Almost sure asymptotics for the continuous parabolic Anderson model*

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Summary. We consider the parabolic Anderson problem $\partial_t u = \kappa\Delta u + \xi(x)u$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with initial condition $u(0,x) = 1$. Here $\kappa > 0$ is a diffusion constant and ξ is a random homogeneous potential. We concentrate on the two important cases of a Gaussian potential and a shot noise Poisson potential. Under some mild regularity assumptions, we derive the second-order term of the almost sure asymptotics of $u(t,0)$ as $t \to \infty$.

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1. Introduction and main result

1.1. The continuous parabolic Anderson problem

We consider the parabolic Anderson problem

$$\begin{aligned} \partial_t u(t,x) &= \kappa\Delta u(t,x) + \xi(x)u(t,x), & (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ u(0,x) &= 1, & x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\kappa > 0$ is a diffusion constant, and $\xi = \{\xi(x); x \in \mathbb{R}^d\}$ is a random homogeneous potential. We shall consider the two cases of a Gaussian field and a shot noise Poisson field. In Gärtner and König [5] (henceforth abbreviated as GK) we investigated the second-order asymptotics of the moments of $u(t,0)$ as t tends to infinity for a more general class of homogeneous potentials. In the present paper we derive the second-order asymptotics for $u(t,0)$ almost surely w.r.t. the field ξ.

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1.2. The Gaussian potential

We assume that $\xi = \{\xi(x); x \in \mathbb{R}^d\}$ is a homogeneous centered Gaussian field on a complete probability space. Probability and expectation w.r.t. ξ will be denoted by $\mathrm{Prob}(\,\cdot\,)$ and $\langle\,\cdot\,\rangle$, respectively. We assume that ξ has a spectral density f satisfying

$$\int_{\mathbb{R}^d} |\lambda|^2 f(\lambda)\, d\lambda < \infty. \tag{1.2}$$

Then the covariance function

$$B(x) = \int_{\mathbb{R}^d} e^{i\,x\cdot\lambda} f(\lambda)\, d\lambda \tag{1.3}$$

is twice continuously differentiable. In particular, ξ has a version which is α-Hölder continuous for each $\alpha \in (0,1)$ and which will be considered throughout. Moreover, B admits the representation

$$B(x) = \int_{\mathbb{R}^d} g(x-y) g(y)\, dy, \tag{1.4}$$

where g is the Fourier transform of $\sqrt{f}$:

$$g(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\,x\cdot\lambda} \sqrt{f(\lambda)}\, d\lambda. \tag{1.5}$$

Clearly, the real-valued function g belongs to $L^2(\mathbb{R}^d)$ and is a.e. symmetric.

In addition to (1.2), we assume that

$$\int_{Q_R^c} g^2(x)\, dx = o\left((\log R)^{-2/3}\right), \qquad R \to \infty. \tag{1.6}$$

(Here and further on we use the notation $Q_R = (-R,R)^d$ for the centered cube in $\mathbb{R}^d$ with side length $2R$.) Finally, we also assume that the Hessian $B''(0) = \{\partial^2_{ij} B(0)\}_{i,j=1}^d$ is nondegenerate.

Essentially, while (1.2) is a requirement about the smoothness of the covariance function B, (1.6) is a mild supposition about the tail behavior of $B(x)$ as $|x| \to \infty$.

1.3. The shot noise Poisson potential

Let $\Phi(dx)$ denote the realizations of a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$. As in the Gaussian case, we denote probability and expectation w.r.t. the Poisson process by $\mathrm{Prob}(\cdot)$ resp. $\langle\,\cdot\,\rangle$.

Given a function $B\colon \mathbb{R}^d \to \mathbb{R}$, we take as potential the superposition of Poisson shifts of the 'cloud' B:

$$\xi(x) = \int_{\mathbb{R}^d} B(x-y)\, \Phi(dy), \qquad x \in \mathbb{R}^d. \tag{1.7}$$

We assume that B is twice continuously differentiable and attains its unique global maximum at 0 with $B(0) > 0$. As in the Gaussian case, we require that the Hessian $B''(0)$ is nondegenerate. Furthermore, we impose the following condition on the tail behavior of B:

$$\max_{Q_R^c} |B| = o\left((\log R)^{-1}\right), \qquad R \to \infty. \tag{1.8}$$

(We abbreviate $\max_{x\in A} f(x)$ by $\max_A f$.)

We also impose the integrability condition

$$\int_{\mathbb{R}^d} \max_{x\in Q_1} |B(x-y)|\,dy < \infty. \tag{1.9}$$

In particular, this ensures that the realizations (1.7) of our potential are well-defined and continuous. For simplicity, to avoid technicalities with PDE's, we assume in addition that ξ is (locally) Hölder continuous by requiring that, for some $\alpha \in (0,1]$,

$$\int_{\mathbb{R}^d} \sup_{x,y\in Q_1, x\neq y} \frac{|B(x-z)-B(y-z)|}{|x-y|^\alpha}\,dz < \infty. \tag{1.10}$$

In Subsection 4.2 of GK we assumed that B is nonnegative and has compact support. But an inspection of that subsection shows that all statements from there remain true under our more general assumptions.

Note that B is not the covariance function here, but it will turn out to play the same role in our asymptotics as the covariance in the Gaussian case.

1.4. Common features of the two fields

It turns out that a substantial part of our results and proofs can be formulated in a unifying notation for the two random potentials. In this subsection we introduce some more notation, simultaneously for both the Gaussian and the Poisson field. In the sequel, the two cases are referred to as 'Gaussian case' resp. 'Poisson case'.

Abbreviate

$$\sigma^2 = B(0) \qquad \text{and} \qquad \Sigma^2 = -B''(0). \tag{1.11}$$

By our assumptions, the symmetric $d \times d$-matrix Σ^2 is positive definite. Hence we may assume that Σ is also symmetric and positive definite.

We next introduce the cumulant generating function

$$H(\varrho) = \log\langle e^{\varrho\xi(0)}\rangle = \begin{cases} \varrho^2\sigma^2/2, & \text{Gaussian case,} \\ \lambda \int (e^{\varrho B(-y)} - 1)\,dy, & \text{Poisson case,} \end{cases} \qquad \varrho > 0. \tag{1.12}$$

Note that H is finite, H' is strictly increasing and $H'(\varrho) \to \infty$ as $\varrho \to \infty$. We also need the one-sided Legendre transform of H given by

$$L(h) = \sup_{\varrho > 0} \left(\varrho h - H(\varrho) \right). \tag{1.13}$$

For large h, the supremum in (1.13) is attained at some $\varrho(h) > 0$ which is the unique solution of

$$H'(\varrho(h)) = h. \tag{1.14}$$

Note that

$$H'(\varrho) = \begin{cases} \varrho\sigma^2, & \text{Gaussian case,} \\ \frac{\lambda\sigma^2}{\det\Sigma} \left(\frac{2\pi}{\varrho}\right)^{d/2} e^{\varrho\sigma^2}(1+o(1)), & \text{Poisson case,} \end{cases} \qquad \varrho \to \infty, \tag{1.15}$$

and therefore

$$\varrho(h) = L'(h) = \begin{cases} h/\sigma^2, & \text{Gaussian case,} \\ (1+o(1))\frac{1}{\sigma^2}\log h, & \text{Poisson case,} \end{cases} \qquad h \to \infty. \tag{1.16}$$

A combination of Theorem 1 in GK with the calculations made in the Subsections 4.1 and 4.2 of that paper yields that

$$\frac{1}{t}\log\langle u(t,0)\rangle = \frac{H(t)}{t} - (\chi + o(1))\sqrt{H'(t)}, \qquad t \to \infty, \tag{1.17}$$

where

$$\chi = \sqrt{\frac{\kappa}{2\sigma^2}}\,\mathrm{tr}\,\Sigma, \tag{1.18}$$

and $\mathrm{tr}\,\Sigma$ denotes the trace of the matrix Σ. In the following, the quantity χ will play an important role.

1.5. The main result

We define a function h_t, $t \gg 1$, as the unique solution of

$$L(h_t) = d\log t. \tag{1.19}$$

Theorem 1.1. *With probability one,*

$$\frac{1}{t}\log u(t,0) = h_t - (\chi + o(1))\sqrt{h_t} \qquad \text{as} \quad t \to \infty. \tag{1.20}$$

As we will explain in Subsection 1.6, this result is closely related to the geometry of high peaks of the potential ξ and corresponding spatial peaks of the solution. The first-order term h_t is roughly equal to $\max_{|x|\leq t}\xi(x)$, a fact that has already been derived by Carmona and Molchanov [3]. The dependence of $u(t,0)$ on the diffusion constant κ enters the second-order term only (via χ).

Note the analogy between (1.20) and (1.17): $H'(t) \approx H(t)/t$ is replaced by h_t.

The numerical value of h_t is

$$h_t = \begin{cases} \sqrt{2d\sigma^2 \log t}, & \text{Gaussian case,} \\ \frac{d\sigma^2 \log t}{\log\log t}(1+o(1)), & \text{Poisson case,} \end{cases} \qquad t \to \infty. \tag{1.21}$$

In the Poisson case this approximation is too rough for the second-order asymptotics. One cannot expect to find an explicit expression replacing h_t in (1.20). This asymptotics does not only depend on σ^2 and Σ, but is sensitive to changes of the shape of the cloud B in $\{x\colon B(x) > \sigma^2/2\}$. Namely, it is easy to see that the last remark applies to the moment asymptotics (1.17) which is crucial for our derivation of the upper bound in (1.20).

1.6. Relation to intermittency

Let us explain some aspects of the philosophy underlying our result. The actual proof given below reflects these heuristics only marginally (see the outline in Subsection 1.8).

The almost sure asymptotics of $u(t,0)$ is closely related to the intermittent behavior of the parabolic Anderson model. Intermittency means that, as $t \to \infty$, the solution exhibits a spatially extremely irregular structure consisting of islands of high peaks which are located far from each other. A simple manifestation of intermittency is that the first moment $\langle u(t,0)\rangle$ grows much faster than the realizations $u(t,0)$ as $t \to \infty$, compare (1.17) with (1.20).

In our context, the key observation is that, by duality,

$$u(t,0) = \int_{\mathbb{R}^d} v(t,x)\,dx, \tag{1.22}$$

where v is the solution of problem (1.1) but with localized initial condition $v(0,x) = \delta_0(x)$ instead of $u(0,x) \equiv 1$. Hence, we in fact study the 'total mass' of the solution $v(t,\cdot)$ as $t \to \infty$. It may be seen, e.g. by use of the Feynman-Kac formula, that this mass is essentially contained in a centered cube Q_t of side length of order roughly t:

$$u(t,0) \approx \int_{Q_t} v(t,x)\,dx. \tag{1.23}$$

But the main contribution to the last integral is believed to come from spatially sparsely distributed high peaks of the solution $v(t,\cdot)$ which are generated locally by corresponding high peaks of the potential $\xi(\cdot)$. One expects that their height and their shape (but not their location) are essentially nonrandom and that their number grows moderately as $t \to \infty$. The logarithmic asymptotics (up to second order) of $u(t,0)$ is therefore fully determined by the mass of one of these local peaks of $v(t,\cdot)$.

A Borel-Cantelli argument shows that, almost surely as $t \to \infty$, the highest peaks of ξ in Q_t have height of order h_t given by (1.19), see also (1.21). As we shall explain later for the Gaussian case, these peaks have approximately a nonrandom parabolic shape

$$V_{t,x_0}(x) = h_t p(x - x_0), \text{ where } p(x) = 1 - \frac{1}{2\sigma^2} |\Sigma x|^2 \text{ and } \Sigma^2 = -B''(0), \quad (1.24)$$

in a microbox of radius $O(h_t^{-1/4})$ around their random center $x_0 \in Q_t$ and $V_{t,x_0}(x) \approx 0$ further away from x_0. Hence, ignoring other peaks of ξ outside this microbox, we may replace locally $v(t,\cdot)$ by $w(t,\cdot)$, where $w(s,x)$ solves

$$\begin{aligned} \partial_s w &= \kappa\Delta w + V_{t,x_0} w, && (s,x) \in (0,\infty) \times \mathbb{R}^d, \\ w(0,x) &= \delta_0(x), && x \in \mathbb{R}^d. \end{aligned} \quad (1.25)$$

A small but sufficient part of the unit mass imposed at the origin at time 0 reaches the microbox around x_0 until time $t/\log^2 t$. From then on, the local mass creation in this microbox dominates the solution, which therefore may be approximated by the first term in the Fourier expansion associated with the harmonic oscillator $\kappa\Delta + h_t p$:

$$w(s,x) \approx e^{\lambda_t s} \mathrm{e}_t(x - x_0), \qquad s \gg t/\log^2 t, \quad (1.26)$$

where λ_t and e_t denote the principal eigenvalue and the corresponding normalized (Gaussian) eigenfunction, respectively. In particular,

$$\lambda_t = h_t - \chi\sqrt{h_t} \quad (1.27)$$

with χ given by (1.18). Thus, we finally arrive at

$$\frac{1}{t} \log \int w(t,x)\, dx \approx h_t - \chi\sqrt{h_t} \quad (1.28)$$

in accordance with Theorem 1.1.

Let us now explain why in the Gaussian case the potential peaks have the parabolic shape (1.24) and why the second derivative of the covariance function B, but not the fourth derivative (which describes the covariances of the field ξ'') enters this formula. For simplicity we restrict ourselves to the one-dimensional case and assume that the realizations of ξ are sufficiently smooth.

Fix $x_0 \in \mathbb{R}$. One easily checks that $\xi(x_0)$, $\xi'(x_0)$ and $\eta(x_0) = \xi''(x_0) - B''(0)\xi(x_0)/\sigma^2$ are independent Gaussian variables. In particular, $\xi(x_0)$ and $\xi''(x_0)$ are highly correlated, and large values of $\xi(x_0)$ enforce large values of $-\xi''(x_0)$. More precisely, given that ξ has a large local maximum $\xi(x_0) = h_t$ at x_0, we have $\xi'(x_0) = 0$, $|\eta(x_0)| \ll h_t$, and

$$\begin{aligned} \xi(x) &\approx \xi(x_0) + \frac{1}{2}\xi''(x_0)(x - x_0)^2 \\ &= \xi(x_0) + \frac{1}{2}\left(\frac{B''(0)}{\sigma^2}\xi(x_0) + \eta(x_0)\right)(x - x_0)^2 \\ &\approx h_t p(x - x_0) \end{aligned} \quad (1.29)$$

in a neighborhood of x_0.

1.7. Bibliographical remarks

For a general discussion of intermittency and related topics we refer to Carmona and Molchanov [2], the lectures by Molchanov [7], and also to the monograph by Sznitman [8] in which 'negative' Poisson clouds are treated thoroughly.

For Gaussian and Poisson fields, rough logarithmic asymptotics for the moments and the almost sure behavior of $u(t,0)$ have been derived by Carmona and Molchanov [3]. In [5], Gärtner and König investigate the second-order term of the moment asymptotics for a larger class of potentials.

For the *spatially discrete* Anderson model with i.i.d. potential $\xi = \{\xi(x); x \in \mathbb{Z}^d\}$, the second-order almost sure asymptotics of $u(t,0)$ and the second-order moment asymptotics $\langle u(t,0)^p \rangle$ with $p \in \mathbb{N}$ have been investigated by Gärtner and Molchanov [6]. The field ξ under investigation there (the so-called double-exponential distribution) does not lead to a spatial scaling (in contrast to the present paper); the main contribution comes from peaks of fixed finite size. The correlation structure of the solution has been studied by Gärtner and den Hollander [4].

1.8. Outline of the paper

In *Section 2* we provide some technical preparations for the proof of Theorem 1.1. In particular, we explain how we shall approximate ξ by potentials with finite radii of correlation, and we put down some notation about initial-boundary value problems and the Feynman-Kac formula. In *Section 3* and *Section 4* we derive the lower and upper bound for the second-order asymptotics (1.20), respectively.

A short outline of our strategy is the following. In Section 3 we indeed show that ξ is bounded below by the parabola $h_t p$ locally in a certain randomly located microbox of size $h_t^{-1/4}$, where as before $p(x) = 1 - |\Sigma x|^2/(2\sigma^2)$ is a second-order approximation of $B/B(0)$. This enables us to bound $t^{-1} \log u(t,0)$ from below by the (non-random) principal eigenvalue $\lambda^{h_t p}(\mathbb{R}^d) = h_t - \chi\sqrt{h_t}$ of the harmonic oscillator $\kappa\Delta + h_t p$ on $\mathbb{R}^d$ which implies the lower bound in Theorem 1.1.

In our proof of the upper bound, we essentially bound $t^{-1} \log u(t,0)$ from above by the principal Dirichlet eigenvalue $\lambda^\xi(Q_t)$ of the random operator $\kappa\Delta + \xi$ in the macrobox $Q_t = [-t,t]^d$. Using a crucial result from GK, we further estimate this eigenvalue from above in terms of the largest Dirichlet eigenvalue of $\kappa\Delta + \xi$ in microboxes of size $h_t^{-1/4}$ that are contained in the box Q_t. Using the moment asymptotics (1.17) from GK, a Borel-Cantelli argument shows that this maximum has indeed the asymptotics that are given on the r.h.s. of (1.20), and this implies the upper bound.

2. Preparations

2.1. Gaussian remainder term

In order to obtain lower bounds for $u(t,0)$ in Section 3 by applying the first Borel-Cantelli lemma, we need to cut the Gaussian field to finite correlation length. We are going to explain how we will do that.

Note that Assumption (1.2) ensures that g in (1.5) has generalized derivatives of first order which belong to $L^2(\mathbb{R}^d)$. As a consequence, differentiating (1.3) twice and using Parseval's identity, one finds that

$$\partial^2_{kl} B(x) = \int_{\mathbb{R}^d} \partial_k g(x-y)\, \partial_l g(y)\, dy, \qquad k,l = 1,\dots,d. \tag{2.1}$$

Let $\psi\colon \mathbb{R}^d \to [0,1]$ be a smooth symmetric function such that $\psi = 1$ on the cube $Q_{1/2}$ and $\psi = 0$ on $Q_1^{\rm c}$. Given $R>0$, we split g into $g = g_R + \widetilde{g}_R$, where

$$g_R(x) = \psi\left(\frac{x}{R}\right) g(x), \qquad x \in \mathbb{R}^d. \tag{2.2}$$

Then g_R and $\widetilde{g}_R$ both belong to $L^2(\mathbb{R}^d)$ and have generalized first order derivatives also belonging to $L^2(\mathbb{R}^d)$.

Given a standard Brownian sheet W on $\mathbb{R}^d$, we may assume that

$$\xi(x) = \int_{\mathbb{R}^d} g(x-y)\, W(dy) \qquad \text{a.s. for each } x \in \mathbb{R}^d. \tag{2.3}$$

Then we may split ξ into $\xi = \xi_R + \widetilde{\xi}_R$ where, correspondingly,

$$\xi_R(x) = \int_{\mathbb{R}^d} g_R(x-y)\, W(dy) \qquad \text{and} \qquad \widetilde{\xi}_R(x) = \int_{\mathbb{R}^d} \widetilde{g}_R(x-y)\, W(dy). \tag{2.4}$$

Both ξ_R and $\widetilde{\xi}_R$ are homogeneous centered Gaussian fields with covariance function

$$B_R(x) = \int_{\mathbb{R}^d} g_R(x-y) g_R(y)\, dy \quad \text{and} \quad \widetilde{B}_R(x) = \int_{\mathbb{R}^d} \widetilde{g}_R(x-y)\widetilde{g}_R(y)\, dy, \tag{2.5}$$

respectively. In particular, ξ_R has a finite correlation radius of order R.

Moreover, using their spectral representations and Parseval's identity, one easily checks that B_R and $\widetilde{B}_R$ are twice continuously differentiable and

$$\partial^2_{kl} \widetilde{B}_R(x) = \int_{\mathbb{R}^d} \partial_k \widetilde{g}_R(x-y)\, \partial_l \widetilde{g}_R(y)\, dy, \tag{2.6}$$

cf. (2.1). In particular, we may and shall assume that ξ_R and $\widetilde{\xi}_R$ are Hölder continuous.

Note that, because of Assumption (1.6),

$$\widetilde{\sigma}_R^2 = \langle \widetilde{\xi}_R(0)^2 \rangle = \int_{\mathbb{R}^d} \widetilde{g}_R^2(y)\, dy \le \int_{Q^{\rm c}_{R/2}} g^2(y)\, dy = o\left((\log R)^{-2/3}\right). \tag{2.7}$$

Lemma 2.1. *For any $K > 0$, there exists $R_0 = R_0(K) > 0$ such that*

$$\operatorname{Prob}\left(\max_{x\in Q_1} |\widetilde{\xi}_R(x)| > \alpha\right) \le \exp\{-K\alpha^2(\log R)^{2/3}\} \tag{2.8}$$

for all $\alpha > 1$ and $R > R_0$.

Proof. Recall that our fields $\widetilde{\xi}_R$ have continuous sample paths. According to Borell's inequality (see e.g. Adler [1], Theorem 2.1), $\langle \max_{Q_1} \widetilde{\xi}_R\rangle$ is finite, and for any $\alpha > \langle \max_{Q_1} \widetilde{\xi}_R\rangle$ we have

$$\operatorname{Prob}\left(\max_{x\in Q_1} |\widetilde{\xi}_R(x)| \ge \alpha\right) \le 4\exp\left\{-\frac{1}{2\widetilde{\sigma}_R^2}\left(\alpha - \left\langle \max_{Q_1} \widetilde{\xi}_R\right\rangle\right)^2\right\} \tag{2.9}$$

In view of (2.7), our assertion will follow if we prove that

$$\lim_{R\to\infty} \left\langle \max_{Q_1} \widetilde{\xi}_R \right\rangle = 0. \tag{2.10}$$

But, according to Adler [1], Corollary 4.15, one has the entropy bound

$$\left\langle \max_{Q_1} \widetilde{\xi}_R \right\rangle \le C \int_0^\infty \left(\log \widetilde{N}_R(\varepsilon)\right)^{1/2} d\varepsilon, \tag{2.11}$$

where C is a positive constant that does not depend on R, and $\widetilde{N}_R(\varepsilon)$ is the smallest number of ε-balls covering the cube Q_1 in the pseudo-metric

$$d_R(x,y) = \left\langle \left|\widetilde{\xi}_R(x) - \widetilde{\xi}_R(y)\right|^2 \right\rangle^{1/2} = \sqrt{2\left(\widetilde{B}_R(0) - \widetilde{B}_R(x-y)\right)} \le \beta_R|x-y| \tag{2.12}$$

with $\beta_R = \max_{x\in Q_1} |\widetilde{B}''_R(x)|^{1/2}$.

By comparison with the Euclidean metric, one finds that $1 \le \widetilde{N}_R(\varepsilon) \le 1 \vee (2\sqrt{d}\beta_R/\varepsilon)^d$ for all $\varepsilon, R > 0$. Therefore the r.h.s. of (2.11) is not bigger than $O(\beta_R)$ as $R \to \infty$.

We derive from (2.6) that, for all x, R and $k, l = 1, \dots, d$,

$$\left|\partial_{kl}^2 \widetilde{B}_R(x)\right| \le \left(\int_{Q_{R/2}^c} (\partial_k \widetilde{g}_R(y))^2\, dy\right)^{1/2} \left(\int_{Q_{R/2}^c} (\partial_l \widetilde{g}_R(y))^2\, dy\right)^{1/2}. \tag{2.13}$$

But

$$\partial_k \widetilde{g}_R(y) = \left(1 - \psi\left(\frac{y}{R}\right)\right) \partial_k g(y) - \frac{1}{R}\partial_k \psi\left(\frac{y}{R}\right) g(y) \tag{2.14}$$

has an L^2-majorant that is independent of $R > 1$. Hence the r.h.s. of (2.13) vanishes as $R \to \infty$. This implies that $\lim_{R\to\infty} \beta_R = 0$, and we arrive at (2.10) which ends the proof. □

Corollary 2.2. *Fix $\gamma > 0$ and define $R_n = e^{\gamma n^{3/4}}$ for $n \in \mathbb{N}$. Then, with probability one,*

$$\max_{Q_{2^n}} |\widetilde{\xi}_{R_n}| \leq o\left(n^{1/4}\right), \qquad n \to \infty. \tag{2.15}$$

Proof. Fix $\delta > 0$ arbitrarily and estimate, for any $n \in \mathbb{N}$,

$$\text{Prob}\left(\max_{Q_{2^n}} |\widetilde{\xi}_{R_n}| > \delta n^{1/4}\right) \leq 2^{nd}\,\text{Prob}\left(\max_{Q_1} |\widetilde{\xi}_{R_n}| > \delta n^{1/4}\right). \tag{2.16}$$

Now apply Lemma 2.1 with $R = R_n$, $\alpha = \delta n^{1/4}$, and sufficiently large $K > 0$ to see that the r.h.s. of (2.16) is summable over $n \in \mathbb{N}$. Hence the first Borel-Cantelli lemma implies our assertion. □

Lemma 2.3. *With probability one, as $R \to \infty$,*

$$\max_{x \in Q_R} |\xi(x)| \leq \sqrt{2d\sigma^2 \log R}(1 + o(1)). \tag{2.17}$$

Proof. The proof is well-known and sketched for completeness only. Recall that our field ξ has continuous paths. Therefore, Borell's inequality (Adler [1], Theorem 2.1) implies that $\langle \max_{Q_1} \xi \rangle$ is finite, and for every $\alpha > \langle \max_{Q_1} \xi \rangle$ we have the inequality

$$\text{Prob}\left(\max_{Q_1} |\xi| > \alpha\right) \leq 4 \exp\left\{-\frac{1}{2\sigma^2}\left(\alpha - \left\langle \max_{Q_1} \xi \right\rangle\right)^2\right\}. \tag{2.18}$$

Now (2.17) follows from a standard application of the first Borel-Cantelli lemma for the sequence of cubes Q_{2^n}. □

2.2. Poisson remainder term

We explain how we will approximate the shot noise Poisson field (1.7) by fields having finite correlation radii. This subsection is superfluous if B is assumed to have compact support.

As in the preceding subsection, let $\psi\colon \mathbb{R}^d \to [0,1]$ be a smooth symmetric function such that $\psi = 1$ on $Q_{1/2}$ and $\psi = 0$ on Q_1^{c}. Split B into $B = B_R + \widetilde{B}_R$, where

$$B_R(x) = \psi\left(\frac{x}{R}\right) B(x), \qquad x \in \mathbb{R}^d. \tag{2.19}$$

Furthermore, split the potential into $\xi = \xi_R + \widetilde{\xi}_R$, where

$$\xi_R(x) = \int_{\mathbb{R}^d} B_R(x - y)\, \Phi(dy), \qquad x \in \mathbb{R}^d. \tag{2.20}$$

Note that the field ξ_R has finite correlation radius of order R. Define

$$\widetilde{\sigma}_R^2 = \max |\widetilde{B}_R|. \tag{2.21}$$

Recall (1.8) to see that

$$\widetilde{\sigma}_R^2 = o\left((\log R)^{-1}\right), \qquad R \to \infty. \tag{2.22}$$

We are going to derive a bound on the a.s. growth of $\widetilde{\xi}_R$ for large R.

Lemma 2.4. *For each $K > 0$, there is an $R_0 = R_0(K) > 0$ such that*

$$\text{Prob}\left(\max_{Q_1} |\widetilde{\xi}_R| > \alpha\right) \leq \exp\{-K\alpha \log\alpha \log R\} \tag{2.23}$$

for all $\alpha > 1$ and $R > R_0$.

Proof. Define $\widehat{B}_R(y) = \max_{x\in Q_1} |\widetilde{B}_R(x-y)|$ for $y \in \mathbb{R}^d$. Then, for all $\beta > 0$, we estimate

$$\begin{aligned}
\text{Prob}\left(\max_{Q_1} |\widetilde{\xi}_R| > \alpha\right) &\leq \text{Prob}\left(\int \widehat{B}_R(y)\, \Phi(dy) > \alpha\right) \\
&\leq e^{-\alpha\beta} \left\langle \exp\left\{\beta \int \widehat{B}_R(y)\, \Phi(dy)\right\}\right\rangle \\
&\leq e^{-\alpha\beta} \exp\left\{\lambda \int (e^{\beta \widehat{B}_R(y)} - 1)\, dy\right\} \\
&= e^{-\alpha\beta} \exp\left\{\lambda \int_0^\beta d\gamma \int e^{\gamma \widehat{B}_R(y)} \widehat{B}_R(y)\, dy\right\} \\
&\leq e^{-\alpha\beta} \exp\left\{\lambda\beta e^{\beta\widetilde{\sigma}_R^2} \int \widehat{B}_R(y)\, dy\right\}.
\end{aligned} \tag{2.24}$$

Now choosing $\beta = \widetilde{\sigma}_R^{-2} \log\alpha$, we obtain that the last line of (2.24) equals

$$\exp\left\{-\frac{\alpha\log\alpha}{\widetilde{\sigma}_R^2}\left(1 - \lambda \int \widehat{B}_R(y)\, dy\right)\right\}. \tag{2.25}$$

Because of (1.9) and (2.19), the last integral vanishes as $R \to \infty$, and our assertion follows from (2.22). □

Corollary 2.5. *Fix $\gamma > 0$ and define $R_n = \exp\{\gamma\sqrt{n \log n}\}$ for $n \in \mathbb{N}$. Then, with probability one,*

$$\max_{Q_{2^n}} |\widetilde{\xi}_{R_n}| \leq o\left(\sqrt{\frac{n}{\log n}}\right), \qquad n \to \infty. \tag{2.26}$$

Proof. Analogous to the proof of Corollary 2.2. □

Lemma 2.6. *With probability one,*

$$\max_{Q_R} |\xi| \leq \frac{d\widehat{\sigma}^2 \log R}{\log\log R}(1 + o(1)), \qquad R \to \infty, \tag{2.27}$$

where $\widehat{\sigma}^2 = \max |B|$.

Proof. Use that $\max_{Q_1} |\xi| \leq \int \widehat{B}(y)\, \Phi(dy)$ where $\widehat{B}(y) = \max_{x\in Q_1} |B(x-y)|$. Now proceed as in (2.24) with $\widetilde{\xi}_R$ replaced by ξ, $\widehat{B}_R$ replaced by $\widehat{B}$ and $\widetilde{\sigma}^2_R$ by $\widehat{\sigma}^2$. Then choose $\beta = \widehat{\sigma}^{-2} \log \frac{\alpha}{\log \alpha}$ to obtain

$$\operatorname{Prob}\left(\max_{Q_1} |\xi| > \alpha\right) \leq \exp\left\{-\frac{\alpha \log \alpha}{\widehat{\sigma}^2}(1+o(1))\right\}, \qquad \alpha \to \infty. \tag{2.28}$$

Now the assertion follows from a standard application of the first Borel-Cantelli lemma for the sequence of cubes Q_{2^n}. $\square$

Let us finally remark that high peaks of the field ξ correspond to areas of high concentration of the underlying Poisson point process Φ. In particular, formulas (2.23), (2.27), and (2.28) can be interpreted in terms of large deviations for Poisson random variables.

2.3. Initial-boundary value problem

We are going to introduce the solution to the parabolic problem (1.1) in a finite box with zero boundary condition, for general potential. Recall that $Q_r = (-r, r)^d$.

Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be an arbitrary Hölder continuous potential. For $r > 0$, let u_r^V be the solution of the initial-boundary value problem for the operator $\kappa\Delta + V$ in the box Q_r with zero boundary condition and initial datum identically equal to one, i.e.,

$$\begin{aligned} \partial_t u_r^V(t,x) &= \kappa\Delta u_r^V(t,x) + V(x) u_r^V(t,x), && (t,x) \in (0,\infty) \times Q_r,\\ u_r^V(0,x) &= 1, && x \in Q_r,\\ u_r^V(t,x) &= 0, && (t,x) \in (0,\infty) \times \partial Q_r. \end{aligned} \tag{2.29}$$

We trivially extend u_r^V to a function on $[0,\infty) \times \mathbb{R}^d$ vanishing outside of $[0,\infty) \times Q_r$. In the sequel, in order to stress the dependence on the potential, we shall write u^ξ instead of u for the solution of (1.1). Note that a.s. for $0 < r < R$ (picking $V = \xi$),

$$u_r^\xi \leq u_R^\xi \leq u^\xi, \qquad \text{on } (0,\infty) \times Q_r. \tag{2.30}$$

Let $\lambda_1 > \lambda_2 \geq \lambda_3 \geq \dots$ be the eigenvalues of the operator $\kappa\Delta + V$ with zero boundary condition in $L^2(Q_r)$. We also write $\lambda_k = \lambda_k^V(Q_r)$ for the k-th eigenvalue to emphasize its dependence on the potential V and the box Q_r. Let $(\mathrm{e}_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(Q_r)$ consisting of corresponding eigenfunctions $\mathrm{e}_k = \mathrm{e}_k^V(Q_r)$.

We also need the fundamental solution $p_r^V(t,\cdot,y)$ of our initial-boundary value problem, i.e., the solution of (2.29) with the initial condition $u_r^V(0,\cdot) = \mathbb{1}$ replaced by $p_r^V(0,\cdot,y) = \delta_y(\cdot)$ for each $y \in Q_r$. We have the Fourier expansion

$$p_r^V(t,x,y) = \sum_{k=1}^{\infty} e^{t\lambda_k} \mathrm{e}_k(x)\, \mathrm{e}_k(y). \tag{2.31}$$

According to Mercer's theorem, this series converges uniformly in $x, y \in Q_r$. In particular, we also have the Fourier expansion

$$u_r^V(t,\cdot) = \sum_{k=1}^{\infty} e^{t\lambda_k} (\mathrm{e}_k, \mathbb{1})_r \mathrm{e}_k(\cdot), \tag{2.32}$$

where we write $(\cdot,\cdot)_r$ for the inner product in $L^2(Q_r)$. Using this, one obtains the estimate

$$\begin{aligned} \big(u_r^V(t,\cdot), \mathbb{1}\big)_r &= \sum_{k=1}^{\infty} e^{t\lambda_k} (\mathrm{e}_k, \mathbb{1})_r^2 \\ &\le e^{t\lambda_1} \sum_{k=1}^{\infty} (\mathrm{e}_k, \mathbb{1})_r^2 = e^{t\lambda_1^V(Q_r)} \|\mathbb{1}\|_r^2 = e^{t\lambda_1^V(Q_r)} (2r)^d, \end{aligned} \tag{2.33}$$

where $\|\cdot\|_r$ is the norm in $L^2(Q_r)$.

2.4. Feynman-Kac formula

We are going to express the solutions u^ξ of (1.1) and u_r^V of the initial-boundary value problem (2.29) in terms of the Feynman-Kac formula. Still we assume that $V\colon \mathbb{R}^d \to \mathbb{R}$ is some Hölder continuous potential.

Let $\{W_t\}_{t\ge 0}$ be Brownian motion in $\mathbb{R}^d$ with generator $\kappa\Delta$. Denote the underlying probability and expectation by $\mathbb{P}_x$ resp. $\mathbb{E}_x$ when $W_0 = x \in \mathbb{R}^d$. Then we have the Feynman-Kac formula

$$u^\xi(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t \xi(W_s)\, ds \right\}, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d. \tag{2.34}$$

In order to represent the solution u_r^V of (2.29) in terms of Brownian motion we introduce the stopping time of the first exit from Q_r:

$$\tau_r = \inf\{t \ge 0\colon W_t \notin Q_r\}. \tag{2.35}$$

Then, for all $r > 0$ and $(t,x) \in (0,\infty) \times \mathbb{R}^d$, we have

$$u_r^V(t,x) = \mathbb{E}_x \exp\left\{ \int_0^t V(W_s)\, ds \right\} \mathbb{1}\{\tau_r > t\}. \tag{2.36}$$

The analogous Feynman-Kac representation for the fundamental solution $p_r^V(t,x,y)$ is in terms of Brownian bridge instead of free Brownian motion:

$$p_r^V(t,x,y) = \mathbb{E}_x \exp\left\{ \int_0^t V(W_s)\, ds \right\} \mathbb{1}\{\tau_r > t\}\, \delta_y(W_t). \tag{2.37}$$

3. The lower bound

In this section we derive the lower bound in (1.20). Our key observation is that, asymptotically as $t \to \infty$, the following holds true with probability one. In the centered macrobox of length $2t/\log t$ one finds a microbox having side length of order $h_t^{-1/4}$ in which the potential ξ is uniformly close to the (correspondingly shifted) function $h_t B/\sigma^2$. Since the microbox is sufficiently small, $B(x)/\sigma^2$ can be replaced by its parabolic approximation $p(x) = 1 - |\Sigma x|^2/(2\sigma^2)$. Then, within the Feynman-Kac formula (2.34), we first force Brownian motion to reach the mentioned microbox in a short proportion of time and then to stay in that box until time t. It will turn out that the fast motion to the microbox is asymptotically negligible. But, after replacing ξ by the shifted parabola $h_t p$, the Feynman-Kac expression in the microbox corresponds to the harmonic oscillator and can easily be calculated. This yields directly the desired asymptotics.

In Subsection 3.1, after introducing some notation, we will start to exploit the Feynman-Kac formula. The details for the proof of existence of a microbox with the mentioned properties will be carried out afterwards in Subsection 3.2.

3.1. Proof of the lower bound

Let $t > 2$ and choose $n = n(t) \in \mathbb{N}$ so that $2^n \le t < 2^{n+1}$; thus $n = \lfloor \log t/\log 2 \rfloor$. Recall that $Q_R = (-R,R)^d$ and that $\varrho(h)$ and h_t are defined by (1.14) and (1.19), respectively.

We fix two parameters $r > 0$ and $\gamma > 0$ arbitrarily. At the very end of the proof we then let $r \to \infty$ and $\gamma \downarrow 0$. Abbreviate

$$R_n = \exp\Big\{\gamma\sqrt{h_{2^n}}\,\varrho(h_{2^n})\Big\}. \tag{3.1}$$

This is the correlation length of the fields ξ_{R_n} by which we will approximate ξ. We will see from (3.14) below that this R_n has the same order of magnitude as the R_n's in Corollaries 2.2 resp. 2.5.

Besides of the 'macrobox' $Q_{2^n/n}$, we also need 'microboxes'

$$Q_{n,z} = z + Q_{rh_{2^n}^{-1/4}} \qquad \text{and} \qquad \widehat{Q}_{n,z} = z + 2Q_{rh_{2^n}^{-1/4}} \tag{3.2}$$

for $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$. Thus, in the macrobox of size $2^n/n$, we have placed about $[2^n/(4R_n n)]^d$ microboxes of size $rh_{2^n}^{-1/4}$ having distance of at least R_n to each other. In (3.14) below it will turn out that the number of our microboxes grows exponentially in n.

We shall use the Feynman-Kac representation and obtain a lower bound by requiring that Brownian motion stays all the time until t in the macrobox Q_{2^n}, reaches soon a suited microbox $Q_{n,z}$, and then stays in its neighborhood $\widehat{Q}_{n,z}$ for the rest of the time until t. In order to formalize this, we introduce

$$\delta_n = \frac{2^n}{n^2}. \tag{3.3}$$

This is the time we give to the Brownian motion to reach a suited microbox $Q_{n,z}$. The stopping time

$$\tau_{n,z} = \inf\{s \geq \delta_n : W_s \notin \widehat{Q}_{n,z}\} \qquad (3.4)$$

is the first exit time from $\widehat{Q}_{n,z}$ after time δ_n. Recall that τ_R denotes the first exit time from Q_R.

Now, applying the Feynman-Kac formula (2.34), we proceed as follows. For every $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$, we have

$$u^\xi(t,0) \geq \mathbb{E}_0\, e^{\int_0^t \xi(W_s)\,ds} \mathbb{1}\{\tau_{2^n} > \delta_n, W_{\delta_n} \in Q_{n,z}\} \mathbb{1}\{\tau_{n,z} > t\}$$

$$\geq e^{-\delta_n \max_{Q_{2^n}} |\xi|}\, \mathbb{P}_0(\tau_{2^n} > \delta_n, W_{\delta_n} \in Q_{n,z}) \inf_{x \in Q_{n,z}} u^\xi_{n,z}(t-\delta_n, x), \qquad (3.5)$$

where, in the last step, we have used the Markov property at time δ_n. We write $u^\xi_{n,z}$ for the solution of (2.29) in $\widehat{Q}_{n,z}$ rather than in some centered box.

With probability one, the first factor on the r.h.s. of (3.5) does not decay exponentially fast toward zero as $t \to \infty$; see Lemma 2.3 resp. 2.6.

The second factor is estimated from below as follows. For large $n \in \mathbb{N}$ resp. t and every $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$,

$$\begin{aligned}
&\mathbb{P}_0(\tau_{2^n} > \delta_n, W_{\delta_n} \in Q_{n,z}) \\
&\quad \geq \mathbb{P}_0(W_{\delta_n} \in Q_{n,z}) - \mathbb{P}_0(\tau_{2^n} \leq \delta_n) \\
&\quad \geq e^{-o(2^n)} \exp\left\{-\frac{d}{4\kappa\delta_n}\left(\frac{2^n}{n}\right)^2\right\} - \exp\left\{-\frac{1}{4\kappa\delta_n}2^{2n}\right\},
\end{aligned} \qquad (3.6)$$

where, in the third line, we have used the reflection principle. Observe that $\delta_n^{-1}(2^n/n)^2 = 2^n \sim t$. This proves that the second factor on the r.h.s. of (3.5) is of order $e^{-O(t)}$ as $t \to \infty$.

In the following lemma, we give the appropriate lower bound on the last factor in (3.5), which is the crucial one. Recall (3.1) and $2^n \leq t < 2^{n+1}$.

Lemma 3.1. *With probability one, for sufficiently large t resp. $n = n(t)$, there is a $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$ such that, as $t \to \infty$,*

$$\frac{1}{t}\log\left\{\inf_{x \in Q_{n,z}} u^\xi_{n,z}(t-\delta_n, x)\right\} \geq h_t - (\chi + c(r,\gamma) + o(1))\sqrt{h_t}, \qquad (3.7)$$

where $c(r,\gamma)$ is a positive number that vanishes as $r \to \infty$ and $\gamma \to 0$. (Note that the l.h.s. of (3.7) depends on r and γ via $Q_{n,z}$ resp. R_n.)

From the proof of this lemma we isolate the following lower bound for the cut-off potential ξ_{R_n} which will be proven in the next subsection. At least partially, the following proposition states that, in one of the microboxes that can be reached by Brownian motion by time δ_n without too much effort, the field ξ looks like the parabola $h_{2^n}p$. In order to formulate it, we introduce the second-order approximation of the function $B/B(0)$,

$$p(x) = 1 - \frac{1}{2\sigma^2}|\Sigma x|^2, \qquad x \in \mathbb{R}^d, \tag{3.8}$$

(recall (1.11)) and the numbers

$$\widehat{h}_n = h_{2^n} - d\gamma\sqrt{h_{2^n}}. \tag{3.9}$$

Recall the definitions of ξ_R in (2.4) resp. (2.20).

Proposition 3.2. *With probability one, for sufficiently large $n \in \mathbb{N}$, there is a $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$ such that*

$$\xi_{R_n}(x) \geq \widehat{h}_n p(x - z), \qquad x \in \widehat{Q}_{n,z}. \tag{3.10}$$

Proof. See Subsection 3.2. □

Before proving Lemma 3.1, let us make some technical remarks about the asymptotic behavior of h_t as $t \to \infty$ which will be used frequently. It is clear from (1.19), $L' = \varrho$, and the monotonicity of ϱ and $h.$ that

$$d[\log t - \log \vartheta(t)] \geq (h_t - h_{\vartheta(t)})\varrho(h_{\vartheta(t)}) \tag{3.11}$$

for any $\vartheta(t) \in (0, t)$. It may be seen from (1.16) and (1.21) that $\varrho(h_t)$ is bounded from below by const $\times \log\log t$ asymptotically as $t \to \infty$. Hence,

$$h_t - h_{\vartheta(t)} \to 0 \text{ if } \frac{t}{\vartheta(t)} \text{ stays bounded} \tag{3.12}$$

and

$$h_t - h_{\vartheta(t)} \text{ remains bounded if } \frac{t}{\vartheta(t)} = O(\log t) \tag{3.13}$$

as $t \to \infty$. Moreover, from (1.16) and (1.21) we deduce that

$$\sqrt{h_{2^n}}\,\varrho(h_{2^n}) = \frac{1 + o(1)}{\sigma^2} \times \begin{cases} (2d\sigma^2 \log 2)^{3/4} n^{3/4}, & \text{Gaussian case,} \\ \sqrt{d\sigma^2 \log 2}\,\sqrt{n \log n}, & \text{Poisson case,} \end{cases} \qquad n \to \infty. \tag{3.14}$$

Proof of Lemma 3.1. Choose n large and pick z as in Proposition 3.2. It is clear from (3.12) that $h_t - h_{t-\delta_n} \to 0$ as $t \to \infty$. Hence, we may and will prove (3.7) with $u^\xi_{n,x}(t - \delta_n, x)$ replaced by $u^\xi_{n,x}(t, x)$.

Because of (3.14), Corollaries 2.2 resp. 2.5 are applicable for our choice (3.1) of R_n. Together with (1.21) they yield almost surely that $\max_{Q_{2^n}} |\widetilde{\xi}_{R_n}| =$

$o(\sqrt{h_t})$ as $t \to \infty$. Hence, with the help of the Feynman-Kac formula, one estimates

$$\inf_{x\in Q_{n,z}} u^{\xi}_{n,z}(t,x) \geq e^{-to(\sqrt{h_t})} \inf_{x\in Q_{n,z}} u^{\xi_{R_n}}_{n,z}(t,x), \qquad t\to\infty. \tag{3.15}$$

Now use (3.10) to get

$$u^{\xi_{R_n}}_{n,z}(t,x) \geq e^{t\widehat{h}_n} u^{\widehat{h}_n V_z}_{n,z}(t,x), \qquad x \in \widehat{Q}_{n,z}, \tag{3.16}$$

where the potential V_z is given by $V_z(x) = -|\Sigma(x-z)|^2/(2\sigma^2)$. Because of (3.12), $h_t - h_{2^n} \to 0$ as $t \to \infty$ and therefore

$$e^{t\widehat{h}_n} = \exp\left\{t\left(h_t - (d\gamma + o(1))\sqrt{h_t}\right)\right\}, \qquad t\to\infty. \tag{3.17}$$

Apply the Brownian scaling property to the Feynman-Kac representation of the second factor on the r.h.s. of (3.16) to get

$$\begin{aligned}\inf_{x\in Q_{n,z}} u^{\widehat{h}_n V_z}_{n,z}(t,x) &= \inf_{x\in Q_{n,0}} u^{\widehat{h}_n V_0}_{n,0}(t,x)\\ &= \inf_{x\in Q_{r+o(1)}} u^{V_0}_{2Q_{r+o(1)}}\left(t\sqrt{\widehat{h}_n}, x\right)\\ &= \inf_{x\in Q_{r+o(1)}} u^{V_0}_{2Q_{r+o(1)}}\left(t\sqrt{h_t}(1+o(1)), x\right), \qquad t\to\infty.\end{aligned} \tag{3.18}$$

As $t \to \infty$, the r.h.s. of (3.18) is equal to $\exp\{t\sqrt{h_t}(\lambda_r + o(1))\}$, where λ_r is the principal eigenvalue of $\kappa\Delta + V_0$ in the cube $2Q_r$ with zero boundary condition. It is well known that λ_r tends to the principal eigenvalue λ_∞ of the harmonic oscillator $\kappa\Delta + V_0$, and $\lambda_\infty = -\chi$.

Summarizing, we obtain (3.7) with $c(r,\gamma) = \lambda_\infty - \lambda_r + d\gamma$, which ends the proof. □

Proof of Theorem 1.1, lower bound. Combining Lemma 3.1 with (3.5), taking into account the remarks below (3.5), and letting $r \to \infty$ and $\gamma \downarrow 0$, we arrive at '$\geq$' in (1.20). □

3.2. Lower bounds for the field

In this subsection we prove Proposition 3.2. First we make a step toward a lower bound for the probability of the event in (3.10). Recall Subsection 1.4.

Lemma 3.3. *For any $r > 0$ and $\delta > 0$, as $h \to \infty$,*

$$\mathrm{Prob}\left(\max_{x\in Q_{rh^{-1/4}}} |\xi(x) - hp(x)| < \delta\sqrt{h}\right) \geq \exp\left\{-L(h) - \varrho(h)o(\sqrt{h})\right\}. \tag{3.19}$$

Proof. Given $h > 0$, we introduce a transformed probability measure Prob_h by

$$\mathrm{Prob}_h(G) = \left\langle 1\!\!1_G \, e^{\varrho(h)\xi(0)} \right\rangle e^{-H(\varrho(h))}. \tag{3.20}$$

We denote expectation w.r.t. Prob_h by $\langle \cdot \rangle_h$. Then we have

$$\langle \xi(0) \rangle_h = H'(\varrho(h)) = h. \tag{3.21}$$

Choosing $\delta' \in (0, \delta)$ arbitrarily, the probability on the left of (3.19) can be estimated from below by

$$\begin{aligned} &\mathrm{Prob}\left(\max_{Q_{rh^{-1/4}}} |\xi - hp| < \delta'\sqrt{h}\right) \\ &= e^{-L(h)} \left\langle 1\!\!1\left\{\max_{Q_{rh^{-1/4}}} |\xi - hp| < \delta'\sqrt{h}\right\} e^{-\varrho(h)(\xi(0)-h)} \right\rangle_h \\ &\geq e^{-L(h)-\delta'\varrho(h)\sqrt{h}}\, \mathrm{Prob}_h\left(\max_{Q_{rh^{-1/4}}} |\xi - hp| < \delta'\sqrt{h}\right). \end{aligned} \tag{3.22}$$

It remains to show that the probability on the r.h.s. stays bounded away from zero as $h \to \infty$. Since δ' can be chosen arbitrarily small, this then ends the proof. For doing this, we distinguish between the Gaussian and the Poisson case. First we turn to the

Gaussian case. Calculating the Laplace transforms of the finite dimensional distributions of ξ under Prob_h, one finds that, under this measure, ξ is a Gaussian field with mean function $hB(\cdot)/\sigma^2$ and covariance function B. In other words, under Prob_h, the field $\xi - hB/\sigma^2$ has the same distribution as ξ under Prob. Note that $\max_{Q_{rh^{-1/4}}} |hp - hB/\sigma^2| = o(\sqrt{h})$ as $h \to \infty$. Therefore the probability on the r.h.s. of (3.22) tends to one as $h \to \infty$.

Poisson case. Let us first remark some features of Prob_h. Note that

$$\left\langle e^{(\Phi,\varphi)} \right\rangle_h = \exp\left\{ \int \left(e^{\varphi(y)} - 1\right) \lambda e^{\varrho(h)B(-y)}\, dy \right\} \tag{3.23}$$

for suitable test functions φ. This shows that, under Prob_h, Φ is an inhomogeneous Poisson process with intensity measure

$$\mu_h(dy) = \lambda e^{\varrho(h)B(-y)}\, dy. \tag{3.24}$$

Using (1.14) and differentiating (1.12) with respect to ϱ, we find that

$$h = H'(\varrho(h)) = \int B(-y)\, \mu_h(dy). \tag{3.25}$$

Since B attains its unique global maximum at zero and $B(0) = \sigma^2$, an application of the Laplace method combined with (3.25) yields

$$\int f(y)\, \mu_h(dy) = \frac{h}{\sigma^2} f(0) + o(h), \qquad h \to \infty, \tag{3.26}$$

for all Lebesgue-integrable continuous functions $f\colon \mathbb{R}^d \to \mathbb{R}$.

Our first step towards the proof that the probability on the r.h.s. of (3.22) stays bounded away from zero as $h \to \infty$ is to show that we may replace hp by $\langle \xi \rangle_h$ in this assertion. For this, we claim that, uniformly in $x \in Q_{rh^{-1/4}}$,

$$\langle \xi(x) \rangle_h = hp(x) + o\left(\sqrt{h}\right) \qquad \text{as } h \to \infty, \tag{3.27}$$

where, as before, $p(x) = 1 + \frac{1}{2\sigma^2}(x, B''(0)x)$. Clearly,

$$\langle \xi(x) \rangle_h = \int B(x-y)\, \mu_h(dy). \tag{3.28}$$

It is immediate from (3.28) and (3.25) that $\langle \xi(0) \rangle_h = h$. Thus, in order to prove (3.27), it remains to check that, uniformly in $x \in Q_{rh^{-1/4}}$,

$$\int [B(x-y) - B(-y)]\, \mu_h(dy) = \frac{h}{2\sigma^2}(x, B''(0)x) + o\left(\sqrt{h}\right), \qquad h \to \infty. \tag{3.29}$$

To this end, we fix a centered cube Q_L that contains the set $\{y\colon B(-y) > \sigma^2/3\}$. Taking into account (1.16) and (1.9), we obtain

$$\int_{Q_L^c} [B(x-y) - B(-y)]\, \mu_h(dy) = o\left(\sqrt{h}\right), \qquad h \to \infty, \tag{3.30}$$

uniformly in $x \in Q_1$. On the other hand, a Taylor expansion of B yields

$$\begin{aligned}\int_{Q_L} &[B(x-y) - B(-y)]\, \mu_h(dy) \\ &= \int_{Q_L} (x, B'(-y))\, \mu_h(dy) + \int_{Q_L} \int_0^1 (1-\vartheta)(x, B''(\vartheta x - y)x)\, d\vartheta\, \mu_h(dy).\end{aligned} \tag{3.31}$$

Together with the definition of Q_L and (1.16), an explicit integration shows that the first integral on the right is of order $o(\sqrt{h})$, uniformly in $x \in Q_1$. By an application of the Laplace method and a comparison with (3.25), one readily verifies that the last integral on the right behaves like

$$\frac{h}{2\sigma^2}(x, B''(0)x) + o\left(\sqrt{h}\right), \tag{3.32}$$

uniformly in $x \in Q_1$. In this way we arrive at (3.29). We thus have proved (3.27).

It remains to show that, for every $\delta > 0$, the probability

$$\mathrm{Prob}_h\left(\max_{x \in Q_{rh^{-1/4}}} |\xi(x) - \langle \xi(x) \rangle_h| < \delta\sqrt{h}\right) \tag{3.33}$$

stays bounded away from zero as $h \to \infty$. This probability may be estimated from below by

$$\mathrm{Prob}_h\left(|\xi(0)-h|<\frac{\delta}{2}\sqrt{h}\right)-\mathrm{Prob}\left(\max_{Q_{rh^{-1/4}}}|\xi-\xi(0)-\langle\xi-\xi(0)\rangle_h|>\frac{\delta}{2}\sqrt{h}\right). \tag{3.34}$$

We shall finish the proof by showing that (i) under Prob_h, $(\xi(0)-h)/\sqrt{h}$ converges in distribution toward a centered normal variable with variance σ^2, and (ii) the second term in (3.34) actually vanishes as $h\to\infty$.

In order to verify (i), we compute the Laplace transform of $(\xi(0)-h)/\sqrt{h}$ to obtain, for any $\alpha\in\mathbb{R}$,

$$\begin{aligned}\log&\left\langle\exp\left\{\alpha\frac{\xi(0)-h}{\sqrt{h}}\right\}\right\rangle_h\\&=H\left(\varrho(h)+\frac{\alpha}{\sqrt{h}}\right)-H(\varrho(h))-\frac{\alpha}{\sqrt{h}}H'(\varrho(h))\\&=\frac{\alpha^2}{h}\int_0^1(1-\vartheta)H''\left(\varrho(h)+\vartheta\frac{\alpha}{\sqrt{h}}\right)d\vartheta.\end{aligned} \tag{3.35}$$

Here we have used again that $h=H'(\varrho(h))$. Using (1.12), we find that

$$H''\left(\varrho(h)+\vartheta\frac{\alpha}{\sqrt{h}}\right)=\int B^2(-y)e^{\vartheta\frac{\alpha}{\sqrt{h}}B(-y)}\,\mu_h(dy). \tag{3.36}$$

A uniform version of (3.26) yields that the last expression behaves like $\sigma^2 h(1+o(1))$ as $h\to\infty$, uniformly in $\vartheta\in(0,1)$. Hence, the r.h.s. of (3.35) converges to $\sigma^2\alpha^2/2$ as $h\to\infty$.

To prove (ii), we first remark that

$$\xi(x)-\xi(0)-\langle\xi(x)-\xi(0)\rangle_h=\int[B(x-y)-B(-y)]\,(\Phi-\mu_h)(dy). \tag{3.37}$$

Similar as in the proof of (3.27), we split the integral into two parts by separately integrating over Q_L^c and Q_L, respectively. But

$$\begin{aligned}\max_{x\in Q_{rh^{-1/4}}}&\left|\int_{Q_L^c}[B(x-y)-B(-y)]\,(\Phi-\mu_h)(dy)\right|\\&\le 2\int_{Q_L^c}\max_{x\in Q_1}|B(x-y)|\,(\Phi-\mu_h)(dy)+4\int_{Q_L^c}\max_{x\in Q_1}|B(x-y)|\,\mu_h(dy).\end{aligned} \tag{3.38}$$

Using the definition of Q_L, (1.16) and (1.9), one finds that the last integral is of order $o(\sqrt{h})$. But, according to Chebyshev's inequality,

$$\begin{aligned} &\mathrm{Prob}_h\left(\int_{Q_L^c} \max_{x\in Q_1} |B(x-y)|\,(\Phi-\mu_h)(dy) > \delta h\right) \\ &\quad\le \frac{1}{\delta^2 h}\int_{Q_L^c} \max_{x\in Q_1} B^2(x-y)\,\mu_h(dy), \end{aligned} \tag{3.39}$$

and the expression on the right also tends to zero as $h\to\infty$ for every $\delta>0$.

Now we use a Taylor expansion for B to obtain

$$\begin{aligned} &\max_{x\in Q_{rh^{-1/4}}}\left|\int_{Q_L}[B(x-y)-B(-y)]\;(\Phi-\mu_h)(dy)\right| \\ &\le rh^{-1/4}\left|\int_{Q_L} B'(-y)\,(\Phi-\mu_h)(dy)\right| + r^2h^{-1/2}\left|\int_{Q_L} B''(-y)\,(\Phi-\mu_h)(dy)\right| \\ &\quad+\frac{1}{2}r^2h^{-1/2}\int_{Q_L}\max_{x\in Q_1}|B''(x-y)-B''(-y)|\,(\Phi-\mu_h)(dy) \\ &\quad+r^2h^{-1/2}\int_{Q_L}\max_{x\in Q_{rh^{-1/4}}}|B''(x-y)-B''(-y)|\,\mu_h(dy). \end{aligned} \tag{3.40}$$

Similarly to (3.26), one easily checks that the last (non-random) summand is of order $o(\sqrt{h})$. For every $\delta>0$, the probability w.r.t. Prob_h of each of the other summands to exceed δh may be estimated from above by use of Chebyshev's inequality. The asymptotics of the associated variances may then be computed with the help of (3.26). In this way one finds that each of the mentioned probabilities tends to zero as $h\to\infty$, and we are done. □

Now one easily sees that also the cut-off field $\xi_R=\xi-\widetilde{\xi}_R$ satisfies a bound like in the preceding lemma. Recall (2.4) resp. (2.20).

Corollary 3.4. *Fix $r,\gamma,\delta>0$, and define $R(h)=\exp\{(1+o(1))\gamma\sqrt{h}\varrho(h)\}$. Then, as $h\to\infty$,*

$$\mathrm{Prob}\left(\max_{x\in Q_{rh^{-1/4}}}\left|\xi_{R(h)}(x)-hp(x)\right|<\delta\sqrt{h}\right)\ge\exp\left\{-L(h)-\varrho(h)o(\sqrt{h})\right\}. \tag{3.41}$$

Proof. Because of Lemma 3.3 (with δ replaced by $\delta/2$) it is clearly enough to show that

$$\mathrm{Prob}\left(\max_{x\in Q_{rh^{-1/4}}}\left|\widetilde{\xi}_{R(h)}(x)\right|>\frac{\delta}{2}\sqrt{h}\right) \tag{3.42}$$

is asymptotically much smaller than the r.h.s. of (3.19). But, according to our choice of $R(h)$, this follows from Lemma 2.1 resp. 2.4 (in the Gaussian case one must choose K large enough). □

Proof of Proposition 3.2. Put

$$\widetilde{h}_n = \widehat{h}_n + \frac{d}{2}\gamma\sqrt{h_{2^n}} = h_{2^n} - \frac{d}{2}\gamma\sqrt{h_{2^n}} \tag{3.43}$$

and observe that p is close to one in $\widehat{Q}_{n,z}$. Therefore, it will be enough to show that, with probability one, for sufficiently large $n \in \mathbb{N}$, there is a $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$ such that

$$\xi_{R_n}(x) \geq \widetilde{h}_n p(x-z) - \frac{\gamma}{4}\sqrt{h_{2^n}}, \qquad x \in \widehat{Q}_{n,z}. \tag{3.44}$$

According to the first Borel-Cantelli lemma, we only need to check that

$$\text{Prob}\left(\min_{z\in 4R_n\mathbb{Z}^d\cap Q_{2^n/n}} \max_{x\in\widehat{Q}_{n,z}} \left|\xi_{R_n}(x) - \widetilde{h}_n p(x-z)\right| > \frac{\gamma}{4}\sqrt{h_{2^n}}\right) \tag{3.45}$$

is summable over $n \in \mathbb{N}$. Note that the correlation length of ξ_{R_n} is $2R_n$, but neighboring microboxes $\widehat{Q}_{n,z}$ have distance of $4R_n$. Thus, the fields $\{\xi_{R_n}(x); x \in \widehat{Q}_{n,z}\}$ are shifts of i.i.d. fields for $z \in 4R_n\mathbb{Z}^d \cap Q_{2^n/n}$. Therefore, using the inequality $1-\alpha \leq e^{-\alpha}$ and Corollary 3.4, we may bound the probability in (3.45) from above by

$$\begin{aligned}&\exp\left\{-\left(\frac{1}{4R_n}\frac{2^n}{n}\right)^d \text{Prob}\left(\max_{\widehat{Q}_{n,0}}\left|\xi_{R_n} - \widetilde{h}_n p\right| \leq \frac{\gamma}{4}\sqrt{h_{2^n}}\right)\right\}\\ &\quad\leq \exp\left\{-2^{nd}\exp\left\{-\gamma d\sqrt{h_{2^n}}\,\varrho(h_{2^n}) - L(\widetilde{h}_n) - \varrho(\widetilde{h}_n)o\left(\sqrt{\widetilde{h}_n}\right)\right\}\right\}. \end{aligned}\tag{3.46}$$

Here we have also used that $R_n = \exp\left\{(1+o(1))\gamma\sqrt{\widetilde{h}_n}\,\varrho(\widetilde{h}_n)\right\}$ in accordance with the definition of $R(\widetilde{h}_n)$.

Use the monotonicity of $L' = \varrho$ (see (1.16)), (1.19) and (3.43) to see that

$$L(\widetilde{h}_n) \leq L(h_{2^n}) - \frac{d}{2}\gamma\sqrt{h_{2^n}}L'(\widetilde{h}_n) = nd\log 2 - \frac{d}{2}\gamma\sqrt{h_{2^n}}\,\varrho(\widetilde{h}_n). \tag{3.47}$$

Since $\widetilde{h}_n/h_{2^n} \to 1$ and $\varrho(\widetilde{h}_n)/\varrho(h_{2^n}) \to 1$ as $n \to \infty$ (use (1.16)) one hence finds that the last line of (3.46) is not bigger than

$$\exp\left\{-\exp\left\{\sqrt{h_{2^n}}\,\varrho(h_{2^n})\left(\gamma\frac{d}{2} - o(1)\right)\right\}\right\} \tag{3.48}$$

which is summable over $n \in \mathbb{N}$ because of (3.14). □

4. The upper bound

In this section we prove the upper bound in (1.20). A rough outline of the proof is as follows. We estimate $t^{-1}\log u(t,0)$ from above by the (random) principal eigenvalue of $\kappa\Delta + \xi$ in a large (t-dependent) box ('macrobox'). Then we estimate this eigenvalue from above by the maximal eigenvalue in the subboxes ('microboxes') of length 2. The almost sure asymptotics for this maximal one will be derived by use of the Borel-Cantelli lemma. To this end we need to estimate the upper tail of the distribution of each such eigenvalue. This will be done with the help of the moment asymptotics (1.17) taken from GK via the exponential Chebyshev inequality. The details will be carried out in reverse order starting with the investigation of the tail behavior of principal eigenvalues.

4.1. Asymptotics of the principal eigenvalue

We derive the second-order asymptotics for the tails of the distribution of the principal eigenvalue $\lambda^\xi(Q_r)$ of $\kappa\Delta+\xi$ in the cubes $Q_r = (-r,r)^d$ for fixed $r > 0$. Using this and the first Borel-Cantelli lemma, we then also obtain an a.s. upper bound for the maximum of many copies of this random variable.

Lemma 4.1. *For any $r > 0$,*

$$\langle e^{\beta\lambda^\xi(Q_r)}\rangle \le \exp\left\{H(\beta) - (\chi - o(1))\beta\sqrt{H'(\beta)}\right\}, \qquad \beta \to \infty. \tag{4.1}$$

Proof. Let $\lambda^\xi(Q_r) = \lambda_1^\xi(Q_r) > \lambda_2^\xi(Q_r) \ge \lambda_3^\xi(Q_r) \ge \dots$ be the eigenvalues of $\kappa\Delta + \xi$ with zero boundary condition in $L^2(Q_r)$. Use the Fourier expansion (2.31) to get

$$\langle e^{\beta\lambda^\xi(Q_r)}\rangle \le \left\langle \sum_{k=1}^\infty e^{\beta\lambda_k^\xi(Q_r)} \right\rangle = \left\langle \int_{Q_r} p_r^\xi(\beta, x, x)\, dx \right\rangle. \tag{4.2}$$

Before applying the Feynman-Kac formula to the r.h.s. of (4.2), we introduce the normalized occupation time measures of Brownian motion:

$$L_\beta(dx) = \frac{1}{\beta}\int_0^\beta 1\!\!1\{W_s \in dx\}\, ds, \qquad \beta > 0. \tag{4.3}$$

Each L_β is a random element of the set $\mathcal{P}_c = \mathcal{P}_c(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ having compact support. We also need the functionals

$$J_\beta(\mu) = -\frac{1}{\beta}\log\frac{\langle e^{\beta\int \xi(x)\,\mu(dx)}\rangle}{\langle e^{\beta\xi(0)}\rangle}, \qquad \mu \in \mathcal{P}_c. \tag{4.4}$$

As a consequence of Jensen's inequality and Fubini's theorem resp. Hölder's inequality, J_β is nonnegative and concave.

Now we may use the Feynman-Kac representation (2.37) and Fubini's theorem to obtain

$$\begin{aligned}
&\left\langle \int_{Q_r} p_r^\xi(\beta,x,x)\,dx \right\rangle \\
&= \left\langle \int_{Q_r} dx\,\mathbb{E}_x \exp\left\{\beta \int_{\mathbb{R}^d} \xi(x)\,L_\beta(dx)\right\} \mathbb{1}\{\tau_r > \beta\}\delta_x(W_\beta) \right\rangle \\
&= \int_{Q_r} dx\,\mathbb{E}_x \exp\{H(\beta) - \beta J_\beta(L_\beta)\}\,\mathbb{1}\{\tau_r > \beta\}\delta_x(W_\beta).
\end{aligned} \tag{4.5}$$

We estimate the r.h.s. of (4.5) as follows. Pick $\alpha = \alpha(\beta) = H'(\beta)^{-1/2}$. Use that $\mathbb{1}\{\tau_r > \beta\} \le \mathbb{1}\{\tau_r > \beta - \alpha\}$. Observing that L_β is a convex combination of $L_{\beta-\alpha}$ and L_α, time-shifted by $\beta - \alpha$ and using the concavity and nonnegativity of J_β, we get

$$-\beta J_\beta(L_\beta) \le -(\beta-\alpha) J_\beta(L_{\beta-\alpha}). \tag{4.6}$$

Taking this into account and applying the Markov property at time $\beta - \alpha$, we find that the r.h.s. of (4.5) is bounded from above by

$$\int_{Q_r} dx\,\mathbb{E}_x \exp\{H(\beta) - (\beta-\alpha) J_\beta(L_{\beta-\alpha})\}\,\mathbb{1}\{\tau_r > \beta-\alpha\}\mathbb{E}_{W_{\beta-\alpha}}\delta_x(W_\alpha). \tag{4.7}$$

In this expression, we want to neglect the last expectation and replace $H(\beta)$ and J_β by $H(\beta-\alpha)$ and $J_{\beta-\alpha}$, respectively. Clearly

$$\mathbb{E}_{W_{\beta-\alpha}}\delta_x(W_\alpha) \le (4\pi\kappa\alpha)^{-d/2} = e^{o\left(\beta\sqrt{H'(\beta)}\right)} \tag{4.8}$$

and

$$H(\beta) - H(\beta-\alpha) \le \alpha H'(\beta) = o\left(\beta\sqrt{H'(\beta)}\right). \tag{4.9}$$

Moreover, in GK, Subsection 4.1 resp. 4.2, we proved that J_β converges to some finite functional J as $\beta \to \infty$, uniformly on compacts in the weak topology of $\mathcal{P}_{\mathrm{c}}$. In particular,

$$J_\beta(L_{\beta-\alpha}) - J_{\beta-\alpha}(L_{\beta-\alpha}) \to 0 \tag{4.10}$$

uniformly on $\{\tau_r > \beta - \alpha\} = \{\operatorname{supp} L_{\beta-\alpha} \subset Q_r\}$ as $\beta \to \infty$. From the last remarks we conclude that the expression (4.7) does not exceed

$$e^{o\left(\beta\sqrt{H'(\beta)}\right)} \int_{Q_r} dx\,\mathbb{E}_x \exp\{H(\beta-\alpha) - (\beta-\alpha) J_{\beta-\alpha}(L_{\beta-\alpha})\}\,\mathbb{1}\{\tau_r > \beta-\alpha\}. \tag{4.11}$$

But an analog of formula (4.5) shows that the integral in (4.11) equals

$$\left\langle \int_{Q_r} u_r^\xi(\beta-\alpha,x)\,dx \right\rangle \le |Q_r| \left\langle u^\xi(\beta-\alpha,0)\right\rangle. \tag{4.12}$$

Putting together all the above estimates (see (4.2), (4.5), (4.7), (4.11), and (4.12)), we find that

$$\left\langle e^{\beta\lambda^\xi(Q_r)}\right\rangle \le e^{o\left(\beta\sqrt{H'(\beta)}\right)}\left\langle u^\xi(\beta-\alpha,0)\right\rangle. \tag{4.13}$$

Now a combination of this with the moment asymptotics (1.17) yields

$$\begin{aligned}&\left\langle e^{\beta\lambda^\xi(Q_r)}\right\rangle\\ &\quad\le \exp\left\{H(\beta-\alpha)-(\chi-o(1))(\beta-\alpha)\sqrt{H'(\beta-\alpha)}+o\left(\beta\sqrt{H'(\beta)}\right)\right\}.\end{aligned} \tag{4.14}$$

Clearly $H(\beta-\alpha)\le H(\beta)$ and $H'(\beta-\alpha)\sim H'(\beta)$, see (1.15). Hence, on the r.h.s. of (4.14) we may replace $\beta-\alpha$ by β to arrive at the desired bound (4.1). □

Corollary 4.2. *Fix $r>0$ arbitrarily. Then, as $\alpha\to\infty$,*

$$\text{Prob}(\lambda^\xi(Q_r)>\alpha)\le \exp\left\{-L(\alpha)-(\chi-o(1))\varrho(\alpha)\sqrt{\alpha}\right\}. \tag{4.15}$$

Proof. For any $\beta>0$, the exponential Chebyshev inequality yields

$$\text{Prob}(\lambda^\xi(Q_r)>\alpha)\le e^{-\alpha\beta}\langle e^{\beta\lambda^\xi(Q_r)}\rangle. \tag{4.16}$$

Our assertion now follows from (4.1) by picking $\beta=\varrho(\alpha)$ and using (1.14). □

Corollary 4.3. *Fix $r>0$. Let $\lambda_1,\lambda_2,\dots$ be a sequence of random variables which are each distributed as $\lambda^\xi(Q_r)$. Then, with probability one,*

$$\max_{i=1,\dots,N^d}\lambda_i\le h_N-(\chi-o(1))\sqrt{h_N},\qquad N\to\infty. \tag{4.17}$$

Proof. Set $N_n=2^n$. Because of the monotonicity of the maximum and the estimate $h_{N_n}-h_{N_{n-1}}=o(\sqrt{h_{N_n}})$ (see (3.13)), it is enough to prove (4.17) for the subsequence $N_n=2^n$. To this end, pick $\varepsilon>0$ arbitrarily small and abbreviate

$$\alpha_{2^n}=h_{2^n}-\chi\sqrt{h_{2^n}}(1-\varepsilon). \tag{4.18}$$

Then, using Corollary 4.2, we obtain

$$\begin{aligned}\text{Prob}\left(\max_{i=1,\dots,2^{nd}}\lambda_i>\alpha_{2^n}\right)&\le 2^{nd}\,\text{Prob}(\lambda^\xi(Q_r)>\alpha_{2^n})\\ &\le 2^{nd}\exp\left\{-L(\alpha_{2^n})-(\chi-o(1))\varrho(\alpha_{2^n})\sqrt{\alpha_{2^n}}\right\}.\end{aligned} \tag{4.19}$$

Using the monotonicity of $L'=\varrho$ and recalling (1.19), we see that

$$\begin{aligned}L(\alpha_{2^n})&\ge L(h_{2^n})-L'(h_{2^n})\,(h_{2^n}-\alpha_{2^n})\\ &=\log(2^{nd})-(1-\varepsilon)\chi\varrho(h_{2^n})\sqrt{h_{2^n}}.\end{aligned} \tag{4.20}$$

Because of (1.16),

$$\varrho(\alpha_{2^n})\sqrt{\alpha_{2^n}} = (1+o(1))\varrho(h_{2^n})\sqrt{h_{2^n}}. \tag{4.21}$$

Hence, the r.h.s. of (4.19) does not exceed

$$\exp\left\{-(1+o(1))\varepsilon\chi\varrho(h_{2^n})\sqrt{h_{2^n}}\right\}. \tag{4.22}$$

But (3.14) shows that this is summable over $n \in \mathbb{N}$, and our assertion follows by an application of the first Borel-Cantelli lemma. □

4.2. Completion of the proof

First we replace $u(t,0)$ by the solution of the initial-boundary value problem in some large, t-dependent macrobox. Then we estimate this from above in terms of the principal eigenvalue of $\kappa\Delta + \xi$ in that box. Using a result from GK, we bound this from above by the maximal principal eigenvalue in the sub-microboxes of fixed length. But its a.s. asymptotics (as $t \to \infty$ or, equivalently, as the number of microboxes increases unboundedly) has been derived in the preceding subsection.

Lemma 4.4. *Put $R(t) = ct\log t$ for some constant $c > 0$. Then, with probability one,*

$$u^\xi(t,0) = u^\xi_{R(t)}(t,0)(1+o(1)) \qquad \text{as} \quad t \to \infty. \tag{4.23}$$

Proof. Define $R_n(t) = nR(t)$. Recall that τ_r is the exit time from the box Q_r. Then the Feynman-Kac formula yields

$$\begin{aligned} 0 \le u^\xi(t,0) - u^\xi_{R(t)}(t,0) &= \sum_{n=1}^\infty \mathbb{E}_0\, e^{\int_0^t \xi(W_s)\,ds} \mathbb{1}\{\tau_{R_n(t)} \le t < \tau_{R_{n+1}(t)}\} \\ &\le \sum_{n=1}^\infty \exp\left\{t \max_{Q_{R_{n+1}(t)}} \xi\right\} \mathbb{P}_0(\tau_{R_n(t)} \le t). \end{aligned} \tag{4.24}$$

As we know from the lower bound in Theorem 1.1, $u^\xi(t,0)$ tends to infinity as $t \to \infty$. Hence, the assertion of the lemma will follow if we show that, with probability one, the r.h.s. of (4.24) vanishes as $t \to \infty$.

We know from Lemma 2.3 resp. 2.6 that there exists a constant $C > 0$ such that, with probability one,

$$\max_{x \in Q_R} \xi(x) \le C \log R \tag{4.25}$$

for all sufficiently large $R > 0$. On the other hand, an application of the reflection principle yields

$$\mathbb{P}_0(\tau_R \le t) \le \exp\left\{-\frac{R^2}{4\kappa t}\right\}, \qquad \frac{R^2}{t} \gg 1. \tag{4.26}$$

Thus, with probability one, for sufficiently large t and all n, we obtain the estimate

$$\begin{aligned}
&\exp\left\{t \max_{Q_{R_{n+1}(t)}} \xi\right\} \mathbb{P}_0(\tau_{R_n(t)} \le t) \\
&\quad \le \exp\left\{Ct\left(\log t + \log\log t + \log(c(n+1))\right) - \frac{c^2 n^2 t}{4\kappa}(\log t)^2\right\} \\
&\quad = e^{-Ct\log t} \exp\left\{-Ct\log t\left(\frac{c^2n^2\log t}{4\kappa C} - 2 - \frac{\log\log t}{\log t} - \frac{\log(c(n+1))}{\log t}\right)\right\}.
\end{aligned} \tag{4.27}$$

For $t > e^{(4\kappa C)\vee 1}$, the term in round brackets is not smaller than $c^2n^2 - 3 - \log(c(n+1))$. Therefore, the second factor on the r.h.s. of (4.27) is summable over n, uniformly in those t. Hence, the r.h.s. of (4.24) indeed vanishes as $t \to \infty$. $\square$

Lemma 4.5. *Fix $R(t) = ct\log t$ for some $c > 0$. Then, with probability one,*

$$u^\xi_{R(t)}(t,0) \le e^{O(\log t)}(u^\xi_{R(t)}(t-1,\cdot), \mathbb{1})_{R(t)}, \qquad t \to \infty. \tag{4.28}$$

Proof. We use the Feynman-Kac formula and apply the Markov property at time 1 to get

$$u^\xi_{R(t)}(t,0) \le \exp\left\{\sup_{x\in Q_{R(t)}} \xi(x)\right\} \int_{Q_{R(t)}} dx\, p_1(x)\, u^\xi_{R(t)}(t-1,x), \tag{4.29}$$

where p_1 is the centered Gaussian kernel with variance 2κ. Now use Lemma 2.3 resp. 2.6 to see that the first factor is not bigger than $e^{O(\log t)}$, a.s. Furthermore, estimate $p_1(x) \le (4\pi\kappa)^{-d/2}$ and absorb this factor in $e^{O(\log t)}$. $\square$

In the following lemma, we write Φ^θ for the shift of a function Φ defined by $\Phi^\theta(x) = \Phi(x-\theta)$.

Lemma 4.6. *There exists a continuous periodic function $\Phi\colon \mathbb{R}^d \to [0,\infty)$ with period 2 in each coordinate such that*

$$\begin{aligned}
&(i) && u^\xi_{R(t)}(t,x) &&\le&& e^{Kt}\frac{1}{|Q_1|}\int_{Q_1} d\theta\, u^{\xi-\Phi^\theta}_{R(t)}(t,x), && \\
&(ii) && \lambda^{\xi-\Phi^\theta}(Q_{R(t)}) &&\le&& \max_{z\in 2\mathbb{Z}^d\cap Q_{R(t)+2}} \lambda^\xi(Q_2+z), && \theta\in Q_1,
\end{aligned} \tag{4.30}$$

for all $t > 0$ and $x \in Q_{R(t)}$, where $K = \int_{Q_1}\Phi(x)\,dx$.

Proof. Assertion (i) is formulated and proved as Step 1 in the proof of Proposition 2 in GK. Part (ii) is taken from Proposition 2 in GK. (In both assertions we replaced the box $Q_r$for some $r \geq 2$ by Q_1.) For the reader's convenience let us describe the nature of assertion (ii) in more detail.

Let $V\colon \mathbb{R}^d \to \mathbb{R}$ be any Hölder continuous potential. Roughly speaking, for any $R > 1$, we derived an upper bound for $\lambda^V(Q_R)$ in terms of the maximum of the eigenvalues $\lambda^V(Q_2+z)$ in the small subboxes Q_2+z of Q_R with $z \in 2\mathbb{Z}^d \cap Q_R$. In order to do this properly, we needed to let these boxes overlap each other slightly, and we needed to lower the potential V in the overlapping area, which will be a neighborhood of the grid $(2\mathbb{Z}^d + \partial Q_1) \cap Q_R$.

More precisely, we constructed a smooth periodic function $\Phi\colon \mathbb{R}^d \to [0,\infty)$ with period 2 in each coordinate whose support is contained in a small neighborhood of the grid $2\mathbb{Z}^d + \partial Q_1$ such that for all $R > 1$ and all Hölder continuous potentials $V\colon \mathbb{R}^d \to \mathbb{R}$,

$$\lambda^{V-\Phi}(Q_R) \leq \max_{z \in 2\mathbb{Z}^d \cap Q_{R+1}} \lambda^V(Q_2 + z). \tag{4.31}$$

□

Now we complete the derivation of the upper bound.

Proof of Theorem 1.1, upper bound. As before, put $R(t) = ct\log t$. As a consequence of the Lemmas 4.4 and 4.5 and since $h_t - h_{t-1} \to 0$ by (3.12), it is sufficient to prove the upper bound for $(u^\xi_{R(t)}(t,\cdot), \mathbb{1})_{R(t)}$ instead of $u^\xi(t,0)$ with the constant c taken slightly larger as in the above lemmas.

Integrate (4.30)(i) over $x \in Q_{R(t)}$ and apply the spectral bound (2.33) (with $r = R(t)$ and $V = \xi - \Phi^\theta$) to obtain

$$\begin{aligned}(u^\xi_{R(t)}(t,\cdot), \mathbb{1})_{R(t)} &\leq e^{Kt} \frac{1}{|Q_1|} \int_{Q_1} d\theta \, \left(u^{\xi-\Phi^\theta}_{R(t)}(t,\cdot), \mathbb{1}\right)_{R(t)} \\ &\leq e^{Kt} R(t)^d \int_{Q_1} d\theta \, \exp\left\{t\lambda^{\xi-\Phi^\theta}(Q_{R(t)})\right\}.\end{aligned} \tag{4.32}$$

An application of the eigenvalue estimate (4.30)(ii) yields that the r.h.s. of (4.32) does not exceed

$$e^{O(t)} \exp\left\{t \max_{z \in 2\mathbb{Z}^d \cap Q_{R(t)+2}} \lambda^\xi(Q_2 + z)\right\}. \tag{4.33}$$

Now apply Corollary 4.3 for $r = 2$ and $N = N_t = \lceil R(t) + 2 \rceil$ to arrive at

$$\left(u^\xi_{R(t)}(t,\cdot), \mathbb{1}\right)_{R(t)} \leq \exp\left\{t\left(h_{N_t} - (\chi - o(1))\sqrt{h_{N_t}}\right)\right\}. \tag{4.34}$$

According to (3.13), $h_{N_t} - h_t$ stays bounded as $t \to \infty$. Therefore we may replace in (4.34) h_{N_t} by h_t which yields the desired upper bound. □

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