rangle = 0. \tag{4.3}$$

The same will then be true for $\lambda^{R,0}(\xi_t(\cdot))$. Since

$$\lambda^{R,\pi}(\xi_t(\cdot)) \le \max_{x \in \mathbb{T}_R^d} \xi_t(x) = \max_{x \in \mathbb{T}_R^d} \xi(x) - \frac{H(t)}{t},$$

we obtain

$$\begin{aligned} &\left\langle e^{t\lambda^{R,\pi}(\xi_t(\cdot))} \mathbb{1}\left(\lambda^{R,\pi}(\xi_t(\cdot)) > M\right)\right\rangle \\ &\qquad \le |\mathbb{T}_R^d|\, e^{-H(t)} \left\langle e^{t\xi(0)} \mathbb{1}\left(\xi(0) > \frac{H(t)}{t} + M\right)\right\rangle. \end{aligned}$$

Applying Chebyshev's exponential inequality and taking into account Assumption (H), we may continue as follows:

$$\begin{aligned} &\le |\mathbb{T}_R^d|\, e^{-2H(t) - Mt} \left\langle e^{2t\xi(0)} \right\rangle \\ &= |\mathbb{T}_R^d| \exp\{-Mt + H(2t) - 2H(t)\} \\ &= \exp\{-Mt + 2(\log 2)\varrho t + o(t)\}. \end{aligned}$$

This clearly proves (4.3). Now an application of the Laplace-Varadhan method for the large deviation principle of Lemma 4.1 yields

$$\left\langle e^{pt\lambda^{R,0}(\xi_{pt}(\cdot))}\right\rangle = \exp\left\{pt\max\left[\lambda^{R,0} - J^{R,\varrho}\right] + o(t)\right\}$$

and

$$\left\langle e^{pt\lambda^{R,\pi}(\xi_{pt}(\cdot))}\right\rangle = \exp\left\{pt\max\left[\lambda^{R,\pi} - J^{R,\varrho}\right] + o(t)\right\}.$$

Substituting this in (4.2), we arrive at the next lemma.

Lemma 4.2. *Let Assumption (H) be satisfied for some $\varrho \in (0,\infty)$. Then for arbitrary $R \in \mathbb{N}$ and $p = 1, 2, \ldots$ we have*

$$\langle u(t,0)^p\rangle \geq \exp\left\{H(pt) + pt\max\left[\lambda^{R,0} - J^{R,\varrho}\right] + o(t)\right\}$$

and

$$\langle u(t,0)^p\rangle \leq \exp\left\{H(pt) + pt\max\left[\lambda^{R,\pi} - J^{R,\varrho}\right] + o(t)\right\}$$

as $t \to \infty$.

Similarly to the proof of Lemma 2.17, we may now use the fact that $\lambda^{R,0}$ and $\lambda^{R,\pi}$ are, respectively, the Legendre transforms of the functionals $\kappa S_d^{R,0}$ and $\kappa S_d^{R,\pi}$ introduced in Section 1.2. Note that this makes sense even for potentials which attain the value $-\infty$. We obtain

$$\begin{aligned}&\max\left[\lambda^{R,0} - J^{R,\varrho}\right]\\ &\quad= \max_{p\in\mathcal{P}(\mathbb{T}_R^d)}\left\{\max_{h\in[-\infty,\infty)^{\mathbb{T}_R^d}}\left[\sum\nolimits_{x\in\mathbb{T}_R^d} p(x)h(x) - J^{R,\varrho}(h(\cdot))\right] - \kappa S_d^{R,0}(p)\right\}.\end{aligned}$$

The inner maximum is attained for

$$h(\cdot) = \varrho\log p(\cdot) + \varrho \tag{4.4}$$

and equals $-\varrho I_d^R(p)$. Hence,

$$\max\left[\lambda^{R,0} - J^{R,\varrho}\right] = -\min\left[\kappa S_d^{R,0} + \varrho I_d^R\right].$$

Correspondingly,

$$\max\left[\lambda^{R,\pi} - J^{R,\varrho}\right] = -\min\left[\kappa S_d^{R,\pi} + \varrho I_d^R\right].$$

This shows that Lemma 4.2 may be combined with Lemma 1.10 to arrive at the assertion of Theorem 1.2 for $0 < \varrho < \infty$.

The above arguments indicate that those peaks of the potential $\xi(\cdot)$ which form the p-th moment have height comparable with

$$\frac{H(pt)}{pt} = \frac{H(t)}{t} + \varrho\log p + o(1).$$

Their shape is given by (4.4) with $p(\cdot)$ being a minimizer of the functional $S_d + \varrho I_d$. In view of Lemma 1.10 and Remark 1.3 c), this means that those peaks which contribute to the p-th moment are of the form

$$\frac{H(t)}{t} + \varrho(\log p + 1) - 2d\kappa\chi(\frac{\varrho}{\kappa}) + 2\varrho \log\left(v_{\varrho/\kappa} \otimes \cdots \otimes v_{\varrho/\kappa}\right).$$

The heights of these peaks differ for different values of p and are much larger than the typical exceedances (2.6) for the almost sure asymptotics. But their profiles coincide.

4.2. Generalization to correlated potentials

Until now we assumed that the potential $\xi(\cdot)$ consists of i.i.d. random variables. In the present section we will explain how our asymptotic results about the moments and the Lifshitz tails may be extended to a large class of dependent random fields. This may be considered as an important step towards the investigation of the spatially continuous situation.

We will assume throughout that $\xi(\cdot)$ is a (not necessarily ergodic) homogeneous random field such that

$$H(t) := \log\left\langle e^{t\xi(0)}\right\rangle < \infty \qquad \text{for all } t \geq 0.$$

As before, we suppose that the initial field $u_0(\cdot)$ is nonnegative, homogeneous, independent of $\xi(\cdot)$, and satisfies (1.2). For general correlated potentials it is unknown whether or not the nonnegative solution of our Cauchy problem is unique a.s. In the following we consider the smallest nonnegative solution which is given by the Feynman-Kac formula (2.4). For a detailed investigation of existence and uniqueness see [9].

Let $\mathcal{P}_f(\mathbb{Z}^d)$ denote the space of probability measures on $\mathbb{Z}^d$ with finite support. Let further δ_z be the Dirac measure at $z \in \mathbb{Z}^d$. We consider shift-invariant functionals $G_t \colon \mathcal{P}_f(\mathbb{Z}^d) \to \mathbb{R}$, $t > 0$, defined by

$$G_t(p) := \log\left\langle \exp\left\{t \sum_{z\in\mathbb{Z}^d} p(z)\xi(z)\right\}\right\rangle.$$

As a consequence of Hölder's inequality, G_t is convex. In particular,

$$G_t(p) \leq G_t(\delta_0) = H(t) < \infty.$$

We impose the following general regularity assumption on the tail behavior of $\xi(\cdot)$ which replaces Assumption (H) of Section 1.

Assumption (G). For each $p \in \mathcal{P}_f(\mathbb{Z}^d)$, the (possibly infinite) limit

$$I(p) := \lim_{t\to\infty} \frac{G_t(\delta_0) - G_t(p)}{t}$$

exists.

As a consequence of the mentioned properties of G_t, the functional I is shift-invariant and concave. Moreover, $0 \leq I \leq \infty$ and $I(\delta_0) = 0$. We next introduce the function

$$\chi_G(\kappa) := \inf_{\mathcal{P}_f(\mathbb{Z}^d)} [\kappa S_d + I], \qquad \kappa > 0,$$

where S_d is the d-dimensional Donsker-Varadhan functional from Section 1.3. Note that χ_G is concave and

$$0 \leq \chi_G(\kappa) \leq 2d\kappa, \qquad \kappa > 0.$$

The left equality is valid if $I \equiv 0$, and the right equality holds if $I(p) = \infty$ for p not being a Dirac measure.

We are now ready to formulate our result about the moments of the solution u to the Cauchy problem (1.1).

Theorem 4.3. *Let Assumption (G) be satisfied. Then*

$$\langle u(t, 0)^p \rangle = \exp\{H(pt) - \chi_G(\kappa)pt + o(t)\}$$

as $t \to \infty$ *for* $p = 1, 2, \ldots$

We next want to illustrate this result by several examples. Its proof will be postponed to the end of this section.

Example 4.4. a) *Uncorrelated potentials.* Theorem 1.2 is a particular case of Theorem 4.3. Indeed, if the random variables $\xi(x)$, $x \in \mathbb{Z}^d$, are independent and identically distributed, then

$$G_t(p) = \sum_z H(tp(z)).$$

Under these circumstances, Assumption (H) implies Assumption (G) with $I = \varrho I_d$, where I_d is the d-dimensional entropy functional introduced in Section 1.3. Hence, taking into account Lemma 1.10, we find that

$$\chi_G(\kappa) = \inf_{\mathcal{P}_f(\mathbb{Z}^d)} [\kappa S_d + \varrho I_d] = 2d\kappa\chi(\frac{\varrho}{\kappa}).$$

b) *Random clouds.* Let $\eta(\cdot)$ be an i.i.d. random field such that

$$H_\eta(t) := \log\left\langle e^{t\eta(0)}\right\rangle < \infty \qquad \text{for all } t \in \mathbb{R}.$$

Given a function $\varphi\colon \mathbb{Z}^d \to \mathbb{R}$ with finite support, consider the random potential

$$\xi(x) := \sum_{y \in \mathbb{Z}^d} \varphi(y - x)\eta(y), \qquad x \in \mathbb{Z}^d.$$

Then the cumulant generating function H of $\xi(0)$ has the form

$$H(t) = \sum_y H_\eta(t\varphi(y)).$$

Let us first consider the case when φ is nonnegative. Suppose that the cumulant generating function H_η satisfies Assumption (H) for some $\varrho \in [0,\infty]$. Then the function H also fulfills this assumption but with parameter $\varrho \sum_y \varphi(y)$. We obtain

$$G_t(p) = \sum_y H_\eta(t(\varphi * p)(y)), \qquad p \in \mathcal{P}_f(\mathbb{Z}^d),$$

where $\varphi * p$ denotes convolution of φ with p. Hence, if ϱ is finite, then the potential $\xi(\cdot)$ satisfies Assumption (G) with

$$I(p) = \varrho \left[I_d(\varphi * p) - I_d(\varphi) \right].$$

Assumption (G) is also fulfilled for $\varrho = \infty$. In this case $I(p) = \infty$ if p is not a Dirac measure. This may be seen from the formula

$$\frac{G_t(\delta_0) - G_t(p)}{t} = \sum_y \left[\frac{\sum_z p(z) H_\eta(t\varphi(y-z)) - H_\eta(t(\varphi * p)(y))}{t} \right]. \tag{4.5}$$

By convexity, the expression in the square brackets is always nonnegative. Now suppose that $|\operatorname{supp} p| \geq 2$. Then one finds $y \in \mathbb{Z}^d$ such that $\operatorname{supp} p$ intersects $y - \operatorname{supp} \varphi$ but is not entirely contained in that set. It will be enough to show that, for this y, the term in the square brackets in (4.5) converges to $+\infty$ as $t \to \infty$. Note that

$$0 < \gamma := \sum_{y - z \in \operatorname{supp} \varphi} p(z) < 1.$$

Hence, again by convexity,

$$\begin{aligned}
\sum_z p(z) H_\eta(t\varphi(y-z)) &= \gamma \sum_{y-z \in \operatorname{supp} \varphi} \frac{p(z)}{\gamma} H_\eta(t\varphi(y-z)) \\
&\geq \gamma H_\eta \left(t \sum_{y-z \in \operatorname{supp} \varphi} \frac{p(z)}{\gamma} \varphi(y-z) \right) \\
&= \gamma H_\eta \left(t \frac{1}{\gamma} (\varphi * p)(y) \right).
\end{aligned}$$

Thus, for our particular lattice site y, the term in the square brackets on the right of (4.5) may be estimated from below by

$$\gamma \, \frac{H_\eta\left(t\gamma^{-1}(\varphi * p)(y)\right) - \gamma^{-1} H_\eta\left(t(\varphi * p)(y)\right)}{t}.$$

But, since $\varrho = \infty$ and $\gamma^{-1} > 1$, this expression converges to infinity as $t \to \infty$.

If φ attains both positive and negative values, then the situation will be more complex. Let us consider in detail the case when the random field $\eta(\cdot)$ is both unbounded from above and below and the cumulant generating functions of $\eta(0)$ and $-\eta(0)$ satisfy Assumption (H) with finite nonnegative parameters ϱ_+ and ϱ_-, respectively. Then we may use the decomposition

$$\begin{aligned}\frac{G_t(\delta_0)-G_t(p)}{t} =& \sum_y \frac{H_\eta(t\varphi(y)) - |\varphi(y)|H_\eta(t\,\mathrm{sign}\,\varphi(y))}{t}\\ &- \sum_y \frac{H_\eta(t(\varphi*p)(y)) - |(\varphi*p)|H_\eta(t\,\mathrm{sign}\,(\varphi*p)(y))}{t}\\ &+ \left[\|\varphi^-\|_1 - \|(\varphi*p)^-\|_1\right]\frac{H_\eta(-t)}{t}\\ &+ \left[\|\varphi^+\|_1 - \|(\varphi*p)^+\|_1\right]\frac{H_\eta(t)}{t},\end{aligned}$$

where $\|\cdot\|_1$ denotes the l^1-norm. As $t\to\infty$, the first sum on the right converges to $-\varrho_- I_d(\varphi^-) - \varrho_+ I_d(\varphi^+)$. The second sum converges to the same expression with φ replaced by $\varphi*p$. The rest is either zero or converges to infinity, since the terms in the square brackets are nonnegative and both $H_\eta(-t)/t$ and $H_\eta(t)/t$ tend to infinity. But the sum of the two terms in the square brackets equals

$$\||\varphi|\|_1 - \||\varphi*p|\|_1$$

which is zero if and only if, for each $y\in\mathbb{Z}^d$, $\mathrm{supp}\,p$ intersects at most one of the two sets $y-\mathrm{supp}\,\varphi^-$ and $y-\mathrm{supp}\,\varphi^+$. Let $\mathcal{D}_\varphi$ denote the set of all probabilities $p\in\mathcal{P}_f(\mathbb{Z}^d)$ with this property. Then the above considerations show that the potential $\xi(\cdot)$ satisfies Assumption (G) with

$$\begin{aligned}I(p) = &\ \varrho_-\left[I_d((\varphi*p)^-) - I_d(\varphi^-)\right]\\ &+ \varrho_+\left[I_d((\varphi*p)^+) - I_d(\varphi^+)\right]\end{aligned}$$

if $p\in\mathcal{D}_\varphi$ and $I(p)=\infty$ otherwise.

Let us finally cast a glance at the particular situation when $\mathrm{supp}\,\varphi^-$ and $\mathrm{supp}\,\varphi^+$ are neighboring in each direction (i.e., for each $e\in\mathbb{Z}^d$ with $|e|=1$, there exist $x,y\in\mathbb{Z}^d$ such that $x-y=e$ and $\varphi(x)\varphi(y)<0$). Then $p\in\mathcal{D}_\varphi$ implies that all connected components of $\mathrm{supp}\,p$ consist of single lattice sites and, therefore, $S_d(p)=2d$. As a consequence, we obtain $\chi_G(\kappa)=2d\kappa$.

c) *Random plateaux.* Let again $\eta(\cdot)$ be a field of i.i.d. random variables with finite cumulant generating function H_η. Given a finite subset V of $\mathbb{Z}^d$, consider the potential

$$\xi(x) := \max_{y\in V}\eta(x-y), \qquad x\in\mathbb{Z}^d.$$

Suppose that H_η satisfies Assumption (H) with $\varrho=\infty$. This is true in particular for i.i.d. Gaussian fields $\eta(\cdot)$. We obtain

$$\max_{m:\ \operatorname{supp} p \to V} \left\langle \exp\left\{ t \sum_z p(z)\eta(z - m(z)) \right\} \right\rangle \le e^{G_t(p)}$$
$$\le \sum_{m:\ \operatorname{supp} p \to V} \left\langle \exp\left\{ t \sum_z p(z)\eta(z - m(z)) \right\} \right\rangle .$$

Note that the upper and lower bounds differ at most by the constant factor $|V|^{|\operatorname{supp} p|}$. Moreover, the maximum on the left may be taken over all $m\colon \mathbb{Z}^d \to V$. For such m,

$$\sum_z p(z)\eta(z - m(z)) = \sum_x p(D_m(x))\eta(x),$$

where $D_m(x) := \{z\colon z - m(z) = x\}$ and $p(D_m(x)) = \sum_{z \in D_m(x)} p(z)$. Taking this into account, we find that

$$G_t(p) = \max_{m\,:\, \mathbb{Z}^d \to V} \sum_x H_\eta(tp(D_m(x)) + O(1)$$

as $t \to \infty$. In particular, $G_t(\delta_0) = H_\eta(t) + O(1)$. Hence,

$$\frac{G_t(\delta_0) - G_t(p)}{t} = \min_{m\,:\, \mathbb{Z}^d \to V} \sum_x \frac{p(D_m(x))H_\eta(t) - H_\eta(tp(D_m(x)))}{t} + o(1).$$

Since $\varrho = \infty$, the minimum on the right is either zero or tends to infinity as $t \to \infty$. It is zero if and only if there exist $m\colon \mathbb{Z}^d \to V$ and $x \in \mathbb{Z}^d$ such that $\operatorname{supp} p \subseteq D_m(x)$. But this happens if and only if $\operatorname{supp} p \subseteq x + V$ for some $x \in \mathbb{Z}^d$. We have therefore shown that Assumption (G) is fulfilled with $I(p) = 0$ if $\operatorname{supp} p \subseteq x + V$ for some x and $I(p) = \infty$ otherwise. Since the functional S_d is shift-invariant, we conclude from this that

$$\chi_G(\kappa) = \kappa \min_{\operatorname{supp} p \subseteq V} S_d(p) = \kappa\lambda^0(V),$$

where $\lambda^0(V)$ denotes the smallest eigenvalue of $-\Delta$ in $l^2(V)$ with Dirichlet boundary condition. If $|V| \ge 2$, then $0 < \lambda^0(V) < 2d$.

As we will see later, our proof of Theorem 4.3 also works for the fundamental solution $q(t, x, y)$ of the Anderson Hamiltonian $\mathcal{H}$. In particular, under Assumption (G),

$$\langle q(t, 0, 0)\rangle = \exp\{H(t) - \chi_G(\kappa)t + o(t)\} \qquad \text{as } t \to \infty. \tag{4.6}$$

This allows a straightforward generalization of our result about the Lifshitz tails in Section 3. To be precise, assume that the potential $\xi(\cdot)$ is homogeneous and ergodic. Suppose further that all exponential moments $\langle e^{t\xi(0)}\rangle$, $t \ge 0$, are finite, the distribution function F of $\xi(0)$ is continuous, and $F(r) < 1$ for all r. Recall that Assumption (F) was introduced in Section 2.1. As before, let $\bar{F}$ and $\bar{N}$ denote the tails of the distribution function F and the spectral distribution function N of our Hamiltonian $\mathcal{H}$, respectively.

Theorem 4.5. *Let Assumption (F) be satisfied for some $\varrho \in [0, \infty)$. Let also Assumption (G) be fulfilled.*

a) If $0 < \varrho < \infty$, then

$$\log \bar{N}(h) \sim \log \bar{F}(h + \chi_G(\kappa)) \qquad \text{as } h \to \infty.$$

b) If $\varrho = 0$, then

$$\bar{F}(h + \chi_G(\kappa) + \delta) \le \bar{N}(h) \le \bar{F}(h + \chi_G(\kappa) - \delta)$$

for arbitrary $\delta > 0$ and all sufficiently large h.

Almost word for word, the proof repeats that of Theorem 3.1 but with the more general asymptotics (4.6) as starting point. In addition, we now have a nontrivial upper bound in assertion b). Its proof is simple and left to the reader.

The rest of this section is devoted to the *proof of Theorem 4.3.* As before, we may and will assume without loss of generality that $u_0 \equiv 1$. Let us begin with a few preliminary remarks after which the proof of the lower bound will turn out to follow closely that for i.i.d. potentials.

First, given $p \in \mathcal{P}_f(\mathbb{Z}^d)$ with $I(p) < \infty$, observe that $I(q) < \infty$ for all q with $\operatorname{supp} q \subseteq \operatorname{supp} p$. Indeed, for such q one finds $\gamma \in (0,1)$ and $r \in \mathcal{P}_f(\mathbb{Z}^d)$ such that

$$p = \gamma q + (1-\gamma) r.$$

Since I is concave and nonnegative, this implies that

$$\infty > I(p) \ge \gamma I(q) + (1-\gamma) I(r) \ge \gamma I(q).$$

Hence, $I(q)$ is finite. Now, if $I(p) < \infty$, then the restriction of the functional I to $\{q \in \mathcal{P}_f(\mathbb{Z}^d)\colon \operatorname{supp} q = \operatorname{supp} p\}$ is finite and continuous. This is obvious from the concavity of I and the observation that the above set is convex and open in its affine hull. In particular, there exists an open neighborhood $U(p)$ of p in $\{q \in \mathcal{P}_f(\mathbb{Z}^d)\colon \operatorname{supp} q \subseteq \operatorname{supp} p\} = \mathcal{P}(\operatorname{supp} p)$ such that I is bounded and continuous on $U(p)$.

Next, we remark that

$$\inf_{\mathcal{P}_f(\mathbb{Z}^d)} [\kappa S_d + I] = \inf_{\mathcal{P}_{f,c}(\mathbb{Z}^d)} [\kappa S_d + I], \tag{4.7}$$

where $\mathcal{P}_{f,c}(\mathbb{Z}^d)$ is the subset of $\mathcal{P}_f(\mathbb{Z}^d)$ consisting of measures with connected support. To see this, fix $p \in \mathcal{P}_f(\mathbb{Z}^d) \setminus \mathcal{P}_{f,c}(\mathbb{Z}^d)$ arbitrarily and let $D_1, \dots, D_m$ denote the connected components of $\operatorname{supp} p$. Then we find $p_1, \dots, p_m \in \mathcal{P}_{f,c}(\mathbb{Z}^d)$

with $\operatorname{supp} p_i = D_i$ for $i = 1, \ldots, m$ and $\gamma_1, \ldots, \gamma_m \in (0,1)$ with $\gamma_1 + \cdots + \gamma_m = 1$ such that

$$p = \sum_{i=1}^{m} \gamma_i p_i.$$

Since I is concave, we have

$$I(p) \geq \sum_{i=1}^{m} \gamma_i I(p_i).$$

Moreover, since the supports of the measures p_i are separated from each other by a distance larger than one, the functional S_d splits into parts:

$$S_d(p) = \sum_{i=1}^{m} \gamma_i S_d(p_i).$$

Hence,

$$\begin{aligned} \kappa S_d(p) + I(p) &\geq \sum_{i=1}^{m} \gamma_i \left[\kappa S_d(p_i) + I(p_i)\right] \\ &\geq \inf_{\mathcal{P}_{f,c}(\mathbb{Z}^d)} \left[\kappa S_d + I\right]. \end{aligned}$$

This clearly proves (4.7).

Because of (4.7), the derivation of the *lower bound* reduces to the following lemma.

Lemma 4.6. *Let Assumption (G) be satisfied. Given $q \in \mathcal{P}_f(\mathbb{Z}^d)$, suppose that $I(q)$ is finite and* $\operatorname{supp} q$ *is connected. Then*

$$\langle u(t,0)^p \rangle \geq \exp\left\{H(pt) - pt\left[\kappa S_d(q) + I(q)\right] - o(t)\right\}$$

as $t \to \infty$ for $p = 1, 2, \ldots$

Proof. Because of shift-invariance, we may assume that $0 \in \operatorname{supp} q$. Let τ be the first time when one of our random walks $x_1(t), \ldots, x_p(t)$ exits $\operatorname{supp} q$. Then

$$\begin{aligned} \langle u(t,0)^p \rangle &= \left\langle \mathbb{E}_0^p \exp\left\{pt \sum_{z \in \mathbb{Z}^d} L_t(z)\xi(z)\right\}\right\rangle \\ &= \mathbb{E}_0^p \exp\left\{G_{pt}(L_t(\cdot))\right\} \\ &\geq e^{H(pt)} \mathbb{E}_0^p \exp\left\{-pt \frac{G_{pt}(\delta_0) - G_{pt}(L_t(\cdot))}{pt}\right\} 1\!\!1(\tau > t). \end{aligned}$$

We know from our preliminary remarks that there exists an open neighborhood $U(q)$ of q in $\{\tilde{q} \in \mathcal{P}_f(\mathbb{Z}^d) \colon \operatorname{supp} \tilde{q} \subseteq \operatorname{supp} q\}$ such that I is bounded and continuous on $U(q)$. Moreover, $U(q)$ may be chosen so that the convergence

$$\frac{G_t(\delta_0) - G_t(\tilde{q})}{t} \to I(\tilde{q}) \qquad \text{as } t \to \infty$$

in Assumption (G) is uniform in $\tilde{q} \in U(q)$. This is again a consequence of concavity. Hence, we obtain

$$\langle u(t,0)^p \rangle \geq e^{H(pt)-o(t)}\, \mathbb{E}_0^p \exp\{-ptI(L_t(\cdot))\}\, 1\!\!1\,(\tau > t, L_t(\cdot) \in U(q))\,.$$

Since $\operatorname{supp} q$ is connected, our assertion now follows by an application of the large deviation principle for the occupation time measures L_t on $\operatorname{supp} q$ with killing on the complement of $\operatorname{supp} q$, cf. assertion a) of Lemma 1.5. □

The proof of the *upper bound* is more subtle, mainly because the method of 'periodization' used in Section 1 breaks down for correlated potentials. Our way out of this dilemma consists in deriving an upper bound for the moments by use of Dirichlet boundary conditions. As in Lemma 1.4, let $u^{R,0}$ denote the solution of our initial boundary value problem on $\mathbb{T}_R^d$ with zero boundary condition and initial datum identically one. Let further

$$\partial \mathbb{T}_R^d := \mathbb{T}_R^d \setminus \mathbb{T}_{R-1}^d$$

denote the boundary of $\mathbb{T}_R^d$. The next lemma provides the key for the derivation of the upper bound.

Lemma 4.7. *For each $R \in \mathbb{N}$ and $p = 1, 2, \ldots$ we have*

$$\langle u(t,0)^p \rangle \leq e^{d\kappa\varepsilon_R pt + o(t)} \sum_{x \in \mathbb{T}_R^d} \left\langle u^{R,0}(t,x)^p \right\rangle$$

as $t \to \infty$, where

$$\varepsilon_R := |\partial \mathbb{T}_R^d| / |\mathbb{T}_R^d| \to 0$$

as $R \to \infty$.

Proof. 1^0 Fix R and p arbitrarily. We consider the periodic frame

$$\Phi := \bigcup_{x \in (2R+1)\mathbb{Z}^d} \left(x + \partial \mathbb{T}_R^d\right).$$

Each lattice site is covered by exactly $|\partial \mathbb{T}_R^d|$ of the shifted frames

$$\Phi_y := y + \Phi, \qquad y \in \mathbb{T}_R^d.$$

From this we conclude that for each $t > 0$ there exists a (random) site $y \in \mathbb{T}_R^d$ such that

$$\sum_{z \in \Phi_y} L_t(z) \leq |\partial \mathbb{T}_R^d| / |\mathbb{T}_R^d| = \varepsilon_R.$$

Because of this, using the probabilistic representation of the moments and the homogeneity of the potential, we get

$$\begin{aligned}\langle u(t,0)^p\rangle &= \left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi(z)\right\}\right\rangle \\ &\le |\mathbb{T}_R^d| \left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi(z)\right\} 1\!\!1\left(\sum_{z\in\Phi} L_t(z)\le \varepsilon_R\right)\right\rangle \\ &\le |\mathbb{T}_R^d| e^{d\kappa\varepsilon_R pt}\left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\}\right\rangle, \end{aligned} \tag{4.8}$$

where the potential $\xi_R(\cdot)$ has been obtained by lowering $\xi(\cdot)$ on the frame Φ by an amount of $d\kappa$:

$$\xi_R(z) := \begin{cases} \xi(z) - d\kappa, & \text{if } z\in\Phi, \\ \xi(z), & \text{otherwise.}\end{cases}$$

2^0 We next introduce the centered cubes

$$V_r := \bigcup_{\substack{z\in(2R+1)\mathbb{Z}^d \\ |z^1|,\dots,|z^d|\le r}} \left(z+\mathbb{T}_R^d\right), \qquad r>0,$$

where $z^1,\dots,z^d$ are the components of z, and denote by $\tau^p(r)$ the first time when one of the random walks $x_1(t),\dots,x_p(t)$ leaves V_r. We set

$$r(t) := t\log t$$

and show that the paths of our random walks may be restricted to the cube $V_{r(t)}$. More precisely, we want to prove that

$$\begin{aligned}&\left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\}\right\rangle \\ &\qquad \le \left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\} 1\!\!1\left(\tau^p(r(t))>t\right)\right\rangle + e^{H(pt)-\gamma t}\end{aligned} \tag{4.9}$$

for arbitrary $\gamma>0$ and all sufficiently large t. With

$$R_n(t) := nr(t), \qquad n\in\mathbb{N},$$

we obtain

$$\begin{aligned}&\left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\}\right\rangle \\ &= \left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\} 1\!\!1\left(\tau^p(r(t))>t\right)\right\rangle \\ &\quad + \sum_{n=1}^\infty \left\langle \mathbb{E}_0^p \exp\left\{pt\sum_z L_t(z)\xi_R(z)\right\} 1\!\!1\left(\tau^p(R_n(t))\le t<\tau^p(R_{n+1}(t))\right)\right\rangle .\end{aligned}$$

The expression under the sum may be estimated from above by

$$\left\langle \exp\left\{pt \max_{z\in V_{R_{n+1}(t)}} \xi(z)\right\}\right\rangle \mathbb{P}_0^p\left(\tau^p(R_n(t))\le t\right)$$
$$\le |V_{R_{n+1}(t)}|\, e^{H(pt)}\, \mathbb{P}_0^p\left(\tau^p(R_n(t))\le t\right).$$

Combining this with assertion a) of Lemma 2.5, one readily checks that the above sum does not exceed

$$e^{H(pt)-\gamma t}$$

for arbitrary $\gamma > 0$ and all sufficiently large t. This proves (4.9).

3^0 Next observe that the first term on the right of (4.9) coincides with the p-th moment of $u^0_{r(t)}(t,0)$, where $u^0_{r(t)}$ is the solution of the initial boundary value problem for our parabolic equation on $V_{r(t)}$ with lowered potential $\xi_R(\cdot)$, Dirichlet boundary condition, and initial datum identically one. Therefore, combining (4.8) with (4.9), we find that

$$\langle u(t,0)^p\rangle \le e^{d\kappa\varepsilon_R pt+o(t)}\left[\left\langle u^0_{r(t)}(t,0)^p\right\rangle + e^{H(pt)-\gamma t}\right]$$

for arbitrary $\gamma > 0$. To get rid of the last exponential term we may take $\gamma > (2+\varepsilon_R)d\kappa p$ and use the trivial bound

$$e^{H(pt)-2d\kappa pt} \le \langle u(t,0)^p\rangle$$

which is obtained from the probabilistic representation of the p-th moment by forcing all random walks to stay at 0 until time t. In this way we arrive at the bound

$$\langle u(t,0)^p\rangle \le e^{d\kappa\varepsilon_R pt+o(t)}\left\langle u^0_{r(t)}(t,0)^p\right\rangle. \tag{4.10}$$

4^0 Now the spectral representation of $u^0_{r(t)}(t,\cdot)$ yields

$$u^0_{r(t)}(t,0) \le |V_{r(t)}|\, e^{t\lambda^0_{r(t)}(\xi_R(\cdot))},$$

where $\lambda^0_{r(t)}(\xi_R(\cdot))$ denotes the principal eigenvalue of the operator

$$\mathcal{H}^{R,t} := \kappa\Delta + \xi_R(\cdot) \qquad \text{in } l^2(V_{r(t)})$$

with zero boundary condition. Hence,

$$\left\langle u^0_{r(t)}(t,0)^p\right\rangle \le e^{o(t)}\left\langle e^{pt\lambda^0_{r(t)}(\xi_R(\cdot))}\right\rangle. \tag{4.11}$$

For each $z \in (2R+1)\mathbb{Z}^d$, let $\lambda^{R,0}(\xi(z+\cdot))$ denote the principal eigenvalue of

$$\mathcal{H}^R_z := \kappa\Delta + \xi(\cdot) \qquad \text{in } l^2(z+\mathbb{T}^d_R)$$

with zero boundary condition. A straightforward computation shows that, because of the special form of the potential $\xi_R(\cdot)$,

$$\mathcal{H}^{R,t} \leq \bigoplus_{\substack{z\in(2R+1)\mathbb{Z}^d\\ |z^1|,\dots,|z^d|\leq r(t)}} \mathcal{H}_z^R$$

in the sense of positive definiteness in $l^2(V_{r(t)})$. Consequently,

$$\lambda_{r(t)}^0(\xi_R(\cdot)) \leq \max_{\substack{z\in(2R+1)\mathbb{Z}^d\\ |z^1|,\dots,|z^d|\leq r(t)}} \lambda^{R,0}(\xi(z+\cdot)).$$

Therefore

$$\left\langle e^{pt\lambda_{r(t)}^0(\xi_R(\cdot))}\right\rangle \leq |V_{r(t)}| \left\langle e^{pt\lambda^{R,0}(\xi(\cdot))}\right\rangle. \tag{4.12}$$

Using the spectral representation of $u^{R,0}(t,\cdot)$, one finds that

$$e^{t\lambda^{R,0}(\xi(\cdot))} \leq \sum_{x\in\mathbb{T}_R^d} u^{R,0}(t,x). \tag{4.13}$$

Thereby one also has to take into account that the eigenfunction corresponding to the eigenvalue $\lambda^{R,0}(\xi(\cdot))$ is positive. Combining (4.10)–(4.13), we finally arrive at the assertion of our lemma. $\square$

Because of the last lemma, it only remains to derive appropriate upper bounds for the moments of $u^{R,0}$. This will be done now.

Lemma 4.8. *Let Assumption (G) be satisfied. Then, for each $R \in \mathbb{N}$, $p = 1,2,\dots,$ and $x \in \mathbb{T}_R^d$, we have*

$$\langle u^{R,0}(t,x)^p\rangle \leq \exp\left\{H(pt) - pt \min_{\substack{p\in\mathcal{P}_f(\mathbb{Z}^d)\\ \operatorname{supp} p\subseteq\mathbb{T}_R^d}} [\kappa S_d(p) + I(p)] + o(t)\right\}$$

as $t\to\infty$.

Proof. Let us assume for simplicity that $x = 0$. Using the probabilistic representation of the p-th moment, we find that

$$\langle u^{R,0}(t,0)^p\rangle = e^{H(pt)}\,\mathbb{E}_0^p \exp\left\{-pt\frac{G_{pt}(\delta_0) - G_{pt}(L_t(\cdot))}{pt}\right\} \mathbb{1}(\tau_R^p > t).$$

Together with Assumption (G) this indicates that our assertion follows from the upper bound in the Laplace-Varadhan method applied for the large deviation principle of Lemma 1.5 a). The expression

$$\frac{G_t(\delta_0) - G_t(p)}{t}$$

is nonnegative and converges to $I(p)$ as $t \to \infty$. In general, this convergence is not uniform and the limiting functional I is not continuous on $\mathcal{P}(\mathbb{T}_R^d)$. Nevertheless, an analysis of the standard proof of the upper bound in the Laplace-Varadhan method (see e.g. Deuschel and Stroock [3], Lemma 2.1.8) shows that it will work if the following assertion is valid. For each $p \in \mathcal{P}(\mathbb{T}_R^d)$ and $\delta \in (0,1)$ there exists a neighborhood $U_\delta(p)$ of p in $\mathcal{P}(\mathbb{T}_R^d)$ such that

$$\liminf_{t\to\infty} \inf_{\tilde{p}\in U_\delta(p)} \frac{G_t(\delta_0) - G_t(\tilde{p})}{t} \geq (1-\delta)I(p). \tag{4.14}$$

To prove this note that

$$U_\delta(p) := \left\{(1-\gamma)p + \gamma r \colon 0 \leq \gamma < \delta,\ r \in \mathcal{P}(\mathbb{T}_R^d)\right\}$$

is a neighborhood of p in $\mathcal{P}(\mathbb{T}_R^d)$ for any $\delta \in (0,1)$. Since the functionals G_t are convex, we obtain

$$\begin{aligned} G_t(\delta_0) - G_t((1-\gamma)p + \gamma r) &\geq (1-\gamma)\left(G_t(\delta_0) - G_t(p)\right) + \gamma\left(G_t(\delta_0) - G_t(r)\right) \\ &\geq (1-\gamma)\left(G_t(\delta_0) - G_t(p)\right). \end{aligned}$$

Hence,

$$\inf_{\tilde{p}\in U_\delta(p)} \frac{G_t(\delta_0) - G_t(\tilde{p})}{t} \geq (1-\delta)\frac{G_t(\delta_0) - G_t(p)}{t} \longrightarrow (1-\delta)I(p)$$

for all $\delta \in (0,1)$. This clearly implies (4.14). $\square$

The proof of Theorem 4.3 is now complete.

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