INFINITESIMAL RIGIDITY
OF FRAMEWORKS AND SURFACES

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Contents

I. Basic concepts

1. **Frameworks.**

§1.1. *An informal definition and examples.* A bar-and-joint framework is a structure made of stiff bars joined at their ends by universal joints. The universality of a joint means that, while one of the adjacent bars is fixed, the other one can be rotated in every direction.

A human arm is an example of a bar-and-joint framework, only that the joints are not universal. Another example is a robotic arm, althouth in this case the joints are also not always universal, and some other types of connections between the bars are possible. An issue in controlling a robotic arm is *motion planning*, that is how to guide a robotic arm so that it performs its tasks effectively. The human brain solves this problem for a human arm almost perfectly, but it is not easy to teach a robot how to use its arms.

An arm is an example of a flexible framework. In some situations there is a need for a rigid framework. For example, the underlying structures of bridges and roofs are often bar-and-joint frameworks, see Figure 1. One can argue that there are no joints in the carrying construction of a bridge, the bars are rigidly attached to each other. However, when a load is applied to the whole structure, these are often the attachment points that break first. Thus, as a first stability test for such a structure can be to imagine the attachment points as universal joints and to analize whether the corresponding framework is rigid or not.

Note that the issue of stability of the real-world constructions is a much more complex one, and cannot be simply reduced to the mathematical theory we are going to study.

§1.2. *A mathematical definition.* Let G be a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We denote the elements of $\mathcal{V}$ by letters i, $j, \ldots$

FIGURE 1. The carrying constructions of bridges and roofs can often be viewed as bar-and-joint frameworks.
Left: General Hertzog Bridge over the Orange River, South Africa. Photograph ©Gregory David Harington, 9 December 2006.
Right: The Great Court of the British Museum. Photograph ©Andrew Dunn, 26 November 2005.

Definition 1. *A framework in* $\mathbb{R}^d$ *with graph* G *is a map*

$$\begin{array}{rccc} P\colon & \mathcal{V} & \to & \mathbb{R}^d, \\ & i & \mapsto & p_i \end{array}$$

such that $p_i \neq p_j$ *whenever* $ij \in \mathcal{E}$.

We depict a framework by drawing the points p_i for all $i \in \mathcal{V}$, and joining by segments those pairs p_i, p_j for which $ij \in \mathcal{E}$, see Figure 2. Note that the segments p_ip_j are allowed to intersect one another, as do the segments p_1p_3 and p_2p_4 on Figure 2.

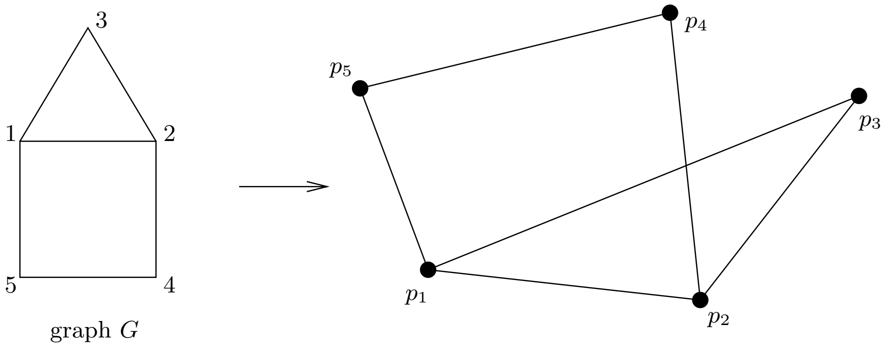

FIGURE 2. A framework in $\mathbb{R}^2$.

2. **Rigidity and infinitesimal rigidity of frameworks.**

§2.1. *Isometries of* $\mathbb{R}^d$.

Definition 2. *An* isometry *of* $\mathbb{R}^d$ *is a bijective map*

$$\Phi\colon \mathbb{R}^d \to \mathbb{R}^d$$

such that

$$\|\Phi(x) - \Phi(y)\| = \|x - y\|,$$

for all pairs of points $x, y \in \mathbb{R}^d$.

The set of all isometries of $\mathbb{R}^d$ *is denoted by* $\mathrm{Iso}(\mathbb{R}^d)$.

Clearly, $\mathrm{Iso}(\mathbb{R}^d)$ is a group. It contains a subgroup $\mathrm{O}(d)$, the *orthogonal group*:

$$\mathrm{O}(d) = \{A \in \mathrm{GL}(d, \mathbb{R}) \mid A^\top A = I\},$$

where a matrix A maps a point $x \in \mathbb{R}^d$ to Ax.

To check that $\mathrm{O}(d) \subset \mathrm{Iso}(\mathbb{R}^d)$, note that $\|v\|^2 = v^\top v$; thus $A^\top A = I$ implies

$$\|Ax-Ay\|^2 = \|A(x-y)\|^2 = (x-y)^\top A^\top \cdot A(x-y) = (x-y)^\top (x-y) = \|x-y\|^2.$$

Every isometry from $\mathrm{O}(d)$ maps the coordinate origin to itself. The converse is also true: every isometry of $\mathbb{R}^d$ that maps the origin to itself is an orthogonal linear transformation. Thus we have:

$$\mathrm{O}(d) = \{\Phi \in \mathrm{Iso}(\mathbb{R}^d) \mid \Phi(0) = 0\}. \tag{1}$$

Parallel translations

$$x \mapsto x + b,$$

where $b \in \mathbb{R}^d$, are also isometries of $\mathbb{R}^d$, and they don't map 0 to itself. It turns out that orthogonal linear transformations and parallel translations generate the group $\mathrm{Iso}(\mathbb{R}^d)$.

Lemma 3. *Every isometry* $\Phi \in \mathrm{Iso}(\mathbb{R}^d)$ *has the form*

$$\Phi(x) = Ax + b,$$

where $A \in \mathrm{O}(d)$*, and* $b \in \mathbb{R}^d$*. Besides, a presentation of* Φ *in this form is unique.*

Proof. Let $\Phi \in \mathrm{Iso}(\mathbb{R}^d)$ be any isometry. Put

$$b := \Phi(0).$$

Then the map $x \mapsto \Phi(x) - b$ is an isometry of $\mathbb{R}^d$ that maps 0 to itself. By (1), this implies that

$$\Phi(x) - b = Ax,$$

for all $x \in \mathbb{R}^d$ and for some $A \in \mathrm{O}(d)$.

In order to prove the uniqueness, assume that

$$\Phi(x) = Ax + b = A'x + b',$$

for all $x \in \mathbb{R}^d$. Then we have

$$b = \Phi(0) = b'.$$

It follows that $Ax = A'x$ for all $x \in \mathbb{R}^d$. Therefore we also have $A = A'$. □

Exercise 1. Show that the set

$$\left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \;\middle|\; A \in \mathrm{O}(d), b \in \mathbb{R}^d \right\} \subset \mathrm{GL}(d+1, \mathbb{R})$$

is a group isomorphic to $\mathrm{Iso}(\mathbb{R}^d)$.

Definition 4. *A* motion *of* $\mathbb{R}^d$ *is a continuous family* Φ_t *of isometries of* $\mathbb{R}^d$ *such that* Φ_0 *is the identity. Usually, we take the time parameter* t *in the segment* $[0, 1]$*. Thus, a motion of* $\mathbb{R}^d$ *is a continuous map*

$$\begin{array}{ccc} [0,1] & \to & \mathrm{Iso}(\mathbb{R}^d), \\ t & \mapsto & \Phi_t \end{array}$$

such that $0 \mapsto \mathrm{id}$.

In fact, the space $\mathrm{Iso}(\mathbb{R}^d)$ is not connected, it has two connected components: orientation preserving isometries $\mathrm{Iso}^+(\mathbb{R}^d)$ and orientation reversing isometries $\mathrm{Iso}^-(\mathbb{R}^d)$. Thus, if $\Phi_0 = \mathrm{id}$, then all Φ_t belong to $\mathrm{Iso}^+(\mathbb{R}^d)$.

Similarly to Lemma 3, any orientation preserving isometry Φ has a representation

$$\Phi(x) = Ax + b,$$

where $A \in \mathrm{SO}(d) = \{A \in \mathrm{O}(d) \mid \det(A) = 1\}$.

§2.2. *Definition of rigidity.* The following definition is similar to Definition 4.

Definition 5. *A* motion *of a framework P is a continuous family of frameworks $P(t)$ for $t \in [0,1]$ such that $P(0) = P$ and*

$$\|p_i(t) - p_j(t)\| = \|p_i - p_j\|,$$

for all $ij \in \mathcal{E}$ and for all $t \in [0,1]$.

That is, each point p_i moves along a trajectory $p_i(t)$ so that the distances between points joined by an edge are preserved.

Definition 6. *A motion $P(t)$ of a framework P is called* trivial *if it is induced by a motion of $\mathbb{R}^d$:*

$$P(t) = \Phi_t \circ P,$$

for some motion Φ_t of $\mathbb{R}^d$.

Another way to state the condition in Definition 6 is

$$p_i(t) = \Phi_t(p_i)$$

for all $i \in \mathcal{V}$. That is, the trajectories $p_i(t)$ arise from moving the whole $\mathbb{R}^d$ as a rigid body.

Definition 7. *A framework is called* rigid *if all of its motions are trivial.*

A framework is called flexible *if it is not rigid.*

That is, a flexible framework P has a motion $P(t)$ that is not "embedded" in a motion of $\mathbb{R}^d$.

§2.3. *Definition of infinitesimal rigidity.* The following definitions mimic those of §2.2. The difference is that, instead of preservation of distances during a continuous deformation, we require first-order preservation of distances during an infinitesimal deformation.

Definition 8. *An* infinitesimal motion *of $\mathbb{R}^d$ is a vector field*

$$\begin{aligned} \xi\colon \mathbb{R}^d &\to \mathbb{R}^d, \\ x &\mapsto \xi(x) \end{aligned}$$

such that moving each point of $\mathbb{R}^d$ along the vector applied at that point does not change the distances in the first order:

$$\left.\frac{d}{dt}\right|_{t=0} \|(x + t\xi(x)) - (y + t\xi(y))\| = 0, \tag{2}$$

for all $x, y \in \mathbb{R}^d$.

Definition 9. *An* infinitesimal motion *of a framework* P *is a map*

$$\begin{array}{rccc} Q\colon & \mathcal{V} & \to & \mathbb{R}^d, \\ & i & \mapsto & q_i \end{array}$$

such that

$$\left.\frac{d}{dt}\right|_{t=0} \|(p_i + tq_i) - (p_j + tq_j)\| = 0, \tag{3}$$

for all $ij \in \mathcal{E}$.

Definition 10. *An infinitesimal motion* Q *of a framework* P *is called* trivial *if it is induced by some infinitesimal motion* ξ *of* $\mathbb{R}^d$*:*

$$Q = \xi \circ P.$$

The last condition can be rewritten as

$$q_i = \xi(p_i),$$

for all $i \in \mathcal{V}$.

Definition 11. *A framework is called* infinitesimally rigid *if all of its infinitesimal motions are trivial.*

A framework is called infinitesimally flexible *if it is not infinitesimally rigid.*

The conditions (2) and (3) can be simplified in the following way.

Lemma 12. *A vector field* ξ *is an infinitesimal motion of* $\mathbb{R}^d$ *if and only if*

$$\langle \xi(x) - \xi(y), x - y \rangle = 0, \tag{4}$$

for all $x, y \in \mathbb{R}^d$.

Similarly, a map Q *is an infinitesimal motion of a framework* P *if and only if*

$$\langle q_i - q_j, p_i - p_j \rangle = 0, \tag{5}$$

for all $ij \in \mathcal{E}$.

Proof. Let us prove the equivalence between (3) and (5).

Equation (3) is equivalent to

$$\left.\frac{d}{dt}\right|_{t=0} \|(p_i + tq_i) - (p_j + tq_j)\|^2 = 0.$$

Since, for any vector v that depends on t, we have

$$\frac{d}{dt}\|v\|^2 = \frac{d}{dt}\langle v, v\rangle = 2\left\langle \frac{d}{dt}v, v \right\rangle,$$

we can compute

$$\frac{d}{dt}\|(p_i + tq_i) - (p_j + tq_j)\|^2 = 2\langle q_i - q_j, (p_i - p_j) + t(q_i - q_j)\rangle$$

which at $t = 0$ equals $2\langle q_i - q_j, p_i - p_j\rangle$. Thus we have

$$\left.\frac{d}{dt}\right|_{t=0} \|(p_i + tq_i) - (p_j + tq_j)\|^2 = 2\langle q_i - q_j, p_i - p_j\rangle.$$

Therefore (3) is equivalent to (5).

The equivalence between (2) and (4) is proved along the same lines. ☐

Condition (5) says that the vectors $q_i - q_j$ and $p_i - p_j$ are orthogonal. Geometrically this means that the signed lengths of the projections of q_i and q_j on the line p_ip_j are the same, see Figure 3, left.

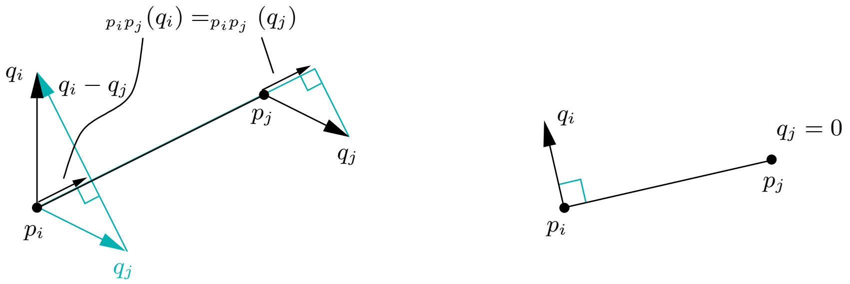

FIGURE 3. A geometric meaning of the condition (5).

Also, if $q_j = 0$, then we have $\langle q_i, p_i - p_j\rangle = 0$ that means $q_i \perp (p_i - p_j)$. The motion of p_i along q_i is called an infinitesimal rotation around p_j, see Figure 3.

Exercise 2. Draw infinitesimal motions of $\mathbb{R}^2$.

3. Relations between rigidity and infinitesimal rigidity.

§3.1. *Rigidity does not imply infinitesimal rigidity.* Below examples are given of frameworks that are rigid, but not infinitesimally rigid.

Example 13. A degenerate triangle in $\mathbb{R}^2$ (or in any $\mathbb{R}^d$ with $d \geq 2$). See Figure 4.

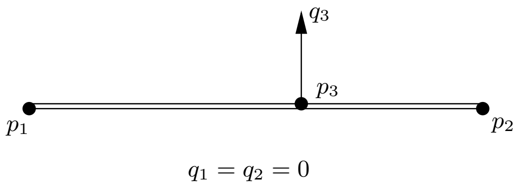

FIGURE 4. A degenerate triangle is rigid, but infinitesimally flexible.

It is easy to prove that a degenerate triangle is rigid.

It requires some work to show that the infinitesimal motion depicted on Figure 4 is non-trivial. Later, Lemma 20 will allow us to check in a routine way whether an infinitesimal motion is trivial or not.

If this example seems too trivial to you, here is another one.

Example 14. See Figure 5.

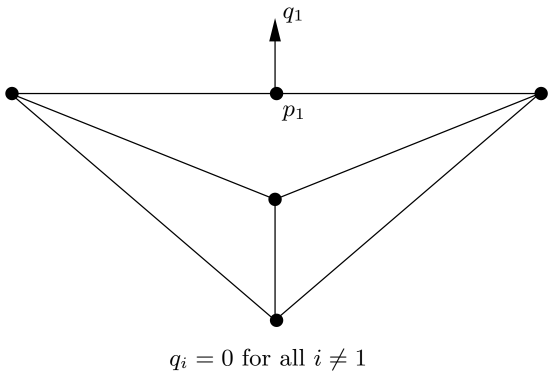

Figure 5. Another infinitesimally flexible rigid framework in $\mathbb{R}^2$.

Actually, Examples 13 and 14 are of similar nature: take a rigid framework, add a new edge to it and subdivide this edge by a new vertex. The resulting framework is rigid, but the new vertex can be infinitesimally moved in the direction orthogonal to the new edge.

Not all infinitesimally flexible rigid frameworks are of this kind. In fact, there are plenty of them and they defy classification.

Example 15. If the lines p_1p_4, p_2p_5, and p_3p_6 on Figure 6 intersect at a point, then the framework is infinitesimally flexible.

Exercise 3. 1) Find a non-trivial infinitesimal motion of the framework on Figure 6.

2) Show that the framework in Figure 6 is rigid. (This is more difficult.)

Example 16 (Twisted octahedron)**.** Here is an infinitesimally flexible framework in $\mathbb{R}^3$.

Take the skeleton of a regular octahedron and rotate one of the triangles, in the plane it spans, by 90° around its center. Figure 7 shows a view from the side of the rotated face. On Figure 8, another view is shown, with the circumcircles of the top and bottom faces drawn.

In fact, instead of taking a regular octahedron, one can take any regular triangular bipyramid, that is, a polyhedron with two horizontal faces whose projection on the horizontal plane looks like Figure 7, left. Moreover, for any n a regular pyramid over an n-gon becomes infinitesimally flexible after twisting one base by 90°.

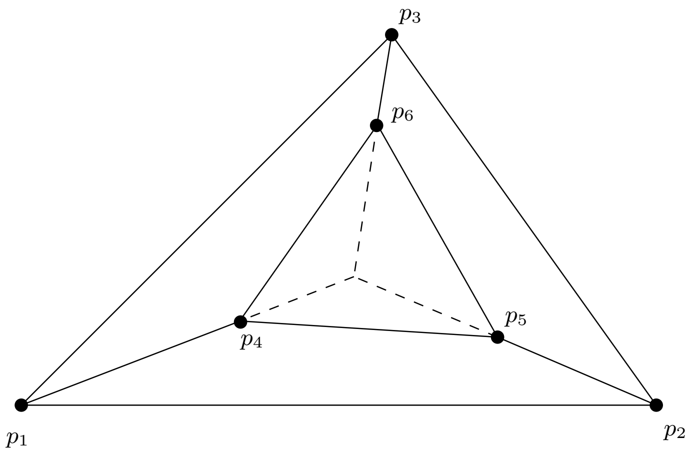

FIGURE 6. A more interesting infinitesimally flexible rigid framework.

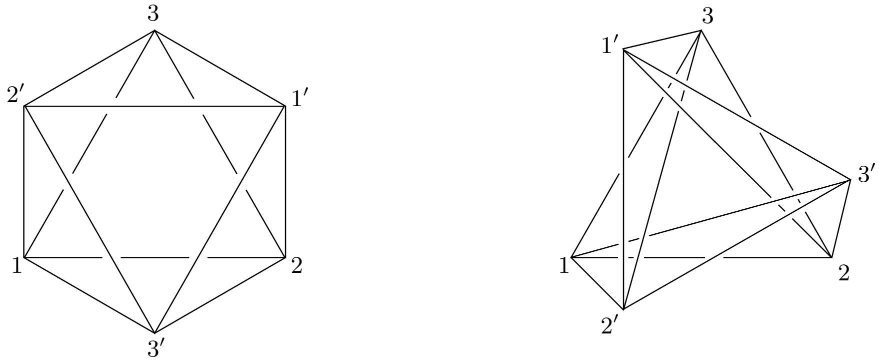

FIGURE 7. On the left, rotate the top face $1'2'3'$ of the octahedron by 90°; this yields the framework on the right.

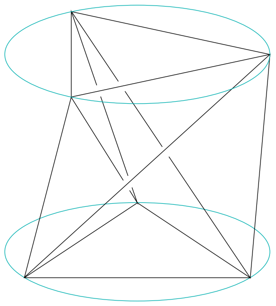

FIGURE 8. Another view of the twisted octahedron.

Exercise 4. Show that a twisted octahedron is rigid, but infinitesimally flexible.

§3.2. *Infinitesimal rigidity implies rigidity.* By comparing the definitions of a motion and of an infinitesimal motion that we gave in Lecture 1, one can get an idea that infinitesimal motions should have something to do with the velocity fields of motions. This is indeed true.

Lemma 17. *The velocity field of a smooth motion (of $\mathbb{R}^d$ or of a framework) is an infinitesimal motion.*

By the velocity field (at the time $t = 0$) of a motion Φ_t of $\mathbb{R}^d$ we mean the vector field

$$\xi(x) = \left.\frac{d}{dt}\right|_{t=0} \Phi_t(x).$$

Similarly, the velocity field of a motion $P(t)$ of a framework P is

$$q_i = \left.\frac{d}{dt}\right|_{t=0} p_i(t). \tag{6}$$

In other words, the velocity field is made of the tangent vectors to the trajectories of the points. We say that a motion is smooth if all trajectories are smooth curves. Note that we consider only the velocities at time $t = 0$.

Proof. Let $P(t)$ be a smooth motion of a framework P. We have to show that the vector field (6) is an infinitesimal motion of P.

By definition of a motion, we have

$$\|p_i(t) - p_j(t)\| = \|p_i - p_j\|$$

for all $ij \in \mathcal{E}$ and all t. It follows that

$$\frac{d}{dt}\|p_i(t) - p_j(t)\|^2 = 0.$$

This means that

$$2\left\langle \frac{d}{dt}p_i(t) - \frac{d}{dt}p_j(t), p_i(t) - p_j(t) \right\rangle = 0$$

holds for all t. By substituting $t = 0$ we obtain

$$2\langle q_i - q_j, p_i - p_j\rangle = 0.$$

Thus the vector field (6) is indeed an infinitesimal motion of the framework P.

For the velocity fields of smooth motions of $\mathbb{R}^d$, the proof goes along the same lines. □

Theorem 1. *Every infinitesimally rigid framework is rigid.*

A sketch of proof. The theorem is equivalent to saying that a flexible framework is also infinitesimally flexible.

Choose a smooth non-trivial motion of the framework. Then its velocity field is an infinitesimal motion by Lemma 17. If this infinitesimal motion is non-trivial, then we are done.

There are two problems. First, one has to show that a *smooth* non-trivial motion exists. Second, it is wrong that the velocity field of a non-trivial smooth motion is a *non-trivial* infinitesimal motion. In particular, every motion can be reparametrized so that its velocity field at $t = 0$ vanishes. Even worse, take a non-trivial smooth motion with zero velocity field and compose it with a smooth trivial motion with non-zero velocity field; you obtain a non-trivial motion whose velocity field does not vanish, but is trivial.

Both problems can be solved. It can be shown (but it is not easy!) that every flexible framework has an analytic non-trivial motion. After that, by a composition with a trivial motion and reparametrization, the analytic non-trivial motion can be modified so that its velocity field becomes a non-trivial infinitesimal motion. □

In Lecture 1, we gave an explicit description of the isometries of $\mathbb{R}^d$ by means of the orthogonal transformations. Let us see what can we learn about the infinitesimal motions of $\mathbb{R}^d$ with this and Lemma 17 at hand.

Lemma 18. *The velocity fields of smooth motions of* $\mathbb{R}^d$ *are vector fields of the form*

$$\xi(x) = Zx + b, \tag{7}$$

where Z *is any antisymmetric* $d \times d$ *matrix:* $Z^\top = -Z$.

Proof. If Φ_t is a smooth motion of $\mathbb{R}^d$, then we have

$$\Phi_t(x) = A_t x + b_t, \quad A_t \in \mathrm{SO}(d),\, b_t \in \mathbb{R}^d.$$

Thus the velocity field of Φ_t has the form

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t(x) = \left(\left.\frac{d}{dt}\right|_{t=0} A_t \right) x + \left.\frac{d}{dt}\right|_{t=0} b_t.$$

The vector $\left.\frac{d}{dt}\right|_{t=0} b_t$ can be just any vector in $\mathbb{R}^d$. Let us show that the matrix $\left.\frac{d}{dt}\right|_{t=0} A_t$ is skew-symmetric. Since $A_t^\top A_t = I$ for all t, we have

$$0 = \frac{d}{dt}\left(A_t^\top A_t\right) = \left(\frac{d}{dt} A_t^\top\right) A_t + A_t^\top \left(\frac{d}{dt} A_t\right)$$

for all t. Since $\Phi_0 = \mathrm{id}$, we have $A_0 = I$ and by substituting $t = 0$ we obtain

$$\left.\frac{d}{dt}\right|_{t=0} A_t^\top + \left.\frac{d}{dt}\right|_{t=0} A_t = 0.$$

In the other direction, let Z be a skew-symmetric matrix. Then $\exp Z$ is an orthogonal matrix:

$$(\exp Z)^\top = \exp(Z^\top) = \exp(-Z) = (\exp Z)^{-1}.$$

Besides, we have

$$Z = \left.\frac{d}{dt}\right|_{t=0} \exp(tZ).$$

Thus the vector field (7) is the velocity field of the motion

$$\Phi_t(x) = \exp(tZ)x + tb.$$

□

Example 19. For $d = 2$, consider the skew-symmetric matrix

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

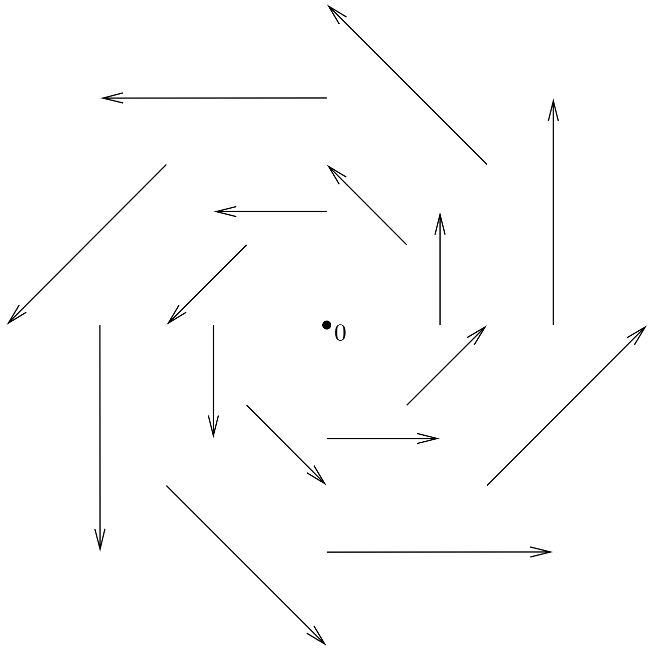

Figure 9. An infinitesimal rotation.

The vector field $\xi(x) = Zx$ is depicted on Figure 9. It is the velocity field of the rotation of $\mathbb{R}^2$ around the origin by the angle t in time t.

The set of all $d \times d$ skew-symmetric matrices is denoted $\mathfrak{so}(d)$. Lemmas 17 and 18 imply that $\mathfrak{so}(d) \oplus \mathbb{R}^d$ is a subspace of the vector space of infinitesimal motions of $\mathbb{R}^d$. The following lemma shows that, in fact, the two spaces coincide.

Lemma 20. *Every infinitesimal motion of* $\mathbb{R}^d$ *has the form* (7).

Proof. Let ξ be an infinitesimal motion of $\mathbb{R}^d$, and let $e_1, e_2, \dots, e_d \in \mathbb{R}^d$ be the standard basis of $\mathbb{R}^d$. We claim that ξ is uniquely determined by its values at e_1, $e_2, \dots, e_d$.

First, let us show how to determine $\xi(0)$. By the geometric characterization of infinitesimal motions given at the end of Lecture 1, the vectors $\xi(0)$ and $\xi(e_i)$ have the same projection to the i-th coordinate axis. Thus, the i-th component of the vector $\xi(0)$ is equal to the ith component of $\xi(e_i)$.

Now, let $x \in \mathbb{R}^d$ be an arbitrary point. If x does not lie in the hyperplane spanned by the points $e_1, e_2, \dots, e_d$, then, similar to the case $x = 0$ just considered, the values $\xi(e_1), \xi(e_2), \ldots, \xi(e_d)$ determine the projections of $\xi(x)$ on d lines in general position, and thus determine $\xi(x)$. If x lies in the hyperplane of $e_1, e_2, \ldots, e_d$, then x does not lie in the hyperplane spanned by 0 and some $d-1$ of the points $e_1, e_2, \ldots, e_d$. So by the above argument we can again determine the value $\xi(x)$. The claim is proved.

In order to prove the lemma, it suffices to show that for every infinitesimal motion ξ there is a pair $(Z, b) \in \mathfrak{so}(d) \oplus \mathbb{R}^d$ such that

$$\xi(e_i) = Ze_i + b$$

holds for all i. (The lemma would follow because both ξ and $x \mapsto Zx + b$ are infinitesimal motions, and by the claim proved above, if they take equal values at $e_1, e_2, \ldots, e_d$, then they are equal identically.)

Put $\xi(e_i) = (\xi_i^1, \xi_i^2, \ldots, \xi_i^d)$. Due to $\langle \xi(e_i) - \xi(e_j), e_i - e_j \rangle = 0$ we have

$$\xi_i^i - \xi_i^j - \xi_j^i + \xi_j^j = 0,$$

for all i, j. It follows that the matrix $Z = (z_i^j)$ with

$$z_i^j = \xi_i^j - \xi_j^j$$

is skew-symmetric and, by putting $b = (\xi_1^1, \xi_2^2, \ldots, \xi_d^d)$ we achieve $\xi(e_i) = Ze_i + b$. The lemma is proved. □

Corollary 21. *The vector space of infinitesimal motions of* $\mathbb{R}^d$ *has dimension* $\frac{d(d+1)}{2}$.

Lemma 20 implies, in particular, that every infinitesimal motion of $\mathbb{R}^d$ is a linear vector field. Thus the infinitesimal motion of a degenerate triangle shown on Figure 4 is non-trivial because it cannot be extended to a linear vector field.

Appendix A. Some rigid frameworks

For some frameworks, our intuition tells us that they are rigid. However, one can easily get stuck when trying to give a formal "by definition" proof of their rigidity. This section aims at a partial reconciling of our need for mathematical rigour with the common sense.

We will deal with frameworks in $\mathbb{R}^2$, but all of the presented results can be generalized to frameworks in $\mathbb{R}^d$.

Lemma A.1. *An orientation preserving isometry of* $\mathbb{R}^2$ *is uniquely determined by the images of two points. Moreover, any pair of points can be mapped to any other pair of points that lie at the same distance, by an orientation preserving isometry.*

An infinitesimal isometry of $\mathbb{R}^2$ *is uniquely determined by its values at two points. Moreover, for any two points* $x_1, x_2 \in \mathbb{R}^2$ *and any two vectors* ξ_1, ξ_2 *such that* $\langle \xi_1 - \xi_2, x_1 - x_2 \rangle = 0$ *there exists an infinitesimal isometry with values* ξ_1 *and* ξ_2 *at* x_1 *and* x_2*, respectively.*

Proof. The second part of the lemma was proved for $\mathbb{R}^d$ during the proof of Lemma 20.

As for the first part, let x, y and x', y' be two pairs of points in $\mathbb{R}^2$ such that

$$\|x - y\| = \|x' - y'\|.$$

For any point $z \in \mathbb{R}^2$, there is a unique point z' such that the triangles xyz and $x'y'z'$ are equal and have the same orientation. (If z lies on the line xy, then the triangle xyz is degenerated and its orientation is not well-defined; but in this case there is just one point z' such that $\|x - z\| = \|x' - z'\|$ and $\|y - z\| = \|y' - z'\|$.) Every orientation preserving isometry Φ must send z to z'. Thus Φ is unique, if exists.

To prove the existence of Φ, choose any point o not on the line xy and a point o' such that the triangles xyo and $x'y'o'$ are equal and have the same orientation. Consider two coordinate systems, one with o as the origin and ox, oy as basis vectors, the other one with o', x', y' in place of o, x, y. Consider a map Φ that sends a point with coordinates (a, b) in the first system to the point with the same coordinates in the second system. It is easy to show that Φ is an isometry. ☐

Lemma A.2 (Pinned vertices)**.** *Let P be a framework in $\mathbb{R}^2$, and let p_1 and p_2 be two vertices of P joined by an edge. Then*

- *P is rigid if and only if every motion that fixes p_1 and p_2 fixes all of the other vertices as well:*

$$p_1(t) = p_1, p_2(t) = p_2 \text{ for all } t \quad \Rightarrow \quad p_i(t) = p_i \text{ for all } i \text{ and } t \tag{8}$$

- *P is infinitesimally rigid if and only if for every infinitesimal motion Q such that $q_1 = q_2 = 0$ we have $q_i = 0$ for all i.*

Proof. In one direction:

Assume that (8) holds, and let $P(t)$ be any motion of the framework P. We have to show that $P(t)$ is trivial. By Lemma A.1, for every t there exists an orientation preserving isometry Φ_t such that

$$\Phi_t(p_1) = p_1(t) \quad \text{and} \quad \Phi_t(p_2) = p_2(t). \tag{9}$$

In particular, $\Phi_0 = \mathrm{id}$. It can be shown that the isometry Φ_t depends on t continuously, thus $t \mapsto \Phi_t$ is a motion of $\mathbb{R}^2$. Consider the motion $(\Phi_t)^{-1} \circ P(t)$ of the framework P. (It shows how the other vertices of P move with respect to the vertices p_1 and p_2.) Due to (9), this motion fixes the vertices p_1 and p_2. Thus, by (8), it fixes all of the vertices of P:

$$(\Phi_t)^{-1}(p_i(t)) = p_i,$$

for all i and t. But this is equivalent to $p_i(t) = \Phi_t(p_i)$ which means that $P(t)$ is induced by a motion of $\mathbb{R}^d$, thus $P(t)$ is a trivial motion of P.

The proof of the second part of the lemma is similar, only instead of composing isometries we add infinitesimal isometries. ☐

Example A.3. A triangle is rigid and infinitesimally rigid. This follows directly from Lemma A.2.

A generalization of Lemma A.2 to frameworks in $\mathbb{R}^d$ would involve any d vertices $p_1, p_2, \ldots, p_d$ that span a hyperplane in $\mathbb{R}^d$ and that are pairwise joined by edges. But not every framework has d pairwise joined vertices, thus the "pinned vertices" criterion cannot always be applied.

One could conjecture that if a framework in $\mathbb{R}^3$ containes no three pairwise joined vertices, then it should anyway be flexible. However, this is false.

Exercise 5. Find a rigid framework in $\mathbb{R}^3$ that doesn't contain a triangle. (It might be not easy, we will get later some hints on how to do it.)

Lemma A.4. *Let P be a rigid (respectively, infinitesimally rigid) framework in $\mathbb{R}^2$. Then each of the following two operations produces a framework which is also rigid (respectively, infinitesimally rigid):*

(a) adding an edge *ij such that i and j are vertices of P previously not connected by an edge;*
(b) edge union*: taking the union $P \cup P'$ with another rigid (respectively, infinitesimally rigid) framework such that P and P' have at least one edge in common.*

Proof. That adding a new edge does not destroy rigidity, is trivial. Note, that every motion (respectively, infinitesimal motion) of the new framework is also a motion (respectively, infinitesimal motion) of the old one.

Let p_1p_2 be a common edge of the frameworks P and P'. Rigidity and infinitesimal rigidity of $P \cup P'$ follow by applying Lemma A.2. ☐

Example A.5. The framework on Figure 10 is rigid and infinitesimally rigid.

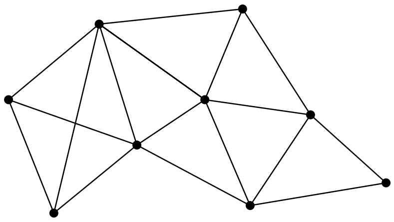

FIGURE 10. This framework is rigid and infinitesimally rigid due to the rigidity of a triangle and Lemma A.4.

§0.3. *Averaging and deaveraging.* Two frameworks P' and P'' with the same graph G are called *isometric* if

$$\|p'_i - p'_j\| = \|p''_i - p''_j\|, \quad \text{for all edges } ij \text{ of } G.$$

The frameworks P' and P'' are called *congruent*, if they are related by an isometry of $\mathbb{R}^d$.

The following simple construction allows to obtain an infinitesimally flexible framework from a pair of isometric non-congruent frameworks.

Theorem 2. *Let P' and P'' be two isometric frameworks. Assume that there is no edge ij such that the segments $p'_ip'_j$ and $p''_ip''_j$ are parallel and oppositely oriented. Then the framework P obtained by "averaging" P' and P'':*

$$p_i = \frac{p'_i + p''_i}{2}$$

has an infinitesimal motion Q given by

$$q_i = \frac{p'_i - p''_i}{2}.$$

Besides, if P' and P'' are not congruent, then the infinitesimal motion Q is non-trivial, and thus P is infinitesimally flexible.

Proof. The condition that there is no corresponding pair of oppositely oriented parallel edges ensures that $p_i \neq p_j$ for all edges ij.

The following simple computation shows that Q is an infinitesimal motion of P provided that P' is isometric to P'':

$$\begin{aligned} \|p'_i - p'_j\|^2 - \|p''_i - p''_j\|^2 &= \big\langle (p'_i - p'_j) - (p''_i - p''_j), (p'_i - p'_j) + (p''_i - p''_j) \big\rangle \\ &= \big\langle (p'_i - p''_i) - (p'_j - p''_j), (p'_i + p''_i) - (p'_j + p''_j) \big\rangle \\ &= 4\big\langle q_i - q_j, p_i - p_j \big\rangle. \end{aligned} \tag{10}$$

Assume that Q is trivial, and let ξ be an infinitesimal motion of $\mathbb{R}^d$ whose restriction to P is Q:

$$q_i = \xi(p_i).$$

We will use ξ to construct an isometry Φ of $\mathbb{R}^d$ such that $\Phi(p'_i) = p''_i$ for all i. Note that

$$\begin{aligned} p'_i &= p_i + q_i, \\ p''_i &= p_i - q_i. \end{aligned}$$

Thus we want Φ to map $p_i + \xi(p_i)$ to $p_i - \xi(p_i)$. So, let us just put

$$\Phi : x + \xi(x) \mapsto x - \xi(x), \tag{11}$$

for all $x \in \mathbb{R}^d$. The same transformations as in equation (10) show that Φ preserves distances, since ξ is an infinitesimal motion:

$$\|(x - \xi(x)) - (y - \xi(y))\| = \|(x + \xi(x)) - (y + \xi(y))\|. \tag{12}$$

But we need to show that equation (11) gives a well-defined map from $\mathbb{R}^d$ to itself. That is, we have to prove that every point in $\mathbb{R}^d$ can be presented in the form $x + \xi(x)$, and that $x + \xi(x) = y + \xi(y)$ implies $x - \xi(x) = y - \xi(y)$.

Recall that every infinitesimal motion of $\mathbb{R}^d$ has the form

$$\xi(x) = Zx + b,$$

where $Z^\top = -Z$. Thus

$$x \mapsto x + \xi(x) \tag{13}$$

is an affine map $x \mapsto (I + Z)x + b$.

Exercise 6. Show that $Z^\top = -Z$ implies $\det(I + Z) \neq 0$.

Exercise 6 implies that the map (13) is one-to-one. Thus equation (11) gives a well-defined map Φ. Theorem 2 is proved. $\square$

Remark 6. If $\xi(x) = Zx + b$, then the map (11) has the form

$$\Phi(x) = (I - Z)(I + Z)^{-1}(x - b) + b.$$

Thus, as a byproduct we have proved that

$$Z^\top = -Z \text{ implies } (I - Z)(I + Z)^{-1} \in O(d). \tag{14}$$

Besides, there is a simple way to prove that Φ is well-defined without using Exercise 6. Since $\xi(x)$ is an affine vector field, it suffices to show that the map (13) is injective. Assume that $x+\xi(x) = y+\xi(y)$ for some $x, y \in \mathbb{R}^d$. Then (12) implies that $x - \xi(x) = y - \xi(y)$, and by addition of these two equations we obtain $x = y$.

Thus, both the result of Exercise 6 and the assertion (14) can be derived from equation (10).

The following example shows that even if P' and P'' are congruent frameworks, Q can be a non-trivial infinitesimal motion of P.

Example 7. Let P' be a triangle $p_1'p_2'p_3'$, and let P'' be the mirror image of P' with respect to the line $p_1'p_2'$. The averaging yields a degenerate triangle and a non-trivial infinitesimal motion.

In fact, it is not hard to show that the isometry Φ constructed in the proof of Theorem 2 is orientation preserving. Thus if P' and P'' are congruent, but only through an orientation reversing isometry, then their average P is infinitesimally flexible. However, one can show that P lies in a hyperplane, and that the infinitesimal motion Q consists of vectors orthogonal to this hyperplane.

The following theorem is a partial converse to Theorem 2. Again, the first part is easy, but the second one causes some problems.

Theorem 1. *Let Q be an infinitesimal motion of a framework P. Then the frameworks P' and P'' defined by*

$$p_i' = p_i + q_i,$$
$$p_i'' = p_i - q_i,$$

are isometric.

Moreover, if Q is a non-trivial infinitesimal motion and if P affinely spans $\mathbb{R}^d$, then the frameworks P' and P'' are not congruent.

Proof. The transformation $(P, Q) \mapsto (P', P'')$ of Theorem 1 is the inverse of the transformation $(P', P'') \mapsto (P, Q)$ of Theorem 2. Thus equation (10) holds and implies that P' and P'' are isometric.

Assume that P' and P'' are congruent. Then we have

$$\|p'_i - p'_j\| = \|p''_i - p''_j\|$$

for all pairs of vertices i, j. Equation (10) implies that

$$\langle q_i - q_j, p_i - p_j \rangle = 0, \tag{15}$$

also for all pairs i, j. Since P affinely spans $\mathbb{R}^d$, there exist $d+1$ vertices $p_0, p_1, \ldots, p_d$ that form an affine basis of $\mathbb{R}^d$. It can be shown that equations (15) imply existence and uniqueness of an infinitesimal motion ξ of $\mathbb{R}^d$ such that

$$\xi(p_i) = q_i \tag{16}$$

for $i \in \{0, 1, \ldots, d\}$. By applying (15) to $i \notin \{0, 1, \ldots, d\}$ and $j \in \{0, 1, \ldots, d\}$, it is easy to show that (16) holds for all i. (See also the proof of Lemma 24.) Theorem is proved. □

Example 8. By deaveraging the framework from Example 16, we obtain two isometric non-congruent frameworks, see Figure 11.

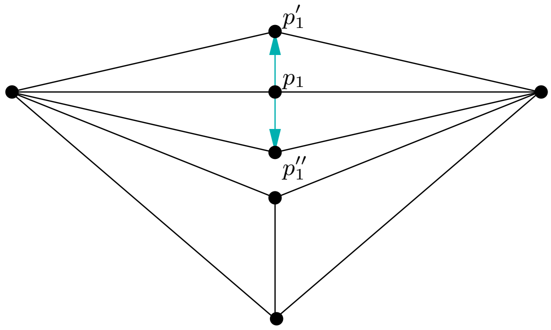

FIGURE 11. Deaveraging an infinitesimally flexible framework.

II. INFINITESIMAL RIGIDITY OF CONVEX POLYHEDRA

1. **Statement of the theorem.** A *convex polyhedron* in $\mathbb{R}^3$ is a bounded intersection of a finite number of half-spaces. The dimension of a polyhedron is the dimension of the affine space it spans. We consider only polyhedra of dimension 3, i. e. those that don't degenerate to a polygon, a segment or a point.

Denote by V, E, F the numbers of vertices, edges, and faces of a convex polyhedron. The following theorem is classical.

Theorem 3 (Euler formula)**.**

$$V - E + F = 2$$

The vertices and edges of a polyhedron form a framework. This framework is not rigid in general, as one can see on the example of a cube. However, if the faces are viewed as rigid panels, then the cube becomes rigid, and it is plausible to assume that so does every convex polyhedron. This is indeed true, and is the main theorem of this section.

Theorem 4. *Let ξ be an infinitesimal motion of a convex polyhedron such that the restriction of ξ to every face is a trivial infinitesimal motion. Then ξ itself is trivial.*

In other words: assume that with every face f_i an infinitesimal motion ξ_i of $\mathbb{R}^3$ is associated so that the motions of any two adjacent faces agree on their common edge. Then all the motions ξ_i are equal:

$$\xi_i|_{f_i \cap f_j} = \xi_j|_{f_i \cap f_j} \text{ for all } i, j \quad \Rightarrow \quad \xi_i = \xi_j.$$

2. **The proof.**

§2.1. *Sign changes around vertices.* Consider an infinitesimal motion of a convex polyhedron K. Mark each edge of K with $+$, $-$, or 0, according to whether the dihedral angle at this edge increases, decreases or remains constant during the motion.

Note that an edge is marked with 0 if and only if the motions of the two adjacent faces coincide. Our goal is to show that all edges are marked with 0.

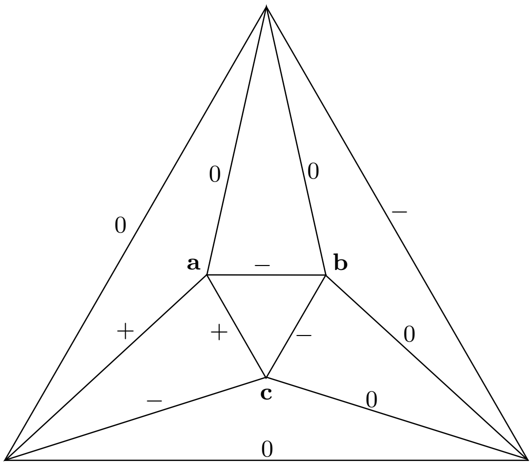

FIGURE 12. The skeleton of K with marked edges.

Lemma 1. *For every vertex of K, either all adjacent edges are marked with 0, or the number of sign changes encountered when going around the vertex is at least 4.*

Lemma 1 will be proved in Section 3.

A sign change is a plus (or minus) sign followed by a (possibly empty) sequence of zeros and then by a minus (respectively plus) sign. For example, on Figure 12 there are two sign changes around each of the vertices **a** and **c** and zero sign changes around the vertex **b**.

The marking on Figure 12 does not satisfy Lemma 1. It is easy to show that the only way to mark the edges of an octahedron so that to satisfy Lemma 1 is to assign all edges zero labels. Now we will show that this is true for the skeleton of every convex polyhedron.

§2.2. *Sign changes on faces.* Given a marking of K, an angle of a face of K is called *bad* if it has one side marked with $+$, and the other side marked with $-$. Let I be the total number of bad angles of all faces of K.

Lemma 2. *The total number of bad angles is at most* $4(E-F)$*:*

$$I \leq 4(E-F).$$

Proof. Every face has an even number of bad angles because as we go along its boundary, the sign must change an even number of times. Thus, a triangular face has at most 2 bad angles, a quadrangular or pentagonal one at most 4, and so on. By summing over all faces, we obtain

$$I \leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \cdots,$$

where F_n denotes the number of n-gonal faces of K.

On the other hand,

$$F_3 + F_4 + F_5 + F_6 + F_7 + \cdots = F,$$

and

$$3F_3 + 4F_4 + 5F_5 + 6F_6 + 7F_7 + \cdots = 2E.$$

Thus we have

$$\begin{aligned} I &\leq 2F_3 + 4F_4 + 4F_5 + 6F_6 + 6F_7 + \cdots \\ &\leq 2F_3 + 4F_4 + 6F_5 + 8F_6 + 10F_7 + \\ &= 2(F_3 + 2F_4 + 3F_5 + 4F_6 + 5F_7 + \cdots) \\ &= 2(2E - 2F). \end{aligned}$$

The lemma follows. ☐

Proof of Theorem 4. Assume that all edges of K are marked with $+$ and $-$, so that there are no zero labels at all.

By Lemma 1, at every vertex there are at least 4 bad angles, thus we have

$$I \geq 4V. \tag{17}$$

On the other hand, Theorem 3 and Lemma 2 imply

$$I \le 4V - 8$$

Thus there must be some edges marked with 0.

Remove all edges marked with 0. We obtain a graph Γ on the sphere, see Figure 13 which is obtained from Figure 12. The graph Γ subdivides the sphere into regions that we, as before, call faces. Note that a face can be non-simply connected, that is it can be homeomorphic to a disc with holes.

Denote by I the total number of bad angles of these newly obtained faces. Then, due to Lemma 1, we have inequality (17). Besides, Lemma 2 still holds, since all arguments in its proof remain valid. Note that now a face, say, with 6 edges can be either a hexagon or a triangle with a triangular hole. Also, an edge can be adjacent not to two different faces, but to a single face; in this case it counts twice in the boundary of this face. For example, on Figure 13 we have two triangular and one hexagonal face.

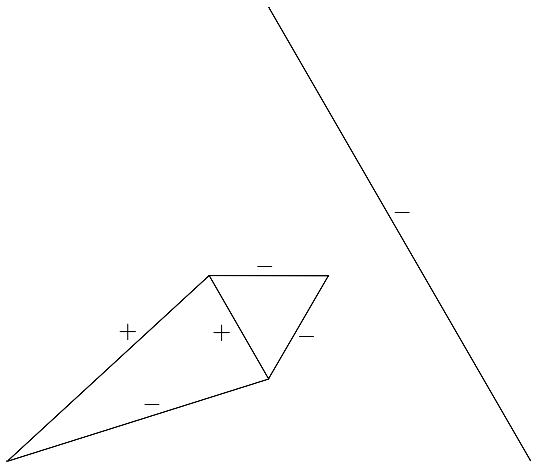

Figure 13. The graph Γ obtained after removing edges marked with 0.

Instead of Theorem 3 we now have

$$V - E + F = C + 1,$$

where C is the number of connected components of the graph Γ. Thus, Lemma 2 implies

$$I \le 4V - 4(C + 1)$$

that contradicts (17), as before. $\square$

3. **Sign changes around vertices: the proof.**

§3.1. *Infinitesimal motions of* $\mathbb{R}^3$. When we defined the marking of edges in §2.1, we spoke about dihedral angles as increasing, decreasing or remaining constant during an infinitesimal deformation. Let's give a precise meaning to this.

Exercise 7. Show that every infinitesimal motion of $\mathbb{R}^3$ has the form

$$\xi(x) = \rho \times x + b,$$

for some vectors $\rho, b \in \mathbb{R}^3$. Here × denotes the vector product.

If $b = 0$, then ξ is an infinitesimal rotation about the line spanned by vector ρ. The direction of the rotation is determined by the direction of ρ (clockwise rotation, if we look in the direction of ρ), the angular velocity is equal to the length of ρ. The vector ρ is called the *angular velocity vector* for the infinitesimal motion ξ.

Remark 3. In general, the vector field $x \mapsto \rho \times x + b$ is an infinitesimal screw motion: a rotation about a line l (which does not need to pass through the coordinate origin) plus a parallel translation along l.

Consider two infinitesimal motions ξ_1 and ξ_2 that agree on a line $l \ni 0$:

$$\xi_1\big|_l = \xi_2\big|_l.$$

Then their difference $\xi_2 - \xi_1$ is an infinitesimal motion that vanishes on l. Thus we have

$$\xi_2(x) - \xi_1(x) = \rho \times x,$$

where the vector ρ spans l.

Let two halfplanes H_1 and H_2 that pass through l be moved by ξ_1 and ξ_2, respectively. Then the vector field $\xi_2 - \xi_1$ is the infinitesimal motion of H_2 relative to H_1. Therefore we can say that H_2 infinitesimally rotates about

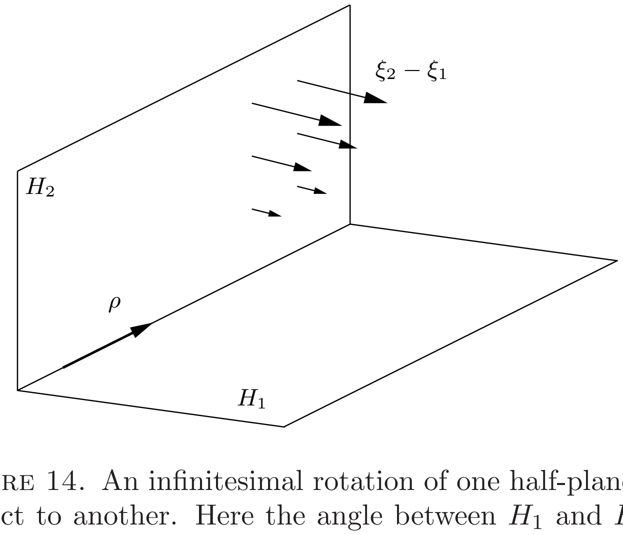

Figure 14. An infinitesimal rotation of one half-plane with respect to another. Here the angle between H_1 and H_2 decreases.

the line l with respect to H_1. The vector ρ shows the angular velocity of this rotation, thus we can tell from its direction whether the angle between H_1 and H_2 increases or decreases. See Figure 14.

§3.2. *Deformations of a polyhedral angle.*

Lemma 4. *Let an infinitesimal motion of a (not necessarily convex) polyhedral angle with edges $l_1, \dots, l_n$ be given. Denote by δ_i the infinitesimal change of the dihedral angle at l_i. Then*

$$\sum_{i=1}^{n} \delta_i v_i = 0,$$

where v_i is a unit vector along the edge l_i so that v_i points away from the angle's vertex.

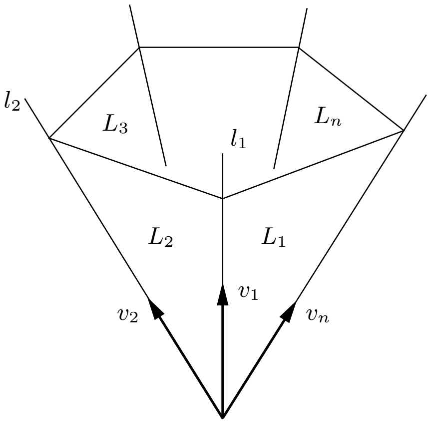

FIGURE 15. A polyhedral angle with unit vectors along the edges.

Proof. Denote by L_i the face of the angle spanned by the edges l_{i-1} and l_i. See Figure 15. Then by the discussion in §3.1,

$$\delta_i v_i = \rho_i$$

is the angular velocity vector of the infinitesimal rotation of the face L_{i+1} relative to the face L_i. This means that

$$\rho_i \times x = \xi_{i+1}(x) - \xi_i(x),$$

where ξ_i is the infinitesimal motion of L_i. By summing this for all i we obtain

$$(\rho_1 + \cdots + \rho_n) \times x = (\xi_2(x) - \xi_1(x)) + \cdots + (\xi_1(x) - \xi_n(x)) = 0,$$

for all x. It follows that

$$\sum_{i=1}^{n} \rho_i = 0,$$

and the lemma is proved. □

§3.3. *Deriving Lemma 1 from Lemma 4.*

Lemma 5. *Let a convex polyhedral angle with edges $l_1, \ldots, l_n$ be given, and let v_i be a unit vector along the edge l_i so that v_i points away from the angle's vertex. Assume that real numbers $a_1, a_2, \ldots, a_n$ are chosen so that*

$$\sum_{i=1}^{n} a_i v_i = 0. \tag{18}$$

Then either $a_i = 0$ for all i, or the numbers a_i change the sign at least four times when we go around the angle's vertex.

Proof. Assume that not all of a_i are zero, and that there are less than four sign changes.

If there are no sign changes, then either all of a_i are nonnegative or all nonpositive. Consider a plane through the angle's vertex such that the angle lies on one side of it. Then all vectors $a_i v_i$ lie on one side on this plane, and equation (18) is impossible.

If there are exactly two sign changes, then we can assume that, for some m, $a_1, a_2, \ldots, a_m \geq 0$ and $a_{m+1}, \ldots, a_n \leq 0$. There exists a plane S that separates the edges $l_1, \ldots, l_m$ from $l_{m+1}, \ldots, l_n$. Then all vectors $a_i v_i$ lie on one side of S, and equation (18) is again impossible. ☐

III. PROJECTIVE INVARIANCE OF INFINITESIMAL RIGIDITY

This section deals with two topics. On one hand, we discuss static rigidity. This concept has a clear physical motivation, and after a closer consideration turns out to be equivalent (more exactly, dual) to the infinitesimal rigidity.

On the other hand, we prove that infinitesimal rigidity is projectively invariant. That is, if a framework is infinitesimally rigid, then its image under any projective transformation is also infinitesimally rigid. This sounds very surprising, as infinitesimal rigidity deals with lengths, and projective transformations don't preserve lengths. (A brief introduction into projective geometry will be given.)

In fact, the first of our topics here, static rigidity, will be used as a tool in proving the projective invariance of infinitesimal rigidity.

1. **Statics.** If you push an object with one hand, the object moves. If you squeeze it between two hands, the object either changes or preserves its form. If the object changes its form, then it is not rigid. If the object is rigid, then it will preserve its form regardless of the direction in which it is squeezed or stretched, and regardless of the number of hands that exert their force on it.

A rigid object preserves its form due to stresses and tensions that arise inside of it. If the object succeeds to create these interior forces so that at every point they compensate the exterior forces, then no part of the object will move. We must, of course, require that the exterior forces are in equilibrium, otherwise the object can move as a rigid body. (For example, if it is pushed with one hand.)

In this section, we formalize these considerations and they will lead us to a definition of *static rigidity.* By establishing a duality between forces and infinitesimal motions, we will show that static rigidity is equivalent to infinitesimal rigidity.

§1.1. *Systems of forces.* Here we formalize the notion of force and define when a system of forces is in equilibrium.

Note that, in our situation, a force is not just a vector, and equilibrium is not just vanishing of the sum of a collection of vectors. For example, two opposite forces applied not along the same line but along two parallel lines are not in equilibrium, as they result in a rotation. (Just imagine two hand that rotate the steering wheel of a car.)

Thus, the application point of a force does matter. On the other hand, two equal forces acting along the same line should be considered as equal: if you pull a wire tied to an object, it does not matter, at which point you hold the wire. Therefore, a force should be considered as a *line-bound vector.*

Definition 1. *A* force *is a pair* $(p, f) \in \mathbb{R}^d \times \mathbb{R}^d$*, where* p *is viewed as a point, and* f *as a vector applied at* p*. A* system of forces *is a formal sum*

$$\sum_i \lambda_i (p_i, f_i) \tag{19}$$

with $\lambda_i \in \mathbb{R}$ *and a finite number of summands, that can be transformed according to the following rules:*

(a) *a zero vector produces a zero force:*

$$(p, 0) \sim 0;$$

(b) *forces applied at the same point can be added and scaled as usual:*

$$\lambda_1(p, f_1) + \lambda_2(p, f_2) \sim (p, \lambda_1 f_1 + \lambda_2 f_2);$$

(c) *a force can be moved along its line of action:*

$$(p, f) \sim (p + \lambda f, f).$$

Sums of the form (19) can be added to each other and multiplied with scalars. It is easy to see that these operations are compatible with the equivalence relations (a), (b), (c). Therefore, the set of all systems of forces in $\mathbb{R}^d$ forms a vector space.

Also note that due to $\lambda(p, f) \sim (p, \lambda f)$, any system of forces can be written in the form

$$\sum_i (p_i, f_i).$$

Exercise 8. Show that every system of forces in $\mathbb{R}^2$ is equivalent to either a single vector (p, f) or to a so called “couple”

$$(p_1, f) + (p_2, -f),$$

where the line $p_1 p_2$ is not parallel to f.

Also show that two couples are equivalent if and only if they have equal "momenta"

$$(p_1 - p_2) \times f \in \mathbb{R}.$$

Here $v \times w$ for $v, w \in \mathbb{R}^2$ denotes the signed area of a parallelogramm spanned by v and w.

As a consequence of Exercise 8, the space of systems of forces in $\mathbb{R}^2$ is three-dimensional, and the couples form a one-dimensional subspace of it.

Lemma 2. *The space of all systems of forces in $\mathbb{R}^d$ has dimension $\frac{d(d+1)}{2}$.*

Proof. Let $e_1, e_2, \ldots, e_d$ be the basis vectors of $\mathbb{R}^d$. We will show that every system of forces φ in $\mathbb{R}^d$ has a unique presentation in the form

$$\varphi \sim (e_1, f_1) + (e_2, f_2) + \cdots (e_d, f_d) \tag{20}$$

such that

$$f_1 \subset \mathbb{R}^1,\ f_2 \subset \mathbb{R}^2,\ \ldots,\ f_d \subset \mathbb{R}^d,$$

where by $\mathbb{R}^k$ we mean the linear space spanned by the first k basis vectors $e_1, e_2, \ldots, e_k$. This implies that the space of systems of forces in $\mathbb{R}^d$ is isomorphic to $\mathbb{R}^1 \oplus \mathbb{R}^2 \oplus \cdots \oplus \mathbb{R}^d \cong \mathbb{R}^{\frac{d(d+1)}{2}}$.

In order to show that every φ has a presentation of the form (20), it suffices to prove this in the case when φ is a single force (p, f). We do this by induction on d.

For $d = 1$ this is obvious: here $(p, f) \sim (e_1, f)$ for all p.

For $d > 1$, consider the subspace $\mathbb{R}^{d-1}$. We will distinguish three cases depending on the position of (p, f) with respect to $\mathbb{R}^{d-1}$.

First, if $p \in \mathbb{R}^{d-1}$ and $f \in \mathbb{R}^{d-1}$, then (p, f) can be viewed as a force in $\mathbb{R}^{d-1}$, and by the induction assumption we have

$$(p, f) \sim (e_1, f_1) + (e_2, f_2) + \cdots + (e_{d-1}, f_{d-1}). \tag{21}$$

Second, assume that f does not lie in $\mathbb{R}^{d-1}$. In this case the line through p spanned by f intersects $\mathbb{R}^{d-1}$ in a unique point p'. We have

$$(p, f) \sim (p', f) \sim (p', f') + (p', f_d) \sim (p', f') + (e_d, f_d),$$

where $f = f' + f_d$ is a decomposition of f into a component parallel to $e_d - p'$ and a component parallel to $\mathbb{R}^{d-1}$, see Figure 16. The force (p', f') has a presentation of the form (21) by the induction assumption.

The third and the last case is when f belongs to $\mathbb{R}^{d-1}$, but p lies outside $\mathbb{R}^{d-1}$. Then we write

$$(p, f) \sim (p, f - e_d) + (p, e_d),$$

and the argument from the second case can be applied to decompose the summands on the right hand side.

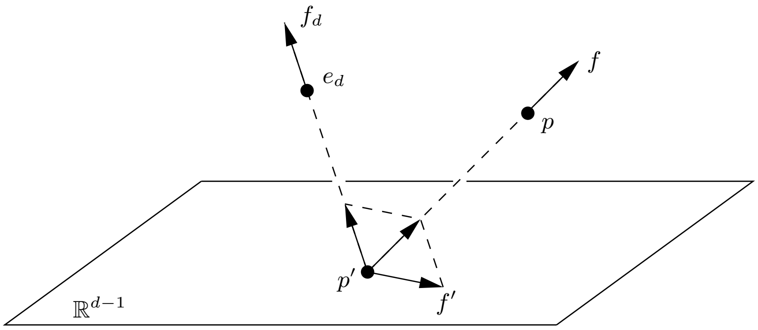

FIGURE 16. Induction step in the proof of Lemma 2.

Thus the existence of a decomposition (20) is proved. To prove the uniqueness, we need to show that

$$(e_1, f_1) + (e_2, f_2) + \cdots + (e_d, f_d) \sim 0 \tag{22}$$

implies $f_i = 0$ for all i. From (22) it follows that

$$(e_1, f_1) + (e_2, f_2) + \cdots + (e_{d-1}, f_{d-1}) \sim (e_d, -f_d)$$

which implies $f_d = 0$ since all forces on the left hand side live in $\mathbb{R}^{d-1}$. By repeating this argument, we obtain $f_{d-1} = 0$, and so on. The lemma is proved. $\square$

§1.2. *Duality between systems of forces and infinitesimal motions.* Let us introduce the following notation:

$$\begin{aligned} \mathcal{M}(\mathbb{R}^d) &:= \{\text{infinitesimal motions of } \mathbb{R}^d\}; \\ \mathcal{F}(\mathbb{R}^d) &:= \{\text{systems of forces in } \mathbb{R}^d\}. \end{aligned}$$

Both $\mathcal{M}(\mathbb{R}^d)$ and $\mathcal{F}(\mathbb{R}^d)$ are vector spaces, besides we know that they have the same dimension $\frac{d(d+1)}{2}$. The next theorem says much more.

Theorem 5. *The space $\mathcal{F}(\mathbb{R}^d)$ is canonically isomorphic to the dual of the space $\mathcal{M}(\mathbb{R}^d)$:*

$$\mathcal{F}(\mathbb{R}^d) \cong (\mathcal{M}(\mathbb{R}^d))^*.$$

Proof. For brevity, let us omit $\mathbb{R}^d$ from the notation.

Consider a canonical pairing (that is, a bilinear map)

$$W : \mathcal{M} \times \mathcal{F} \to \mathbb{R},$$

$$W(\xi, \varphi) = \sum_i \langle \xi(p_i), f_i \rangle,$$

where $\varphi = \sum_i (p_i, f_i)$. Let us show that W is well-defined, that is the value $W(\xi, \varphi)$ does not depend on the choice of a representation of φ as a sum of forces. For this, we need to show that $\sum_i \langle \xi(p_i), f_i \rangle$ does not change when

$\sum_i (p_i, f_i)$ is transformed according to the rules (a), (b), (c) from Definition 1. Invariance with respect to (a) and (b) is obvious. For (c), we need to show

$$\langle \xi(p), f \rangle = \langle \xi(p + \lambda f), f \rangle.$$

This follows immediately from $\langle \xi(x) - \xi(y), x - y \rangle = 0$ by substituting $x = p + \lambda f$, $y = p$.

The pairing W defines a canonical linear map

$$\begin{array}{rccc} \widehat{W}: & \mathcal{M} & \to & \mathcal{F}^*, \\ & \xi & \mapsto & W(\xi, \cdot), \end{array}$$

where $W(\xi, \cdot)$ is a functional on $\mathcal{F}$ that sends φ to $W(\xi, \varphi)$. We will show that $\widehat{W}$ is one-to-one.

First, let us show that $\widehat{W}$ is injective: for every $\xi \neq 0$ the functional $W(\xi, \cdot)$ is not zero. That is, we need to show that for every $\xi \neq 0$ there exists $\varphi \in \mathcal{F}$ such that $W(\xi, \varphi) \neq 0$. This is indeed true, since $\xi \neq 0$ means that there is $p \in \mathbb{R}^d$ such that $\xi(p) \neq 0$. So we can put $\varphi = (p, \xi(p))$ and then we have

$$W(\xi, \varphi) = \langle \xi(p), \xi(p) \rangle \neq 0.$$

Since the dimensions of $\mathcal{M}$ and $\mathcal{F}$ coincide, the injectivity of $\widehat{W}$ implies that it is one-to-one. Thus $\widehat{W}$ is a canonical linear isomorphism. Its adjoint is a canonical linear isomorphism between $\mathcal{F}$ and $\mathcal{M}^*$. The theorem is proved. □

Remark 3. There is no canonical isomorphism between $\mathcal{F}(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$. This can be seen on the example of $d = 2$. The only natural choice would be to identify a single force with a translation. But the map that sends (p, f) to the constant vector field $\xi(x) = f$ is not injective. Also, if we extend it by linearity, the couple $(p_1, f) + (p_2, -f)$ will be mapped to 0.

Since $\mathcal{F}(\mathbb{R}^d)$ and $\mathcal{M}(\mathbb{R}^d)$ are canonically dual, the absence of a canonical isomorphism between them is equivalent to the absence of a canonical scalar product on each of them. The isomorphism

$$\mathcal{M}(\mathbb{R}^d) \cong \mathfrak{so}(d) \oplus \mathbb{R}^d$$

defines a scalar product on $\mathcal{M}(\mathbb{R}^d)$, but this scalar product depends on the choice of coordinate origin in $\mathbb{R}^d$. We view the Euclidean space $\mathbb{R}^d$ as an empty space without distinguished points.

To be more exact, we have two versions of the Euclidean space: one withouth the origin, this is the space where the frameworks live, and whose infinitesimal motions we consider. This is the space of points and should rather be denoted by $\mathbb{E}^d$. Sometimes we choose a coordinate origin in $\mathbb{E}^d$, in order to write formulas like $\xi(x) = Zx + b$. The other one is the standard Euclidean space $\mathbb{R}^d$ which is a linear space with a positively definite scalar product. This is the space where the vectors of infinitesimal motions and those of forces take values.

Exercise 9. In the linear space of infinitesimal motions, consider the subspace of translations:

$$\mathcal{M} \supset \mathcal{T} = \{\xi \mid \xi(x) = \text{const}\}.$$

Show that

$$\mathcal{T}^\perp = \Big\{\varphi = \sum_i (p_i, f_i) \mid \sum_i f_i = 0\Big\}.$$

Exercise 10. Let l be a line in $\mathbb{R}^3$ passing through 0. Denote by $\mathcal{R}_l \subset \mathcal{M}$ the vector space of infinitesimal rotations about l. Show that

$$\mathcal{R}_l^\perp = \Big\{\varphi = \sum_i (p_i, f_i) \mid \sum_i p_i \times f_i \parallel l\Big\}.$$

Exercise 11. Show that in $\mathbb{R}^3$,

$$\sum_i (p_i, f_i) \sim 0 \quad \Longleftrightarrow \quad \sum_i f_i = 0 \text{ and } \sum_i p_i \times f_i = 0.$$

§1.3. *Static rigidity.* Let P be a framework in $\mathbb{R}^d$.

Definition 4. *A* load *on the framework P is a collection of forces applied at the vertices of a framework. Formally, a load is a map*

$$\begin{array}{rccl} F: & \mathcal{V} & \to & \mathbb{R}^d, \\ & i & \mapsto & f_i. \end{array}$$

A load is called an equilibrium load*, if the system of forces $\sum_{i\in\mathcal{V}}(p_i, f_i)$ is equivalent to zero.*

Example 5. Let P be a triangle in $\mathbb{R}^2$. A load that consists of three nonzero forces f_1, f_2, f_3 is an equilibrium load if and only if the lines along which the forces act intersect at a point and $f_1 + f_2 + f_3 = 0$. See Figure 17.

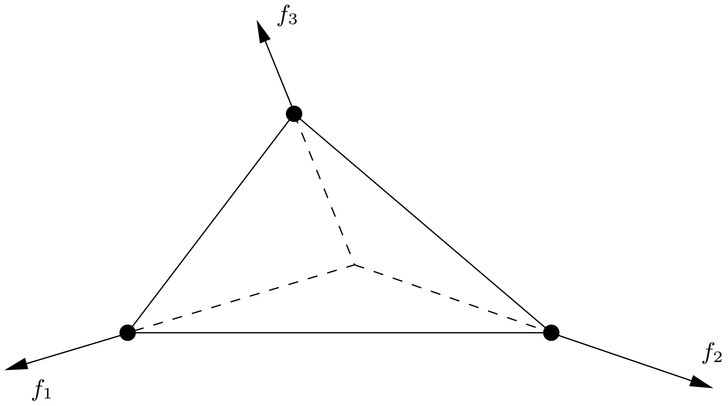

FIGURE 17. An equilibrium load on a triangle framework in $\mathbb{R}^2$.

A rigid body responds to an equilibrium load by interior forces that cancel the exterior forces of the load.

Definition 6. *A* stress *on the framework P is a map*

$$\begin{array}{rccc} \Omega: & \mathcal{E} & \to & \mathbb{R}, \\ & ij & \mapsto & \omega_{ij}. \end{array}$$

The stress Ω is said to resolve *the load F, if*

$$f_i + \sum_{j \in \mathcal{V}} \omega_{ij}(p_i - p_j) = 0 \text{ for all } i \in \mathcal{V}, \tag{23}$$

where we assume $\omega_{ij} = 0$ for all $ij \notin \mathcal{E}$.

The inequality $\omega_{ij} > 0$ means that the edge ij is under compression, so that it pushes the vertices i and j apart. The inequality $\omega_{ij} < 0$ means that the edge ij is under tension, so that it pulls the vertices i and j towards each other.

Example 7. Figure 18 shows a stress on a triangle framework and a load resolved by this stress. The interior forces $\omega_{ij}(p_i - p_j)$ are shown in grey.

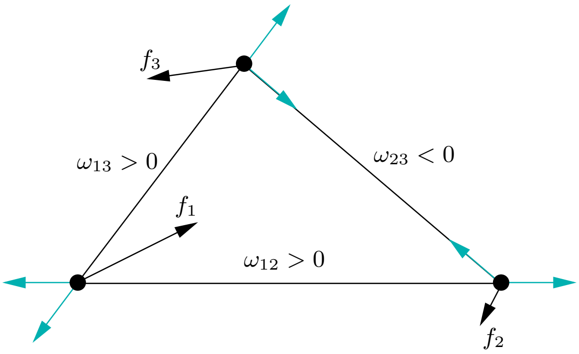

FIGURE 18. A resolvable load on a triangle framework.

Lemma 8. *Every resolvable load is an equilibrium load.*

Proof. We have

$$\begin{aligned} \sum_i (p_i, f_i) &= \sum_i \Big(p_i, \sum_j \omega_{ij}(p_j - p_i)\Big) \\ &= \sum_{ij} \Big(p_i, \omega_{ij}(p_j - p_i)\Big) + \Big(p_j, \omega_{ij}(p_i - p_j)\Big) = 0. \end{aligned}$$

□

Definition 9. *The framework P is called* statically rigid, *if every equilibrium load on P can be resolved.*

Example 10. A quadrilateral is not statically rigid. Indeed, consider two forces that act on two opposite vertices along the diagonal and cancel each other, see Figure 19. This defines an equilibrium load. Assume that there is a stress that resolves this load. Then the stresses on all edges are uniquely determined by the condition that they should compensate the given forces. But at the other two vertices, where zero exterior forces are applied, the stresses create non-zero interior forces. Thus the equation (23) does not hold there.

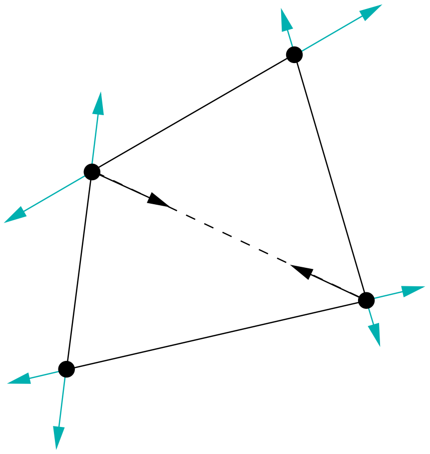

FIGURE 19. An equilibrium non-resolvable load on a quadrilateral.

2. **Equivalence between infinitesimal and static rigidity.**

§2.1. *Main theorem.* With every framework P the following vector spaces are associated:

$$\begin{array}{lcl} \mathcal{Q} & = & \{\text{infinitesimal motions of } P\}; \\ \mathcal{Q}_0 & = & \{\text{trivial infinitesimal motions of } P\}; \\ \mathcal{L} & = & \{\text{equilibrium loads on } P\}; \\ \mathcal{L}_0 & = & \{\text{resolvable loads on } P\}. \end{array}$$

We have inclusions

$$\begin{array}{lcl} \mathcal{Q}_0 & \subset & \mathcal{Q}; \\ \mathcal{L}_0 & \subset & \mathcal{L}. \end{array}$$

When the first inclusion is an equality, the framework is infinitesimally rigid, when the second inclusion is an equality, the framework is statically rigid. More generally, we can form quotient vector spaces $\mathcal{Q}/\mathcal{Q}_0$ and $\mathcal{L}/\mathcal{L}_0$ and call $\dim \mathcal{Q}/\mathcal{Q}_0$ the *number of kinematic degrees of freedom* and $\dim \mathcal{L}/\mathcal{L}_0$ the *number of static degrees of freedom.*

Theorem 6. *A framework is infinitesimally rigid if and only if it is statically rigid.*

More generally, there is a canonical isomorphism

$$\mathcal{Q}/\mathcal{Q}_0 \cong (\mathcal{L}/\mathcal{L}_0)^*.$$

In particular, for every framework the number of kinematic degrees of freedom is equal to the number of static degrees of freedom.

Proof. Consider the space $(\mathbb{R}^d)^{\mathcal{V}}$ of "vector fields" on P:

$$(\mathbb{R}^d)^{\mathcal{V}} = \{X : \mathcal{V} \to \mathbb{R}^d\}.$$

(Recall that $\mathcal{V}$ denotes the set of vertices of the underlying graph G of P.) In particular, the vector spaces $\mathcal{Q}$ and $\mathcal{L}$ are subspaces of $(\mathbb{R}^d)^{\mathcal{V}}$. On $(\mathbb{R}^d)^{\mathcal{V}}$, there is a canonical scalar product

$$\langle X, Y\rangle = \sum_{i\in\mathcal{V}} \langle x_i, y_i\rangle. \tag{24}$$

By applying Lemma 12 to the subspaces

$$\mathcal{L}_0 \subset \mathcal{L} \subset (\mathbb{R}^d)^{\mathcal{V}}$$

and taking into account Lemma 11, we obtain

$$\mathcal{Q}/\mathcal{Q}_0 = \mathcal{L}_0^\perp/\mathcal{L}^\perp \cong (\mathcal{L}/\mathcal{L}_0)^*.$$

□

Lemma 11. *With respect to the scalar product* (24)*, we have*

(a) $\mathcal{Q} = \mathcal{L}_0^\perp$;
(b) $\mathcal{Q}_0 = \mathcal{L}^\perp$.

Proof. The space $\mathcal{L}_0$ of resolvable loads is spanned by the loads $(F^{ij})_{ij\in\mathcal{E}}$ with components

$$\begin{array}{rcl} f_i^{ij} & = & p_i - p_j; \\ f_j^{ij} & = & p_j - p_i; \\ f_k^{ij} & = & 0 \quad \text{for } k \ne i, j. \end{array}$$

Indeed, a load resolved by a stress Ω can be written as

$$F = -\sum_{ij\in\mathcal{E}} \omega_{ij} F^{ij}.$$

It follows that

$$Q \in \mathcal{L}_0^\perp \iff \langle Q, F^{ij}\rangle = 0 \text{ for all } ij \in \mathcal{E}.$$

On the other hand, for every $Q \in (\mathbb{R}^d)^{\mathcal{V}}$ we have

$$\langle Q, F^{ij}\rangle = \langle q_i, p_i - p_j\rangle + \langle q_j, p_j - p_i\rangle = \langle q_i - q_j, p_i - p_j\rangle.$$

Thus we have $\langle Q, F^{ij}\rangle = 0$ for all edges ij if and only if Q is an infinitesimal motion. This proves the assertion (a) of the lemma.

In order to prove (b), first consider an equilibrium load F and a trivial infinitesimal motion Q. By definition, the system of forces corresponding to F is equivalent to zero:

$$\varphi = \sum_i (p_i, f_i) \sim 0,$$

and the vector field Q can be extended to an infinitesimal motion of $\mathbb{R}^d$:

$$\exists \xi \in \mathcal{M}(\mathbb{R}^d) \text{ such that } q_i = \xi(p_i) \text{ for all } i.$$

Thus we have

$$\langle Q, F\rangle = W(\xi, \varphi) = W(\xi, 0) = 0.$$

Here W is the pairing introduced in §1.2, and we use that W is well-defined. It follows that $\mathcal{Q}_0 \subset \mathcal{L}^\perp$.

In order to prove the inverse inclusion $\mathcal{L}^\perp \subset \mathcal{Q}_0$, consider an arbitrary vector field $Q \in \mathcal{L}^\perp$. Then Q is orthogonal to all loads F^{ij}, where i, j can now be an arbitrary pair of vertices, not necessarily joined by an edge. It follows that

$$\langle q_i - q_j, p_i - p_j\rangle = 0 \text{ for all } i, j.$$

If the framework P does not lie in a hyperplane, then this implies that Q can be extended to an infinitesimal motion of $\mathbb{R}^d$, thus $Q \in \mathcal{Q}_0$. If P does lie in a hyperplane, an extra argument is needed that uses forces not contained in the affine span of P. $\square$

Lemma 12. *For every pair of subspaces of* $\mathbb{R}^N$

$$W \subset V \subset \mathbb{R}^N,$$

the scalar product on $\mathbb{R}^N$ *incuces an isomorphism*

$$W^\perp / V^\perp \cong (V/W)^*.$$

Proof. Any element of $W^\perp/V^\perp$ can be written as $x + V^\perp$ for some $x \in W^\perp$, and any element of V/W can be written as $y + W$ for some $y \in V$. The pairing

$$\langle x + V^\perp, y + W\rangle = \langle x, y\rangle$$

is well-defined because of $V^\perp \subset W^\perp$. Also, this pairing is non-degenerate since for every $x \notin V^\perp$ there exists $y \in V$ such that $\langle x, y\rangle \neq 0$. $\square$

§2.2. *Rigidity matrix.*

Definition 13. *The rigidity matrix of a framework* P *is*

$$R(P) = \begin{array}{cc} & \begin{array}{ccc} & ij & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \left(\begin{array}{ccc} & \vdots & \\ \cdots & p_i - p_j & \cdots \\ & \vdots & \\ \cdots & p_j - p_i & \cdots \\ & \vdots & \end{array}\right) \end{array}.$$

That is, the rows and columns of $R(P)$ are indexed by the vertices, respectively edges, of P. The only non-zero entries in the column ij are on its intersection with rows i and j.

For brevity, we will usually write R instead of $R(P)$.

The entries of R are elements $\mathbb{R}^d$, so that R should be viewed as the matrix of a linear map

$$R : \mathbb{R}^{\mathcal{E}} \to (\mathbb{R}^d)^{\mathcal{V}}.$$

The transpose $R^\top$ maps $(\mathbb{R}^d)^{\mathcal{V}}$ to $\mathbb{R}^{\mathcal{E}}$: an element X of $(\mathbb{R}^d)^{\mathcal{V}}$ is a "vector with vector components", and we use the scalar product in $\mathbb{R}^d$ when multiplying entries of R with components of X.

Lemma 14. *The following equalities hold:*

(a) $\mathcal{L}_0 = \operatorname{im} R$;
(b) $\mathcal{Q} = \ker R^\top$.

Proof. The columns of R are the basic resolvable loads F^{ij} defined in the first part of the proof of Lemma 11. Thus, Lemma 14 was actually proved there: it is just another way of saying that $\mathcal{L}_0$ is spanned by $\{F^{ij}\}$ and that orthogonality with respect to F^{ij} means preserving the length of the edge p_ip_j in the first order. ☐

The following lemma shows how the rigidity matrix can be used to determine whether a given framework is infinitesimally rigid.

Lemma 15. *Let P be a framework in $\mathbb{R}^d$ that does not lie in a hyperplane. Then P is infinitesimally rigid if and only if*

$$\operatorname{rk} R = dV - \frac{d(d+1)}{2}, \tag{25}$$

where V is the number of vertices of P.

Proof. The rank of a linear map is equal to the dimension of the domain minus the dimension of the kernel. By applying this to the transpose $R^\top$, we obtain

$$\operatorname{rk} R = \operatorname{rk} R^\top = dV - \dim \ker R^T = dV - \dim \mathcal{Q}.$$

Infinitesimal rigidity is equivalent to the equality $\mathcal{Q} = \mathcal{Q}_0$. Since P does not lie in a hyperplane, every trivial infinitesimal motion of P has a unique extension to an infinitesimal motion of $\mathbb{R}^d$. Thus we have

$$\dim \mathcal{Q}_0 = \dim \mathcal{M}(\mathbb{R}^d) = \frac{d(d+1)}{2}.$$

By combining this with the previous equality, we obtain (25). ☐

Lemma 16. *For every infinitesimally rigid framework in $\mathbb{R}^d$, we have*

$$E \geq dV - \frac{d(d+1)}{2},$$

where V is the number of vertices, and E is the number of edges of P.

Proof. If a framework is infinitesimally rigid, then it does not lie in a hyperplane. Thus Lemma 15 can be applied, and equation (25) holds. On the other hand, we have

$$\operatorname{rk} R \leq E,$$

because E is the number of columns of R. The lemma follows. □

Definition 17. *A* self-stress *is a stress that resolves a zero load:*

$$\sum_j \omega_{ij}(p_i - p_j) = 0 \quad \textit{for all } i.$$

Physically, a self-stress means that if we replace each positively stressed edge by a spring, each negatively stressed edge by a cable, and omit the non-stressed edges, then the resulting construction will be in equilibrium.

Example 18. Every quadrilateral with diagonals can be self-stressed. See Figure 20.

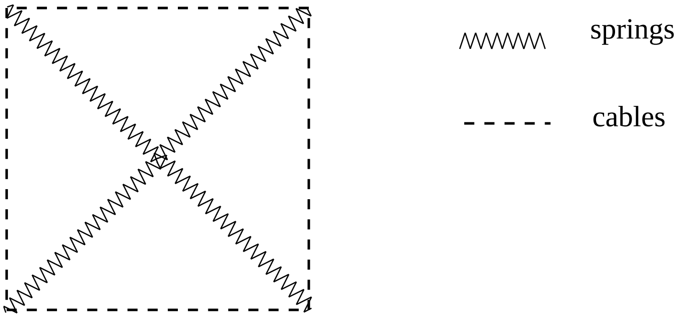

FIGURE 20. A self-stressed framework.

Lemma 19. *Let P be a framework in $\mathbb{R}^d$ that does not lie in a hyperplane and such that*

$$E = dV - \frac{d(d+1)}{2}. \tag{26}$$

Then P is infinitesimally rigid if and only if P has no self-stresses.

Proof. Note that a self-stress is an element of the kernel of the linear map

$$R : \mathbb{R}^{\mathcal{E}} \to (\mathbb{R}^d)^{\mathcal{V}}.$$

Lemma 15 and equation (26) imply that P is infinitesimally rigid if and only if R has rank E, thus if and only if the kernel of R is empty. The lemma is proved. □

Exercise 12. Consider the planar framework on Figure 21. Show that if the lines p_1p_4, p_2p_5, p_3p_6 intersect, then the framework has a self-stress.

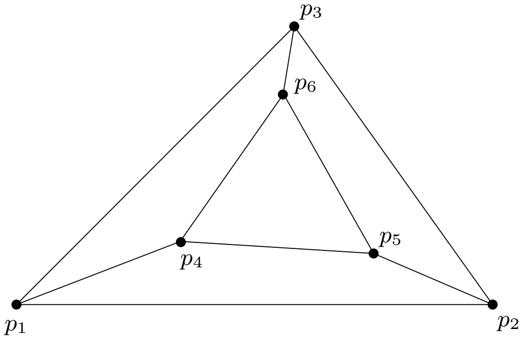

FIGURE 21

§2.3. *Static rigidity of convex polyhedra.* Although the following theorem follows from the infinitesimal rigidity of convex polyhedra and from the equivalence between infinitesimal and static rigidity, it is interesting to look at its direct proof.

Theorem 7. *Let K be a convex polyhedron in $\mathbb{R}^3$ such that all its faces are triangles. Then the skeleton of K is statically rigid.*

Proof. First, we need the following lemma.

Lemma 20. *For a convex polyhedron in $\mathbb{R}^3$ with all faces triangles, we have*

$$E = 3V - 6. \tag{27}$$

Proof. By the Euler formula, we have

$$V - E + F = 2.$$

Besides, since every face has three edges and every edge belongs to two faces, we have

$$3F = 2E.$$

By substituting $F = \frac{2}{3}E$ in the Euler formula, we obtain (27). ☐

As we have $d = 3$, Lemma 19 implies that the skeleton of K is statically rigid if and only if it has no self-stresses.

Lemma 21. *Let Ω be a self-stress on the skeleton of K. Then for every vertex of K, the stresses on the adjacent edges are either all zero or change the sign at least four times as we go around the vertex.*

Proof. Consider the vertex p_i of the polyhedron K. Let $v_{ij} = \frac{p_j - p_i}{\|p_i - p_j\|}$ be the unit vector at p_i directed along the edge $p_i p_j$. Then by definition of a self-stress, we have

$$\sum_j \omega_{ij} \|p_i - p_j\| \cdot v_{ij} = 0.$$

Lemma 5 from Chapter II implies that the numbers $\omega_{ij}\|p_i - p_j\|$ (and hence the numbers ω_{ij}, too) either are all zero or change the sign at least four times. The lemma is proved. □

Now, it remains to apply the argument from Section 2 of Chapter II to show that all stresses must be zero. Thus the skeleton of K has no self-stresses, and Theorem 7 is proved. □

For an infinitesimally flexible non-convex polyhedron, there is the following surprising relation between its infinitesimal motions and self-stresses of its skeleton.

Lemma 22. *Given an infinitesimal motion of a polyhedron, let δ_{ij} be the infinitesimal change of the dihedral angle at the edge p_ip_j. Then the numbers*

$$\omega_{ij} = \frac{\delta_{ij}}{\|p_i - p_j\|}$$

form a self-stress of the skeleton of the polyhedron.

Proof. This is a direct consequence of the equation

$$\sum_j \delta_{ij} v_{ij} = 0$$

proved in Lemma 4 of Chapter II. □

§2.4. *Affine invariance of infinitesimal rigidity.*

Definition 23. *An* affine transformation *of $\mathbb{R}^d$ is a map*

$$\Phi : \mathbb{R}^d \to \mathbb{R}^d$$

given by

$$\Phi(x) = Ax + b,$$

where $A \in \mathrm{GL}(d, \mathbb{R})$, $b \in \mathbb{R}^d$.

Remark 24. The linear transformation A is called the *linear part* of Φ. It is easy to see that A does not depend on the choice of a coordinate origin in $\mathbb{R}^d$.

Remark 25. Affine transformations map lines to lines. This is their characteristic property: for $d \geq 2$, every homeomorphism of $\mathbb{R}^d$ that maps lines to lines is an affine transformation.

Theorem 8. *Let P be a framework in $\mathbb{R}^d$, and let Φ be an affine transformation of $\mathbb{R}^d$. Then P is statically rigid if and only if the framework $\Phi \circ P$ is statically rigid.*

Proof. Let $A \in \mathrm{GL}(d, \mathbb{R})$ be the linear part of Φ. For every load F on P, consider the load AF on $\Phi \circ P$. That is, a force f applied at a point p is transformed into a force Af at $\Phi(p)$, see Figure 22.

Lemma 26 implies that the transformation $F \mapsto AF$ maps $\mathcal{L}(P)$ onto $\mathcal{L}(\Phi \circ P)$ and $\mathcal{L}_0(P)$ onto $\mathcal{L}_0(\Phi \circ P)$. Thus $\mathcal{L}(P) = \mathcal{L}_0(P)$ holds if and only if holds $\mathcal{L}(\Phi \circ P) = \mathcal{L}_0(\Phi \circ P)$, and the theorem is proved. □

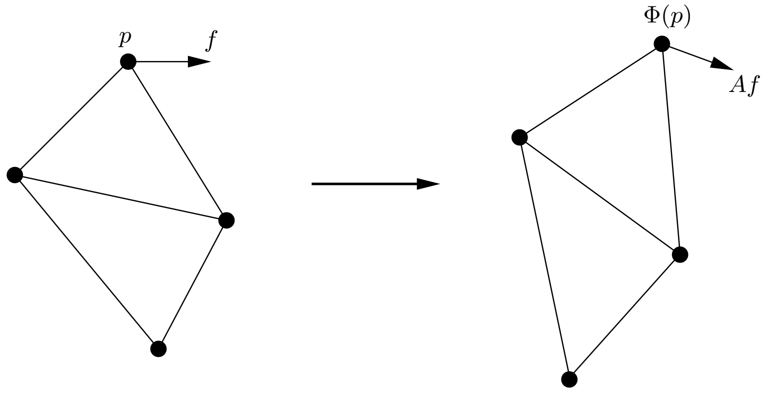

FIGURE 22. Transformation of forces under an affine map.

Lemma 26. *Let P be a framework in $\mathbb{R}^d$, and let Φ be an affine transformation of $\mathbb{R}^d$ with linear part A. Then*

(a) *F is an equilibrium load on P if and only if AF is an equilibrium load on $\Phi \circ P$;*
(b) *F is a resolvable load on P if and only if AF is a resolvable load on $\Phi \circ P$.*

Proof. The linear part of Φ^{-1} is A^{-1}. Therefore it suffices to prove only the implications in one direction, that is if F is in equilibrium, respectively resolvable, then AF is equilibrium, respectively resolvable.

For the first statement of the lemma, we need to prove

$$\sum_i (p_i, f_i) \sim 0 \quad \Rightarrow \quad \sum_i (\Phi(p_i), Af_i) \sim 0.$$

The equivalence $\sum_i (p_i, f_i) \sim 0$ means that the expression $\sum_i (p_i, f_i)$ can be transformed to 0 by means of the elementary equivalences (a), (b), (c) from Definition 1 in §1.1. Therefore it suffices to show that if to both sides of an elementary equivalence the transformation $(p, f) \mapsto (\Phi(p), Af)$ is applied, then we obtain two equivalent expressions.

For equivalences (a) and (b) this follows directly from the linearity of the map A. Let us check this for (c):

$$(\Phi(p + \lambda f), Af) \sim (A(p + \lambda f) + b, Af) \sim (\Phi(p) + \lambda Af, Af) \sim (\Phi(p), Af).$$

Thus the first statement is proved.

To prove the second statement of the lemma, we just show that if a load F on P is resolved by a stress Ω, then the same Ω resolves the load AF on $\Phi \circ P$. Indeed, if we have

$$f_i + \sum_j \omega_{ij}(p_i - p_j) = 0$$

for all i, then we have

$$\begin{aligned} Af_i + \sum_j \omega_{ij}(\Phi(p_i) - \Phi(p_j)) &= Af_i + \sum_{ij} \omega_{ij} A(p_i - p_j) \\ &= A(f_i + \sum_{ij} \omega_{ij}(p_i - p_j)) = 0. \end{aligned}$$

□

Theorem 9. *Let P be a framework in $\mathbb{R}^d$, and let Φ be an affine transformation of $\mathbb{R}^d$. Then P is infinitesimally rigid if and only if the framework $\Phi \circ P$ is infinitesimally rigid.*

Proof. This is a direct consequence of Theorem 8 and of the equivalence between static and infinitesimal rigidity, Theorem 6. □

Remark 27. It is not surprising that the static rigidity is affinely invariant: the definition of the static rigidity uses only linear operations on vectors, and the proof of Theorem 8 just consequently uses this. By contrast, the affine invariance of the infinitesimal rigidity is a surprise, since the definition of the infinitesimal rigidity uses the notion of the distance (or that of the Euclidean scalar product), and the distances are not preserved under affine transformations.

The infinitesimal motions of an affine image $\Phi \circ P$ of the framework P can be deduced from the infinitesimal motions of P, as the next lemma shows. Duality between forces and motions gives the key to the formula.

Lemma 28. *Let Q be an infinitesimal motion of a framework P, and let Φ be an affine transformation of $\mathbb{R}^d$ with a linear part A. Then $(A^{-1})^\top Q$ is an infinitesimal motion of $\Phi \circ P$. Besides, if Q is a trivial infinitesimal motion, then so is $(A^{-1})^\top Q$.*

Proof. Let us use the equation $\mathcal{Q}(P) = \mathcal{L}_0^\perp(P)$. Thus, if Q is an infinitesimal motion of P, then we have

$$\langle Q, F \rangle = 0$$

for all resolvable loads F on P. Therefore

$$\begin{aligned} \big\langle (A^{-1})^\top Q, AF \big\rangle &= \sum_i \big\langle (A^{-1})^\top q_i, Af_i \big\rangle = \sum_i (q_i^\top A^{-1})(Af_i) \\ &= \sum_i q_i^\top f_i = \sum_i \langle q_i, f_i \rangle = \big\langle Q, F \big\rangle = 0. \end{aligned}$$

Since $\mathcal{L}_0(\Phi \circ P) = A\mathcal{L}_0(P)$, it follows

$$(A^{-1})^\top Q \in \mathcal{L}_0(\Phi \circ P)^\top = \mathcal{Q}(\Phi \circ P).$$

To show that triviality of Q implies triviality of $(A^{-1})^\top Q$, one applies the same argument with $\mathcal{L}$ in place of $\mathcal{L}_0$. □

For example, if a framework in $\mathbb{R}^2$ undergoes an affine transformation $(x, y) \mapsto (2x, y)$, then the vectors of an infinitesimal motion must be transformed according to the rule $(z, t) \mapsto (\frac{z}{2}, t)$.

Exercise 13. Prove Lemma 28 directly, by using the $\langle q_i - q_j, p_i - p_j\rangle = 0$ description of infinitesimal motions.

3. Projective invariance of infinitesimal rigidity.

§3.1. *Projective geometry.* The development of the projective geometry was motivated by the artists' desire to make their paintings look real. To our eye, distant objects seem smaller that those close to us, angles become distorted. In order to reproduce the perspective distortion on the paper or canvas in a reliable way, a study of projective geometry was needed.

Let us consider a simple example. The rails of a rail track (of an ideal rail track) are two parallel lines. However, for our eye they look like two lines that intersect on the horizon. See Figure 23.

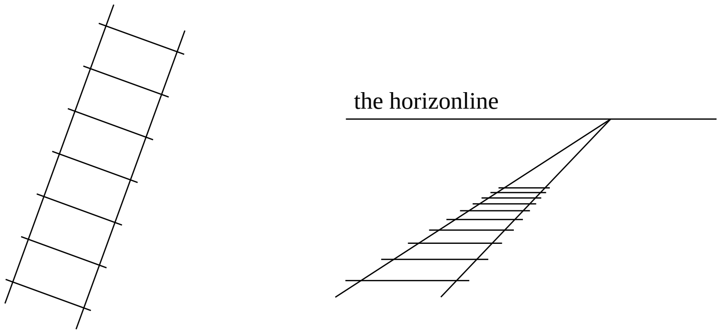

Figure 23. A railroad and how do we see it.

A simplified explanation of what happens is as follows. The light rays pass through the eye lens and project the picture on the retina. Thus the image that we see is the central projection of the object. If, for simplicity, we consider only flat objects lying on the earth surface and assume that the retina is flat, then we deal with a central projection from a plane to a plane, see Figure 24.

Example 29. The image of a circle under a central projection is an ellipse, or a parabola, or a hyperbola. Indeed, the lines Ox, where x lies on a circle in the plane Π, form a cone (an elliptic cone in general). The image of the circle is the intersection of this cone with the plane Π', thus a conic section.

If the planes Π and Π' are parallel, then the central projection is just a similarity: it increases all distances by the same factor. Otherwise, the central projection is defined not over all the plane Π and its image is not the whole plane Π'. Indeed, let $l \subset \Pi$ be a line parallel to Π', and $l' \subset \Pi'$

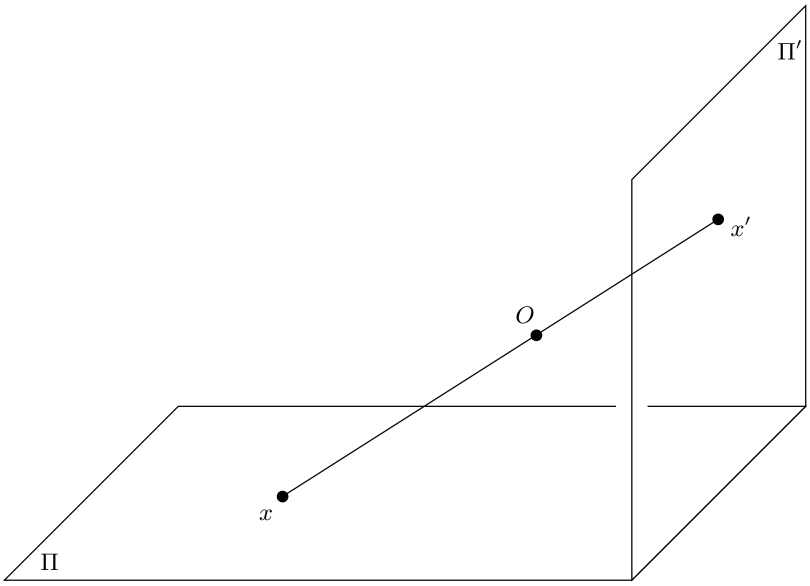

FIGURE 24. The central projection with center O.

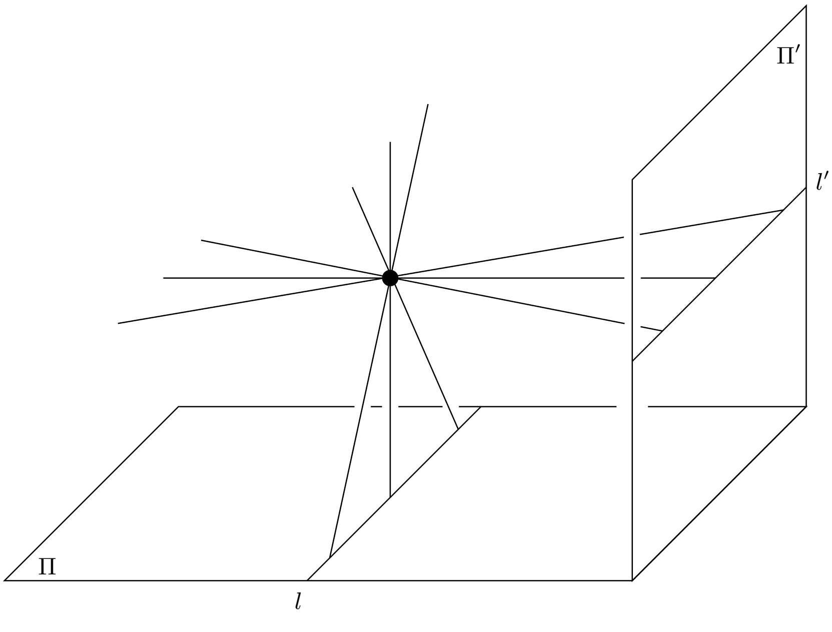

FIGURE 25. The central projection $\Pi \to \Pi'$ is not defined on l, and its image does not contain l'.

be a line parallel to Π. Then all points of l have no images in Π', and all points of l' have no preimages in Π. See Figure 25.

This property is somewhat unpleasant, and it becomes really annoying if we want to consider compositions of central projections. Let us try to find a remedy.

The points on Π that lie close to the line l are mapped to points on Π' that lie "far away". Similarly, the points on Π' close to l' have preimages in the outskirts of the plane Π. This suggests an idea that an extended central projection should map the points of l to certain points "at infinity" of the plane Π', and the points "at infinity" of the plane Π are mapped to the line l' on the plane Π'. Since l and l' are lines, it is natural to call the set of all points at infinity *the line at infinity.*

Definition 30. *A* projective plane *is a plane together with its line at infinity.*

With this definition, central projections become bijections between two projective planes. Besides, they have the following property.

Lemma 31. *Central projections map lines to lines.*

Proof. The lines drawn through O and through points of a line in Π form a plane. The intersection of this plane with Π' is the image line. Besides, one of the lines on Π is mapped to the line at infinity of Π', and the line at infinity of Π is mapped to a line on Π'. ☐

Note that up to now we have only a vague idea of the points at infinity. Now we will make this idea to a strict definition.

Consider a point $x \in \Pi$, and consider the set of all lines in Π that pass through x. (This set is called the *line pencil* through x.) If $x' \in \Pi'$ is the image of x, then all lines through x are mapped to lines through x'. Assume that x has no image in the finite part of Π'. All lines through x are still mapped to lines on Π', and all the image lines are now parallel. Indeed, if two of them would intersect at a point $y' \in \Pi'$, then this y' would be the image of x. Thus, the line pencil through x is mapped to a pencil of parallel lines. Let us put by definition that all these lines intersect "in the infinity". Their point of intersection is the image of x.

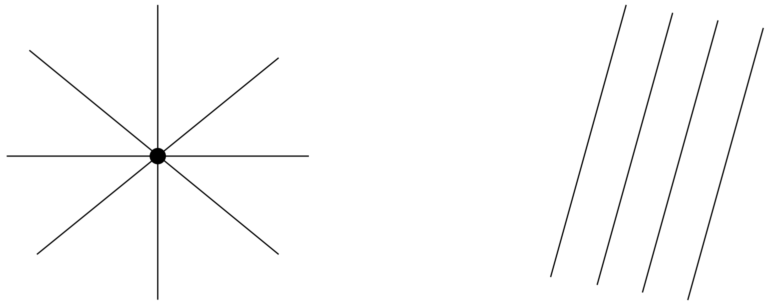

FIGURE 26. Line pencils.

(In fact, we can *define* a point at infinity as a pencil of parallel lines. Every "finite" point is identified with the line pencil through it. A line is said to pass through a point, if it belongs to the points' pencil. This puts the geometry of projective plane on a solid base and allows to develop it in an axiomatic way. For example, Euclidean axioms imply that, on the projective plane, there is a line through any two points, and that any two lines intersect at one point.)

Example 32. In the example of a rail track, see Figure 23, the horizon line is the image of a line at infinity. That is why the images of the parallel rails intersect at the horizon.

The railroad ties form a family of parallel segments, thus they define a point at infinity. The image of this point lies again on the horizon, thus the lines that contain the railroad ties should all pass through one point on the horizon, see Figure 27.

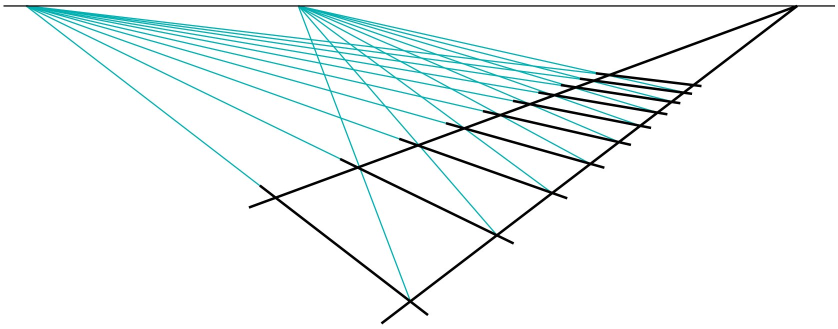

Figure 27. A truly correct picture of a railroad.

Furthermore, the railroad ties are equally spaced, and if we want to reproduce this on the picture, we can observe that the diagonals of the rectangles bounded by rails and ties yield again two families of parallel lines. It suffices to choose a point on the horizon where the images of one of these families intersect, in order to draw the images of equally spaced ties, see Figure 27.

The pencil of parallel lines that correspond to a given point at infinity indicates the "direction" in which the point lies. Note that as the lines are not oriented, two opposite directions are identified. That is, traveling to infinity you will find yourself coming back from the other side of the plane. This can be well illustrated on the example of hyperbola.

Example 33. Consider a central projection that maps a circle $C \subset \Pi$ to a hyperbola on Π'. Then the line $l \in \Pi$ that is mapped to infinity intersects C in two points a and b. The directions in which the images a', b' of a and b lie are given by the asymptotes of the hyperbola. This makes the hyperbola to a closed curve. On Figure 28, note the correspondence between the four

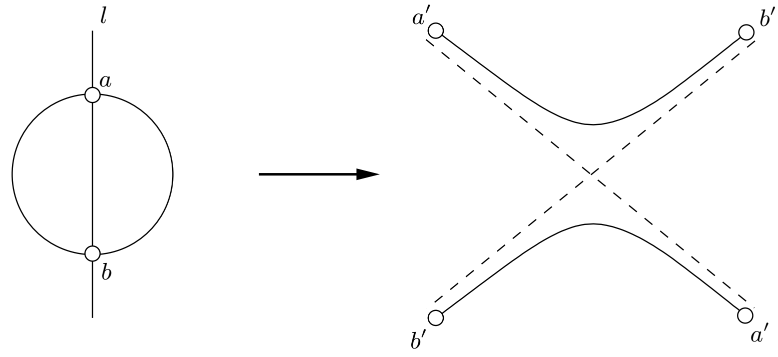

FIGURE 28. Hyperbola on the projective plane.

ends of a hyperbola. Also, note that the hyperbola encloses a disk that is the image of the disk bounded by the preimage circle.

If you travel to infinity inside the region enclosed by the hyperbola, then after intesecting the line at infinity you come back with a reversed orientation. Also, it can be seen that the complement to the region enclosed by a hyperbola is a Möbius band.

We will now give another definition. This definition makes the projective plane more easy-to-handle, in particular, we will be able to introduce coordinates and make computations.

Consider the set of all lines through the point O, the projection center. These lines are in a one-to-one correspondence with the points of the plane Π (augmented with its line at infinity). Indeed, to every point $x \in \Pi$, there is the line Ox, and to every point at infinity of Π, there is a line through O parallel to Π and showing the direction where the point at infinity lies. The central projection from Π to Π' can be viewed as a composition of two maps: for $x \in \Pi$, choose the corresponding line through O, then choose the point $x' \in \Pi'$ that corresponds to the same line.

This motivates the idea of declaring the lines through O to the elements of a projective plane. Let O be the coordinate origin of $\mathbb{R}^3$.

Definition 34. *The* projective plane *is the set of all lines in* $\mathbb{R}^3$ *passing through the coordinate origin. Any plane* Π *not containing the origin defines an* affine model *of a projective plane: lines through* 0 *that intersect* Π *become "finite points" of the projective plane, lines parallel to* Π *become "points at infinity".*

The new definition erases the distinction between "finite points" and "points at infinity": points that are finite in one model can be at infinity in another model.

Now we want also to have a good set of transformations of the projective plane. Of course, we would like that these transformations had something to do with central projections of a plane to a plane. The next definition might seem unrelated.

Definition 35. *Let* $A \in \mathrm{GL}(3,\mathbb{R})$ *be a linear transformation of* $\mathbb{R}^3$. *As* A *maps lines through the origin to lines through the origin, it defines a map* Φ *of the projective plane to itself. Every map* Φ *arising in this way is called a* projective transformation *of the projective plane.*

Let $x \in \mathbb{R}^3$ be a point different from the origin. It determines a line through the origin, and thus a point of the projective plane. Let us denote this point by $[x]$. Then Definition 35 can be rephrased as follows:

$$\Phi[x] = [Ax], \text{ for any } A \in \mathrm{GL}(3,\mathbb{R}), \text{ is called a projective map.}$$

Every plane L through the origin determines a *line* l on the projective plane: l consists of all lines through 0 that lie in L. The next lemma is obvious.

Lemma 36. *Projective transformations map lines to lines.*

This lemma provides a link between central projections and projective transformations.

Theorem 10. *Let* Φ *be a composition of a central projection* $\Pi \to \Pi'$ *with an isometry* $\Pi' \to \Pi$. *Then* Φ *is a (restriction of a) projective transformation of* Π *to itself.*

Sketch of the proof. By Lemma 31, the map Φ maps lines to lines. It can be shown that a map that maps lines to lines is uniquely determined by the images of four points no three of which lie on a line. On the other hand, it is easy to prove that a linear map $A \in \mathrm{GL}(3,\mathbb{R})$ can map any four lines through the origin such that no three of them lie in a plane to any other four lines with these properties. Thus Φ arises from A in a way described in Definition 35. ☐

Conversely, it can be shown that every projective transformation can be realized as a composition of several central and parallel projections. Moreover, either a central projection followed by an isometry or a parallel projection followed by a similarity will suffice.

§3.2. *Projective geometry: formal definitions.*

Definition 37. *The* d*–dimensional projective space* $\mathbb{R}\mathrm{P}^d$ *is the set of all lines in* $\mathbb{R}^{d+1}$ *passing through the coordinate origin.*

We can identify $\mathbb{R}\mathrm{P}^d$ *with*

$$(\mathbb{R}^{d+1} \setminus \{0\}) \,/\, \sim, \tag{28}$$

where $x \sim \lambda x$ *for all* $\lambda \in \mathbb{R}\setminus\{0\}$. *The identification is achieved by associating with the point* $x \in \mathbb{R}^{d+1} \setminus \{0\}$ *the line through* 0 *and* x.

An element of (28), *the equivalence class of* x, *is denoted by* $[x]$.

Recall that every hyperplane Π in $\mathbb{R}^{d+1}$ that does not pass through 0 provides an *affine model* of $\mathbb{R}\mathrm{P}^d$:

$$\mathbb{R}\mathrm{P}^d = \Pi \cup l_\infty.$$

Namely, if a line through 0 intersects Π, then we associate with it the intersection point $p \in \Pi$. That is, we have an embedding

$$\begin{array}{ccc} \Pi & \subset & \mathbb{R}\mathrm{P}^d, \\ p & \mapsto & [p]. \end{array}$$

The lines through 0 that are parallel to Π correspond to the points "at infinity" of Π. The set of all points at infinity is the line at infinity l_∞.

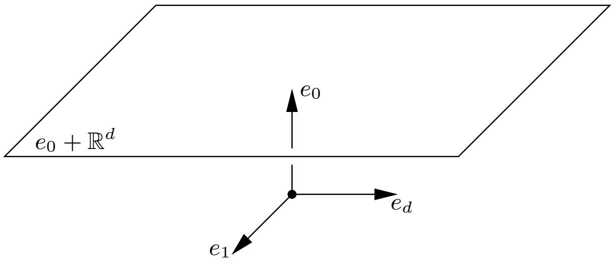

FIGURE 29. The plane $\Pi = e_0 + \mathbb{R}^d$ used as an affine model for the projective space $\mathbb{R}\mathrm{P}^d$.

Here is the standard affine model that we will use. Let $(e_0, e_1, \ldots, e_d)$ be a basis of $\mathbb{R}^{d+1}$. Let $\mathbb{R}^d$ be the linear span of the vectors $e_1, \ldots, e_d$. Put

$$\Pi = e_0 + \mathbb{R}^d = \{x \in \mathbb{R}^{d+1} \mid x^0 = 1\},$$

see Figure 29.

Definition 38. *A* projective transformation *is a map* $\mathbb{R}\mathrm{P}^d \to \mathbb{R}\mathrm{P}^d$ *induced by a linear transformation of* $\mathbb{R}^{d+1}$*: every* $C \in \mathrm{GL}(d+1, \mathbb{R})$ *induces the map*

$$[x] \mapsto [Cx]. \tag{29}$$

In other words, since C *is linear, it maps lines to lines, so we map every line through* 0 *to its image under* C*.*

Two different maps $C, C' \in \mathrm{GL}(d+1, \mathbb{R})$ can induce the same projective transformation. For example, this happens when $C' = \lambda C$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. The next lemma shows that this is the only possibility.

Lemma 39. *If two linear transformations* $C, C' \in \mathrm{GL}(d+1, \mathbb{R})$ *induce the same projective transformation of* $\mathbb{R}\mathrm{P}^d$*, then there is* $\lambda \in \mathbb{R} \setminus \{0\}$ *such that* $C' = \lambda C$*.*

Proof. We have $[Cx] = [C'x]$ for all $x \in \mathbb{R}^{d+1} \setminus \{0\}$. By definition, this means

$$C'x = \lambda_x \cdot Cx,$$

with $\lambda_x \in \mathbb{R} \setminus \{0\}$ for all x. We want to show that λ_x is constant.

Let $x, y \in \mathbb{R}^{d+1} \setminus \{0\}$. If x and y are linearly dependent, then $\lambda_x = \lambda_y$ follows trivially. Otherwise, consider the image of the vector $x+y$ under C':

$$C'(x+y) = \lambda_{x+y} \cdot C(x+y) = \lambda_{x+y} \cdot Cx + \lambda_{x+y} \cdot Cy,$$
$$C'(x+y) = C'x + C'y = \lambda_x \cdot Cx + \lambda_y \cdot Cy.$$

Linear independence of x and y implies linear independence of Cx and Cy. Therefore the two above representations of the vector $C'(x+y)$ as a linear combination of Cx and Cy must be identical. It follows that

$$\lambda_x = \lambda_{x+y} = \lambda_y.$$

☐

Thus, the group of projective transformations of $\mathbb{RP}^d$ is the quotient group

$$\mathrm{PGL}(d+1, \mathbb{R}) = \mathrm{GL}(d+1, \mathbb{R}) / \mathbb{R} \cdot \mathrm{Id}\,.$$

Let us show that every affine transformation of $\mathbb{R}^d$ is also a projective transformation.

Lemma 40. *Affine transformations of* $\mathbb{R}^d$ *are exactly those projective transformations of* $\mathbb{RP}^d$ *that map the line at infinity* l_∞ *to itself.*

More exactly, we claim the following. Identify $\mathbb{R}^d$ with the hyperplane $\Pi = e_0 + \mathbb{R}^d$ viewed as an affine model of $\mathbb{RP}^d$. Then there is a natural bijection between affine transformations of Π and those projective transformations of $\mathbb{RP}^d$ that preserve the line at infinity l_∞ of Π.

Proof. We have

$$l_\infty = \{[x] \mid x^0 = 0\}.$$

Consider a linear transformation $C \in \mathrm{GL}(d+1, \mathbb{R})$ such that the corresponding projective transformation $[C]$ preserves l_∞. That is, for every $x \in \mathbb{R}^{d+1}$ such that $x^0 = 0$ the vector $y = Cx$ also satisfies $y^0 = 0$. This occurs if and only if the matrix C has the form

$$C = \left(\begin{array}{c|c} c & 0 \cdots 0 \\ \hline b & A \end{array} \right),$$

where $c \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$, $A \in \mathrm{GL}(d, \mathbb{R})$. As $[C] = [\lambda C]$, we can assume $c = 1$. Thus, for every $p \in \mathbb{R}^d$ we have

$$C : (1, p) \mapsto (1, Ap + b).$$

Since $p \mapsto Ap + b$ is an affine transformation, and every affine transformation has this form, the lemma is proved. ☐

§3.3. *Bivector interpretation of forces.*

Definition 41. *The* exterior square *of a vector space V is the quotient space*

$$\Lambda^2 V = V \otimes V / \langle x \otimes y - y \otimes x \rangle.$$

The equivalence class of $x \otimes y$ is denoted $x \wedge y$ ("wedge product").

Thus, every element of $\Lambda^2 V$ can be written as a sum

$$x_1 \wedge y_1 + \dots + x_n \wedge y_n,$$

where the wedge product should be treated as a bilinear and antisymmetric operation. Note also that the antisymmetry implies $x \wedge x = 0$ for all x.

Let $V = \mathbb{R}^{d+1}$, and let $e_0, e_1, \dots, e_d$ be a basis of $\mathbb{R}^{d+1}$. Then it can be shown that the bivectors

$$\{e_i \wedge e_j \mid i < j\}$$

form a basis of $\Lambda^2 \mathbb{R}^{d+1}$. In particular, we have

$$\dim \Lambda^2 \mathbb{R}^{d+1} = \binom{d+1}{2} = \frac{d(d+1)}{2}.$$

Definition 42. *A bivector $\alpha \in \Lambda^2 V$ is called* decomposable*, if it can be written as*

$$\alpha = x \wedge y,$$

for some $x, y \in V$.

Example 43. In $\Lambda^2 \mathbb{R}^3$, every bivector is decomposable. Indeed, we have

$$\begin{aligned} ae_0 \wedge e_1 + be_0 \wedge e_2 + ce_1 \wedge e_2 &= e_0 \wedge (ae_1 + be_2) + ce_1 \wedge e_2 \\ &= e_0 \wedge (ae_1 + be_2) + ce_1 \wedge \left(\frac{a}{b}e_1 + e_2\right) \\ &= \left(e_0 + \frac{c}{b}e_1\right) \wedge (ae_1 + be_2), \end{aligned}$$

provided that $b \neq 0$. If $b = 0$, then we can do similar transformation by exchanging the roles of basis vectors, or simply write

$$ae_0 \wedge e_1 + ce_1 \wedge e_2 = e_1 \wedge (-ae_0 + ce_2).$$

In $\Lambda^2 \mathbb{R}^4$, there are indecomposable bivectors, for example

$$e_0 \wedge e_1 + e_2 \wedge e_3.$$

In fact, $\alpha \in \Lambda^2 V$ is decomposable if and only if $\alpha \wedge \alpha = 0$. Here $\alpha \wedge \alpha \in \Lambda^4 V$.

Lemma 44. *Let $\alpha = x \wedge y \in \Lambda^2 V$ be a non-zero decomposable bivector. Let (x', y') be any other basis of the two-dimensional subspace of V spanned by x and y, so that*

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy. \end{aligned}$$

Then

$$x' \wedge y' = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \wedge y.$$

Proof. A direct computation. □

Recall that we denote the basis vectors of $\mathbb{R}^{d+1}$ by $(e_0, e_1, \ldots, e_d)$, and put $\mathbb{R}^d$ to be the span of $e_1, \ldots, e_d$. For every $p \in \mathbb{R}^d$, denote

$$\widetilde{p} = e_0 + p. \tag{30}$$

Theorem 11. *Let $\mathcal{F}(\mathbb{R}^d)$ be the space of systems of forces in $\mathbb{R}^d$. The map*

$$(p, f) \mapsto \widetilde{p} \wedge f \tag{31}$$

can be extended by linearity to an isomorphism

$$\mathcal{F}(\mathbb{R}^d) \to \Lambda^2 \mathbb{R}^{d+1}.$$

Proof. Let us check that the extension by linearity is well-defined. For this we have to show that for every equivalent to 0 system of forces

$$\sum_i \lambda_i (p_i, f_i) \sim 0 \tag{32}$$

the corresponding bivector vanishes:

$$\sum_i \lambda_i (\widetilde{p_i} \wedge f_i) = 0.$$

Since every equivalence (32) is a consequence of basic ones, we need to check only that the three basic equivalences between systems of forces become valid identities when forces are replaced by bivectors according to (31). This is done as follows:

(a) $\widetilde{p} \wedge 0 = 0$, since $\widetilde{p} \wedge 0 = [\widetilde{p} \otimes 0] = [0]$;
(b) $\lambda_1(\widetilde{p} \wedge f_1) + \lambda_2(\widetilde{p} \wedge f_2) = \widetilde{p} \wedge (\lambda_1 f_1 + \lambda_2 f_2)$ because of bilinearity of the wedge product;
(c) $\widetilde{p + \lambda f} \wedge f = \widetilde{p} \wedge f$, since $\widetilde{p + \lambda f} = \widetilde{p} + \lambda f$ and $f \wedge f = 0$.

Further, we have

$$\dim \mathcal{F}(\mathbb{R}^d) = \frac{d(d+1)}{2} = \dim \Lambda^2 \mathbb{R}^{d+1}.$$

Thus, in order to show that the linear extension of (31) is an isomorphism, it suffices to show that it is surjective. For this, we need only to show that every decomposable bivector $x \wedge y \in \Lambda^2 \mathbb{R}^{d+1}$ is the image of a system of forces in $\mathbb{R}^d$. We consider two cases.

First, let not both of the vectors x, y lie in $\mathbb{R}^d$. Then the vector space $\operatorname{span}\{x, y\}$ intersects the hyperplane $e_0 + \mathbb{R}^d$. Let x', y' be another basis of $\operatorname{span}\{x, y\}$ such that $x' \in e_0 + \mathbb{R}^d$ and $y \in \mathbb{R}^d$. By Lemma 44, we have

$$x \wedge y = \lambda(x' \wedge y'),$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. Thus, the bivector $x \wedge y$ corresponds to the force (p, f), where

$$p = x' - e_0, \quad f = \lambda y'.$$

Second, let $x, y \in \mathbb{R}^d$. Then we have

$$x \wedge y = (x + e_0) \wedge y - e_0 \wedge y = \widetilde{x} \wedge y + \widetilde{0} \wedge (-y).$$

It follows that the bivector $x \wedge y$ corresponds to the system of forces

$$(x, y) + (0, -y).$$

$\square$

Exercise 14. Show that a system of forces $\sum_i (p_i, f_i)$ is equivalent to 0 if and only if

$$\sum_i f_i = 0 \quad \text{and} \quad \sum_i \det \begin{pmatrix} p_i^s & p_i^t \\ f_i^s & f_i^t \end{pmatrix} = 0, \quad \text{for all } 1 \le s < t \le d.$$

Here $p_i = (p_i^1, \ldots, p_i^d)$ and $f_i = (f_i^1, \ldots, f_i^d)$ are coordinate representations of p_i and f_i with respect to a basis of $\mathbb{R}^d$.

§3.4. *Main theorem.*

Theorem 12. *Let P be a framework in $\mathbb{R}^d$. Identify $\mathbb{R}^d$ with an affine model of $\mathbb{R}\mathrm{P}^d$, so that $\mathbb{R}\mathrm{P}^d = \mathbb{R}^d \cup l_\infty$. Let*

$$\Phi : \mathbb{R}\mathrm{P}^d \to \mathbb{R}\mathrm{P}^d$$

be a projective transformation such that $\Phi(p_i) \notin l_\infty$, for all vertices p_i of the framework P. Then the framework $\Phi \circ P$ is infinitesimally rigid if and only if P is infinitesimally rigid.

More generally,

$$\dim \big(\mathcal{Q}(\Phi \circ P)/\mathcal{Q}_0(\Phi \circ P)\big) = \dim \big(\mathcal{Q}(P)/\mathcal{Q}_0(P)\big) \tag{33}$$

— the space of infinitesimal motions of $\Phi \circ P$ modulo trivial ones has the same dimension as the space of infinitesimal motions of P modulo trivial ones.

A couple of remarks before we proceed to the proof.

First, exactly as it was the case with Theorem 9, we will use the equivalence between infinitesimal and static rigidity. That is, we will prove that the static rigidity is projectively invariant, more generally that the dimensions of the spaces of equilibrium loads modulo resolvable ones coincide for a framework and for its projective image.

Second, the affine invariance of infinitesimal, respectively static, rigidity is a special case of the projective invariance, since every affine transformation can be viewed as a projective transformation, see Lemma 40.

Third and the last, unlike the affine invariance, the projective invariance is not straightforward for static rigidity. Indeed, the definition of static rigidity uses linear operations on vectors, and projective transformations don't treat them well. As we will see, the proof is based on the correspondence between forces and bivectors established in Theorem 11.

Proof. Identify $\mathbb{R}^d$ with $e_0+\mathbb{R}^d$ via $p \mapsto \widetilde{p}$ as in (30). Let $C \in \mathrm{GL}(d+1,\mathbb{R})$ be a representative of the projective map Φ. For every $p \in \mathbb{R}^d$, we will construct a linear isomorphism $C_p^{\mathrm{stat}} \in \mathrm{GL}(d,\mathbb{R})$ such that for every load $F=\{(p_i,f_i)\}$ on the framework P, the load

$$C^{\mathrm{stat}}(F)=\{(\Phi(p_i), C_{p_i}^{\mathrm{stat}} f_i)\}$$

is in equilibrium, respectively resolvable, if and only if F is in equilibrium, respectively resolvable.

Note that we have

$$C\widetilde{p}=\lambda(p)\cdot\widetilde{\Phi(p)}, \tag{34}$$

with $\lambda(p) \in \mathbb{R}\setminus\{0\}$, see Figure 30.

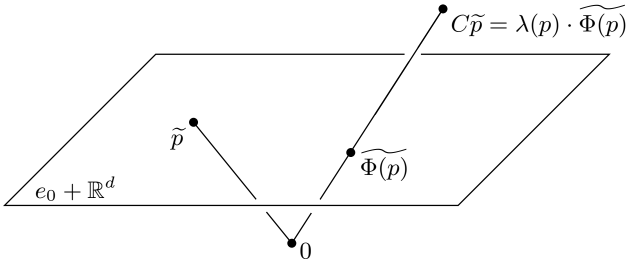

FIGURE 30. Representing a projective map Φ by a linear map C.

The map C induces a linear isomorphism

$$\begin{array}{rccc} C_*: & \Lambda^2\mathbb{R}^{d+1} & \to & \Lambda^2\mathbb{R}^{d+1}, \\ & \sum_i x_i\wedge y_i & \mapsto & \sum_i Cx_i \wedge Cy_i. \end{array}$$

Due to (34), C_* maps bivectors divisible by $\widetilde{p}$ to bivectors divisible by $\widetilde{\Phi(p)}$. Therefore, for every $p,f \in \mathbb{R}^d$, we have $C_*(\widetilde{p}\wedge f)=\widetilde{\Phi(p)}\wedge y$ for some $y \in \mathbb{R}^{d+1}$, and by performing a basis change in $\mathrm{span}\{\widetilde{\Phi(p)},y\}$ we find an $f' \in \mathbb{R}^d$ such that

$$C_*(\widetilde{p}\wedge f)=\widetilde{\Phi(p)}\wedge f'.$$

Put

$$C_p^{\mathrm{stat}}(f):=f'.$$

Since f' depends on f linearly, we have $C_p^{\mathrm{stat}} \in \mathrm{GL}(d,\mathbb{R})$.

Let F be a load on P. By Theorem 11, F is an equilibrium load if and only if

$$\sum_i \widetilde{p_i}\wedge f_i=0. \tag{35}$$

Since we have

$$C_*\Big(\sum_i \widetilde{p_i}\wedge f_i\Big)=\sum_i C_*(\widetilde{p_i}\wedge f_i)=\sum_i \widetilde{\Phi(p_i)}\wedge C_{p_i}^{\mathrm{stat}}(f_i),$$

the equation (35) is equivalent to $\sum_i \widetilde{\Phi(p_i)} \wedge C^{\mathrm{stat}}_{p_i}(f_i) = 0$ which says that $C^{\mathrm{stat}}(F)$ is an equilibrium load. Thus F is in equilibrium if and only if $C^{\mathrm{stat}}(F)$ is in equilibrium.

Now assume that F is a resolvable load on P. By definition, there exist numbers ω_{ij}, for all edges ij of P, such that

$$f_i + \sum_j \omega_{ij}(p_i - p_j) = 0 \tag{36}$$

holds for all i. By taking an exterior product of (36) with $\widetilde{p_i}$, we obtain

$$\widetilde{p_i} \wedge f_i + \sum_j \omega_{ij} \cdot \widetilde{p_i} \wedge (p_i - p_j) = 0.$$

Since $p_i - p_j$ is equal to $\widetilde{p_i} - \widetilde{p_j}$, this can be rewritten as

$$\widetilde{p_i} \wedge f_i = \sum_j \omega_{ij} \cdot \widetilde{p_i} \wedge \widetilde{p_j}.$$

Apply to the both sides of the last equation the map C_*. Because of (34), we will have

$$\widetilde{\Phi(p_i)} \wedge C^{\mathrm{stat}}_{p_i} f_i = \sum_j \lambda(p_i)\lambda(p_j)\omega_{ij} \cdot \widetilde{\Phi(p_i)} \wedge \widetilde{\Phi(p_j)}.$$

In other words, the stress $ij \mapsto \lambda(p_i)\lambda(p_j)\omega_{ij}$ resolves the load $C^{\mathrm{stat}}(F)$. Since the above arguments can be reversed, we obtain that F is a resolvable load if and only if $C^{\mathrm{stat}}(F)$ is resolvable.

As a result, the map C^{stat} establishes isomorphisms between the vector spaces $\mathcal{L}(\Phi \circ P)$ and $\mathcal{L}(P)$ as well as between $\mathcal{L}_0(\Phi \circ P)$ and $\mathcal{L}_0(P)$. Because of the duality between forces and infinitesimal motions, this induces isomorphisms between corresponding spaces of infinitesimal motions, which results in the equality (33). Theorem is proved. $\square$

§3.5. *Examples and applications.* Due to the projective invariance of infinitesimal rigidity, the criterium when a framework P with a given graph G is infinitesimally flexible always has a projective nature. That is, the conditions on the framework P that imply its infinitesimal flexibility can be expressed through properties that are preserved by projective transformations. Examples of such properties are collinearity of a set of points or concurrence of certain lines or planes. By contrast, conditions on values of angles and lengths of edges, or parallelity of lines don't have a projective nature, since angles and lengths are not preserved by projective transformations, and parallel lines can be mapped to intersecting ones.

Let us illustrate this on two examples.

Example 45. Consider the framework shown on Figure 31. It is infinitesimally flexible if and only if the lines p_1p_4, p_2p_5, and p_3p_6 have a common point. As projective transformation map lines to lines, the concurrence of three lines is projectively invariant.

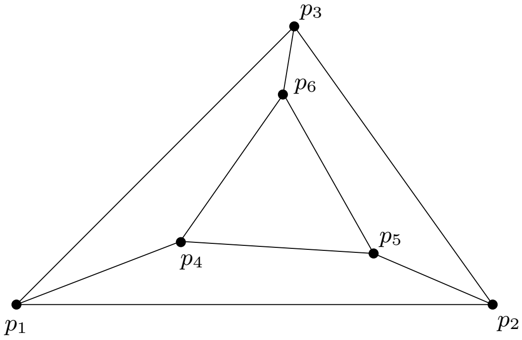

FIGURE 31. This planar framework is infinitesimally flexible if and only if the lines p_1p_4, p_2p_5, p_3p_6 intersect at a point.

Example 46. Consider an octahedron in $\mathbb{R}^3$. Color each of its faces black or white so that adjacent faces have different colors. The octahedron is infinitesimally flexible if and only if all of its black faces intersect at a point. This means also that the four white faces intersect at a point if and only if the four black faces do.

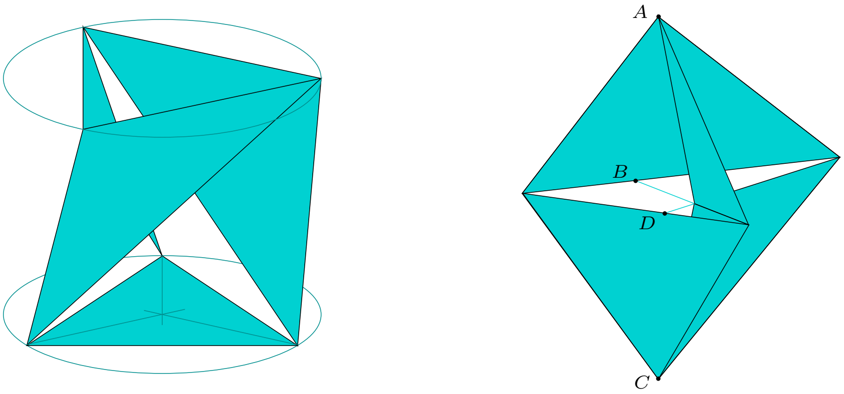

FIGURE 32. Examples of infinitesimally flexible octahedra. Left: antiprism twisted by 90°. Right: the points A, B, C, D lie in one plane.

The projective invariance can also be applied to prove criteria for infinitesimal flexibility. Here is an example.

Definition 47. *A graph G is called* bipartite*, if its vertex set can be presented as a disjoint union $\mathcal{V} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ so that every edge of G joins a vertex from $\mathcal{V}_1$ with a vertex from $\mathcal{V}_2$ (that is there are no edges joining two vertices of $\mathcal{V}_1$ and no edges joining two vertices of $\mathcal{V}_2$).*

Theorem 13. *Let P be a bipartite framework in $\mathbb{R}^2$ inscribed in a conic section. If P has more than two vertices, then P is infinitesimally flexible.*

Proof. Apply a projective transformation that transforms the conic section through the vertices of P into the unit circle with center 0. Consider the vector field on the vertices of P given by

$$q_i = \begin{cases} p_i, & \text{for } i \in \mathcal{V}_1; \\ -p_i, & \text{for } i \in \mathcal{V}_2. \end{cases}$$

This is an infinitesimal motion, since for $i \in \mathcal{V}_1$ and $j \in \mathcal{V}_2$ we have

$$\langle q_i - q_j, p_i - p_j \rangle = \langle p_i + p_j, p_i - p_j \rangle = \|p_i\|^2 - \|p_j\|^2 = 1 - 1 = 0.$$

On the other hand, for $i, j \in \mathcal{V}_1$ (if $|\mathcal{V}_1| = 1$, then take $i, j \in \mathcal{V}_2$) we have

$$\langle q_i - q_j, p_i - p_j \rangle = \|p_i - p_j\|^2 \neq 0.$$

Therefore $\{q_i\}$ is a non-trivial infinitesimal motion, and the theorem is proved. ☐

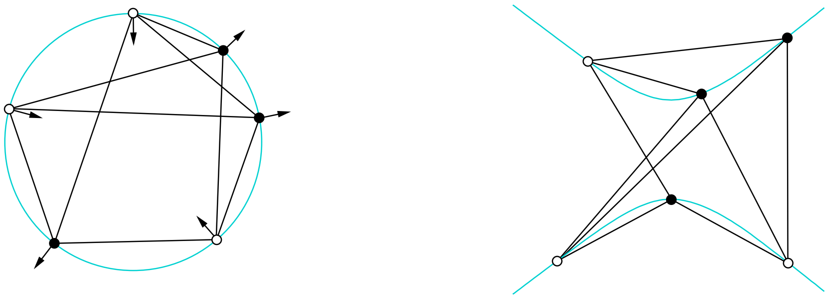

FIGURE 33. Bipartite frameworks inscribed in a conic section.

In the same way, a bipartite framework in $\mathbb{R}^d$ is infinitesimally flexible, if it is inscribed in a quadric which is projectively equivalent to the sphere. In $\mathbb{R}^3$ these are ellipsoid, elliptic paraboloid, and two-sheeted hyperboloid. In fact, the framework will be infinitesimally flexible if its vertices lie on any non-degenerate quadric. This can be proved by taking complexification, since in the complex projective space all non-degenerate quadrics are equivalent.

For a necessary and sufficient condition when a complete bipartite framework is infinitesimally flexible, as well as for applications of this for satellite networks and structural engineering, see the paper of Walter Whiteley "Infinitesimal motions of a bipartite framework".

§3.6. *Formulas for transformations of forces and motions.* Recall that during the proof of Theorem 12 we associated with a linear isomorphism $C \in \mathrm{GL}(d+1, \mathbb{R})$ (where C represents a projective transformation $\Phi : \mathbb{RP}^d \to \mathbb{RP}^d$) a family of linear isomorphisms

$$C_p^{\mathrm{stat}} : \mathbb{R}^d \to \mathbb{R}^d$$

for all $p \in \mathbb{R}^d$ such that $\Phi(p) \notin l_\infty$. Isomorphisms C_p^{stat} have the property that the maps

$$(p_i, f_i) \mapsto \big(\Phi(p_i), C_{p_i}^{\mathrm{stat}}(f_i)\big)$$

send equilibrium loads on P to equilibrium loads on $\Phi \circ P$, and resolvable loads to resolvable ones.

Any other representative of the projective transformation Φ has the form λC for some $\lambda \in \mathbb{R} \setminus \{0\}$. From the construction of C_p^{stat} it is clear that a scaling of C results in a scaling of C_p^{stat} by the same factor. Thus we can put

$$\Phi^{\mathrm{stat}} = [C^{\mathrm{stat}}],$$

where Φ^{stat} is a family of linear isomorphisms Φ_p^{stat} determined up to a common factor independent of the point p.

Example 48. If Φ is an affine transformation $p \mapsto Ap + b$, then for all p we have

$$\Phi_p^{\mathrm{stat}} = A,$$

the linear part of Φ. Respectively, Φ_p^{stat} can be any multiple of A.

The following theorem computes Φ_p^{stat} for every projective transformation Φ.

Theorem 14. *Up to a scalar factor independent of* p,

$$\Phi_p^{\mathrm{stat}}(f) = h_l(p) h_l(p+f) \cdot (\Phi(p+f) - \Phi(p)). \tag{37}$$

Here $l \subset \mathbb{R}^d$ *is the hyperplane mapped by* Φ *to the hyperplane at infinity, and* $h_l(x)$ *is the signed distance from the point* x *to* l.

To define the signed distance from l, we have to choose a positive half-space with respect to l. Note that the value of the right hand side in (37) does not depend on this choice.

Proof. Recall that in order to represent a projective transformation Φ by a linear transformation $C \in \mathrm{GL}(d+1, \mathbb{R})$ we identify the plane $\mathbb{R}^d$ with the hyperplane $e_0 + \mathbb{R}^d \subset \mathbb{R}^{d+1}$ via $p \mapsto \widetilde{p} = e_0 + p$. In order to simplify the notations, we will write p instead of $\widetilde{p}$ during this proof.

For every $C \in \mathrm{GL}(d+1, \mathbb{R})$, the map C_p^{stat} is uniquely determined by the equation

$$Cp \wedge Cf = \Phi(p) \wedge C_p^{\mathrm{stat}}(f) \quad \text{for all } f \in \mathbb{R}^d, \tag{38}$$

and by the condition $C_p^{\mathrm{stat}}(f) \in \mathbb{R}^d$. The vector Cp is a multiple of $\Phi(p)$:

$$Cp = \lambda(p)\Phi(p). \tag{39}$$

The factor $\lambda(p)$ is computed in Lemma 50. Scale C so that we have $\mu = 1$ in (40). Then we have

$$Cp = h_l(p)\Phi(p),$$

for all $p \in e_0 + \mathbb{R}^d \setminus l$. As $f \in \mathbb{R}^d$, this formula is not applicable to f directly, but we can make the following transformations:

$$\begin{aligned} Cp \wedge Cf = Cp \wedge C(p+f) &= h_l(p)h_l(p+f) \cdot \Phi(p) \wedge \Phi(p+f) \\ &= h_l(p)h_l(p+f) \cdot \Phi(p) \wedge (\Phi(p+f) - \Phi(p)). \end{aligned}$$

Since $\Phi(p+f) - \Phi(p) \in \mathbb{R}^d$, by comparing this with (38) we obtain formula (37). □

Remark 49. It is not obvious that the right hand side of the formula (37) depends linearly on f. However, linearity of C_p^{stat} (hence of every representative of Φ_p^{stat}) follows from the formula (38).

For $p + f \in l$, the right hand side of (37) is not defined. In this case, the value of $\Phi_p^{\mathrm{stat}}(f)$ can be determined by linearity.

Lemma 50. *For every $C \in \mathrm{GL}(d+1, \mathbb{R})$ there exists $\mu \in \mathbb{R} \setminus \{0\}$ such that for all $p \in e_0 + \mathbb{R}^d \setminus l$ we have*

$$\lambda(p) = \mu \cdot h_l(p), \tag{40}$$

where $\lambda(p)$ is the scalar factor in (39).

Proof. We have

$$\lambda(p) = \langle Cp, e_0 \rangle,$$

because the number $\langle Cp, e_0 \rangle$ is the 0th coordinate of Cp, and the 0th coordinate of $\Phi(p)$ equals 1.

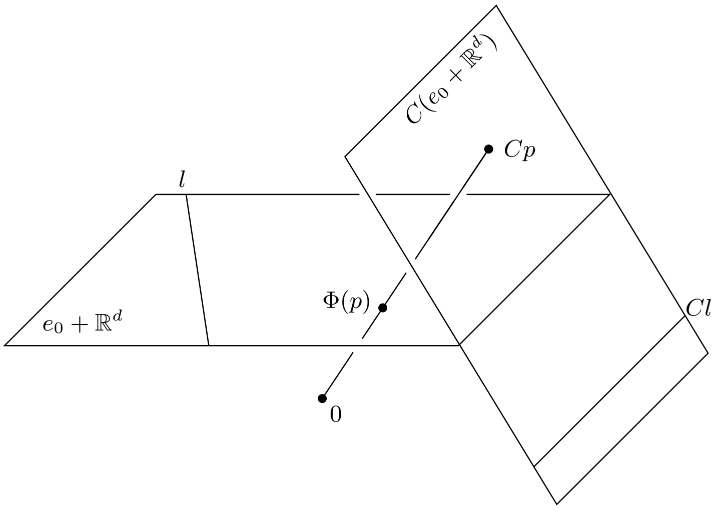

FIGURE 34. Computing the factor $\lambda(p)$ in (39).

Consider the hyperplane $C(e_0 + \mathbb{R}^d) \subset \mathbb{R}^{d+1}$, which is the image of the hyperplane $e_0 + \mathbb{R}^d$, see Figure 34. Since $\Phi(l) = l_\infty$, the affine subspace

$Cl \subset C(e_0 + \mathbb{R}^d)$ is contained in $\mathbb{R}^d$. Therefore we have

$$h_{Cl}(Cp) = \mu_1 \cdot \langle Cp, e_0 \rangle,$$

for some constant μ_1 that depends on the angle between the hyperplanes $e_0 + \mathbb{R}^d$ and $C(e_0 + \mathbb{R}^d)$. (If Φ is not an affine transformation, then these hyperplanes are not parallel, and we have $\mu_1 \neq 0$.) Also, because of linearity of C, we have

$$h_{Cl}(Cp) = \mu_2 \cdot h_l(p)$$

for all $p \in e_0 + \mathbb{R}^d$, with a non-zero constant μ_2. By combining all three equations, we obtain

$$\lambda(p) = \frac{\mu_2}{\mu_1} \cdot h_l(p),$$

and the lemma is proved. □

Theorem 15. *Equation* (37) *is equivalent to*

$$\Phi_p^{\text{stat}} = h_l^2(p) \cdot \mathrm{d}\Phi_p, \tag{41}$$

where $\mathrm{d}\Phi_p$ *is the differential at* p *of the map* Φ.

Proof. By using the linearity of Φ_p^{stat}, we find

$$\Phi_p^{\text{stat}}(f) = \frac{1}{t}\Phi_p^{\text{stat}}(tf) = h_l(p)h_l(p+tf) \cdot \frac{\Phi(p+tf) - \Phi(p)}{t}.$$

As $t \to 0$, the right hand side converges to

$$h_l^2(p) \cdot \frac{\partial \Phi}{\partial f} = h_l^2(p) \cdot \mathrm{d}\Phi(f),$$

and the theorem is proved. □

Theorem 16. *Let* Φ *be a projective transformation of* $\mathbb{R}\mathrm{P}^d$ *and let* l *be the hyperplane mapped to the hyperplane at infinity. Let* P *be a framework in* $\mathbb{R}^d \subset \mathbb{R}\mathrm{P}^d$ *such that* $p_i \notin l$ *for all vertices* p_i *of* P. *Put*

$$\Phi_p^{\text{kin}} = h_l^{-2}(p)(\mathrm{d}\Phi_p^{-1})^*. \tag{42}$$

Then for every infinitesimal motion $Q = (q_i)_{i\in\mathcal{V}}$ *of* P *the vector field*

$$\Phi^{\text{kin}}(Q) = (\Phi_{p_i}^{\text{kin}}(q_i))_{i\in\mathcal{V}}$$

is an infinitesimal motion of the framework $\Phi \circ P$. *Besides, if* Q *is trivial then so is* $\Phi^{\text{kin}}(Q)$.

Proof. The proof is similar to that of Lemma 28. The formula (42) is designed in such a way that

$$\langle \Phi_p^{\text{kin}}(q), \Phi_p^{\text{stat}}(f) \rangle = \langle q, f \rangle$$

holds for all $q, f \in \mathbb{R}^d$, where the formula (41) for Φ_p^{stat} is used. □

IV. SMOOTH SURFACES

1. **Basic notions.**

§1.1. *Isometric deformations.* Let M be an abstract smooth surface, that is a topological surface equipped with a C^∞-structure. Let

$$p : M \to \mathbb{R}^3$$

be an immersion, that is a smooth map such that the differential

$$dp_x : T_xM \to T_{p(x)}\mathbb{R}^3$$

has rank 2 for all $x \in M$. Then p induces a Riemannian metric on M given by

$$\langle X, Y\rangle := \langle dp(X), dp(Y)\rangle,$$

for all $X, Y \in T_xM$ and all $x \in M$. (We will often omit the subscript x at dp.)

Definition 51. *An* isometric deformation *of an immersion* $p : M \to \mathbb{R}^3$ *is a smooth map*

$$P : [0, 1] \times M \to \mathbb{R}^3$$

such that

(a) *for all* $t \in [0, 1]$, *the map*

$$\begin{array}{rcl} p^t : M & \to & \mathbb{R}^3, \\ p^t(x) & = & P(t, x) \end{array}$$

is an immersion;

(b) $p^0 = p$;

(c) *for all* $t \in [0, 1]$, *the immersions* p^t *induce the same Riemannian metric on* M:

$$\langle dp^t(X), dp^t(Y)\rangle = \langle dp(X), dp(Y)\rangle$$

for all $X, Y \in T_xM$ *and all* $x \in M$.

Definition 52. *An isometric deformation* P *of an immersion* p *is called* trivial, *if*

$$p^t = \Phi^t \circ p,$$

for all $t \in [0, 1]$, *where* Φ^t *is an isometry of* $\mathbb{R}^3$.

An immersion p *is called* rigid, *if it has no non-trivial isometric deformations. A non-rigid immersion is called* flexible.

By abuse of terminology, we will also say that the surface $p(M)$ is rigid, if the immersion p of M is rigid.

Example 53. Every bounded planar surface is flexible, for example it can be rolled into a cylinder. Let's, say, take a stripe

$$M = [0, 2\pi] \times (-\infty, +\infty) \subset R^2$$

embedded into $\mathbb{R}^3$ via $p : (x, y) \mapsto (x, y, 0)$. The family of maps

$$\begin{array}{rcl} p^t(x, y) & = & \left(\frac{\sin tx}{t}, y, \frac{1-\cos tx}{t}\right) \text{ for } t \neq 0, \\ p^0(x, y) & = & (x, y, 0). \end{array}$$

forms an isometric deformation of p. At the time t, all segments $[0, 2\pi] \times \{y\}$ are mapped to circular arcs of radius $\frac{1}{t}$.

§1.2. *Infinitesimal isometric deformations.* Before giving the definition, let us make the following observation. Let

$$q : M \to \mathbb{R}^3 \tag{43}$$

be an arbitrary smooth map. Then the map

$$\begin{array}{rcl} M & \to & \mathbb{R}^3, \\ x & \mapsto & p(x) + tq(x) \end{array} \tag{44}$$

is an immersion for all sufficiently small t. Indeed, if dp maps basis vectors of T_xM to two linearly independent vectors in $\mathbb{R}^3$, then the image vectors remain linearly independent after a small perturbation of p.

The family of maps (44) is a linear deformation of the immersion p with the velocity q. Every smooth (not necessarily isometric) deformation p^t has a velocity field, thus it can be approximated in the first order by a linear deformation. The definition of an infinitesimal isometric deformation arises by requiring that a given first-order (i. e. linear) deformation preserves the Riemannian metric in the first order.

Definition 54. *A smooth map* (43) *is called an* infinitesimal isometric deformation *of an immersion p if the Riemannian metrics induced by immersions* (44) *for small t are constant in the first order at $t = 0$:*

$$\left.\frac{d}{dt}\right|_{t=0} \langle dp(X) + tdq(X), dp(Y) + tdq(Y)\rangle = 0$$

for all $X, Y \in T_xM$ and all $x \in M$.

Definition 55. *An infinitesimal isometric deformation q of an immersion p is called* trivial*, if there exists an infinitesimal isometry ξ of $\mathbb{R}^3$ such that*

$$q(x) = \xi(p(x))$$

for all $x \in M$.

An immersion p is called infinitesimally rigid*, if it has no non-trivial infinitesimal isometric deformations. If an immersion is not infinitesimally rigid, then it is called* infinitesimally flexible.

Lemma 56. *A vector field q is an infinitesimal deformation of an immersion p if and only if one of the following two equivalent conditions are satisfied:*

(a) *for all $X \in T_xM$,*

$$\langle dp(X), dq(X)\rangle = 0;$$

(b) *for all $X, Y \in T_xM$ and all $x \in M$,*

$$\langle dp(X), dq(Y)\rangle + \langle dp(Y), dq(X)\rangle = 0.$$

Proof. Denote $p^t(x) = p(x) + tq(x)$. An easy computation shows that

$$\left.\frac{d}{dt}\right|_{t=0} \langle dp^t(X), dp^t(Y)\rangle = \langle dp(X), dq(Y)\rangle + \langle dp(Y), dq(X)\rangle.$$

Thus, the condition (b) is necessary and sufficient for q to be an infinitesimal isometric deformation.

The condition (b) implies (a) by substituting $Y = X$. In the opposite direction, (a) implies (b) by subtracting from the equation

$$\langle dp(X+Y), dq(X+Y)\rangle = 0$$

the equations

$$\langle dp(X), dq(X)\rangle = 0 \quad \text{and} \quad \langle dp(Y), dq(Y)\rangle = 0.$$

□

Example 57. Let $M \subset \mathbb{R}^2$ be embedded into $\mathbb{R}^3$ via $p : (x, y) \mapsto (x, y, 0)$. Then the embedding p has lots of infinitesimal isometric deformations, most of which are non-trivial. Namely, for every smooth function $f : M \to \mathbb{R}$, the map

$$q(x, y) = (0, 0, f(x, y))$$

is an infinitesimal isometric deformation. Indeed, for every $X \in TM$, the vector $dp(X)$ belongs to the xy-plane, and the vector $dq(X)$ is parallel to the z-axis. Therefore $\langle dp(X), dq(X)\rangle = 0$.

It follows that every smooth surface in $\mathbb{R}^3$ that contains a planar piece is infinitesimally flexible. As an (obviously non-trivial) infinitesimal isometric deformation one can take a vector field that vanishes everywhere except in a small disk inside the planar piece where it is orthogonal to the surface.

§1.3. *Relations between infinitesimal isometries and isometries.*

Lemma 58. *The velocity field of an isometric deformation is an infinitesimal isometric deformation.*

Proof. The velocity field of an isometric deformation $\{p^t\}$ is

$$q(x) = \left.\frac{d}{dt}\right|_{t=0} p^t(x).$$

By Definition 51 we have

$$\frac{d}{dt}\langle dp^t(X), dp^t(Y)\rangle = 0.$$

On the other hand,

$$\left.\frac{d}{dt}\right|_{t=0} \langle dp^t(X), dp^t(Y)\rangle = \langle p(X), q(Y)\rangle + \langle p(Y), q(X)\rangle,$$

because of $\frac{d}{dt}(dp^t) = d(\frac{dp^t}{dt})$. The lemma follows. □

It is also easy to see that the velocity field of a trivial isometric deformation is a trivial infinitesimal isometric deformation.

Lemma 58 suggests an idea that infinitesimal rigidity should imply rigidity: if a framework has a non-trivial isometric deformation, then the velocity field of this deformation is an infinitesimal isometric deformation that has good chances to be non-trivial. Of course, it can happen to be trivial as well: for example, the initial velocity of a non-constant deformation can be zero. If the immersion and its deformation are analytic, then the deformation can be reparametrized so that its initial velocity is not zero and, moreover, is not a trivial infinitesimal isometric deformation. But in the general C^∞-case the existence of a reparametrization or of another, reparametrizable, deformation is not proved. The problem whether infinitesimal rigidity of a smooth surface implies its rigidity remains open.

The following lemma shows how an infinitesimal isometric deformation leads to a pair of isometric immersions.

Lemma 59. *Let q be an infinitesimal isometric deformation of an immersion $p : M \to \mathbb{R}^3$. Let $t \in \mathbb{R}$ be such the maps $p+tq$ and $p-tq$ are immersions (in particular, this holds for all sufficiently small t). Then $p+tq$ and $p-tq$ induce the same Riemannian metric on M.*

Besides, if $p(M)$ is not contained in a plane, then the map

$$p(x) + tq(x) \mapsto p(x) - tq(x)$$

cannot be extended to an isometry of $\mathbb{R}^3$.

Exercise 15. Prove Lemma 59.

Remark 60. There are many similarities between the notions and theorems in the infinitesimal rigidity of frameworks and surfaces. The formula $\langle dp(X), dq(X)\rangle = 0$ from Lemma 56 is similar to the formula $\langle p_i - p_j, q_i - q_j\rangle = 0$ from Lemma 13 in Chapter I; Lemma 59 is an analog of Theorem 2'.

2. **Rotation field of an infinitesimal isometric deformation.**

§2.1. *Definition and existence.*

Lemma 61. *Let q be an infinitesimal isometric deformation of an immersion $p : M \to \mathbb{R}^3$. Then there exists a smooth map*

$$\eta : M \to \mathbb{R}^3$$

such that $dq = \eta \times dp$. That is,

$$dq_x(X) = \eta(x) \times dp_x(X)$$

for all $X \in T_xM$ and $x \in M$. Besides, η is unique.

The map η is called the rotation field *of q.*

Proof. For a fixed $x \in M$, the map

$$f : dp(X) \mapsto dq(X)$$

is a linear map from a 2-dimensional space $T_{p(x)}p(M)$ to $\mathbb{R}^3$ (note that f can have a non-trivial kernel, because q is not necessarily an immersion). On the other hand, every vector $\eta \in \mathbb{R}^3$ determines a linear map

$$r_\eta : dp(X) \mapsto \eta \times dp(X).$$

We need to show that there exists a unique η such that $f = r_\eta$.

Let X, Y be a basis of $T_x(M)$. A linear map $T_{p(x)}p(M) \to \mathbb{R}^3$ is uniquely determined by the images of $dp(X)$ and of $dp(Y)$, thus the space of all such maps has dimension 6. By Lemma 56, we have

$$\left\{ \begin{array}{l} \langle dp(X), dq(X)\rangle = 0; \\ \langle dp(Y), dq(Y)\rangle = 0; \\ \langle dp(X), dq(Y)\rangle + \langle dp(Y), dq(X)\rangle = 0. \end{array} \right. \tag{45}$$

Consider this as a system of linear equations with the coordinates of $dq(X)$, $dq(Y)$ as unknowns and the coordinates of $dp(X)$ and $dp(Y)$ as coefficients. Since the vectors $dp(X)$ and $dp(Y)$ are linearly independent, the system has rank 3. Thus the solution space has dimension 3. On the other hand, for every $\eta \in \mathbb{R}^3$, the pair $(r_\eta(dp(X)), r_\eta(dp(Y)))$ solves the system (45), and for different η the solutions are different. It follows that there exists a unique η such that $dq(X) = \eta \times dp(X)$ and $dq(Y) = \eta \times dp(Y)$.

The vector $\eta = \eta(x)$ depends on x smoothly, because the coefficients of the system (45) do. □

Intuitively, Lemma 61 means that, under an infinitesimal isometric deformation, an infinitely small piece of the surface $p(M)$ around the point $p(x)$ undergoes an infinitesimal rotation with the angular velocity vector $\eta(x)$.

Remark 62. Besides an infinitesimal rotation, each small piece of $p(M)$ undergoes a translation, but at the moment we ignore this.

Lemma 63. *An infinitesimal isometric deformation q is trivial if and only if its rotation field is constant.*

Proof. If q is trivial, then there exists an infinitesimal isometry ξ of $\mathbb{R}^3$ such that $q(x) = \xi(p(x))$ for all $x \in M$. From the classification of infinitesimal isometries of $\mathbb{R}^3$ (Lemma 24 in Chapter I and Exercise 6 in Chapter II) it follows that

$$q(x) = \eta_0 \times p(x) + b_0, \tag{46}$$

for some $\eta_0, b_0 \in \mathbb{R}^3$. This implies

$$dq_x = \eta_0 \times dp_x, \tag{47}$$

that is $\eta(x) = \eta_0$ for all $x \in M$.

In the opposite direction, assume that (47) holds for all $x \in M$. By integrating (47) along a path on M that joins x and y, we obtain

$$q(x) - q(y) = \eta_0 \times (p(x) - p(y)),$$

for all $x, y \in M$. It follows that there exists $b_0 \in \mathbb{R}^3$ such that (46) holds, that is q is trivial. □

§2.2. *Local properties of the rotation field.* We will work in local coordinates. Every point $x \in M$ has a neighborhood W which is diffeomorphic to a subset U of the plane. By identifying W with U, an immersion p of M can locally be represented as

$$\begin{array}{rccl} p: & U & \to & \mathbb{R}^3, \\ & (u,v) & \mapsto & p(u,v), \end{array}$$

where u, v are coordinates in $\mathbb{R}^2 \supset U$. We use the classical notation

$$p_u := \frac{\partial p}{\partial u}, \quad p_v := \frac{\partial p}{\partial v}.$$

For every point $x \in U$, the vectors p_u and p_v are the images of the standard basis under the map dp_x:

$$p_u = dp_x(1,0), \quad p_v = dp_x(0,1).$$

Therefore they form a basis of the tangent space $T_{p(x)}p(U)$.

We use a similar notation for the maps $q : U \to \mathbb{R}^3$ and $\eta : U \to \mathbb{R}^3$. Note that, as q and η are not necessarily immersions, the pairs of vectors q_u, q_v and η_u, η_v can be linearly dependent.

Lemma 64. *At every point $x \in U$, we have*

$$\eta_u \times p_v = \eta_v \times p_u. \tag{48}$$

Proof. By substituting in the equation $dq(X) = \eta \times dp(X)$ the vectors of the standard basis we obtain

$$\begin{array}{rcl} q_u & = & \eta \times p_u, \\ q_v & = & \eta \times p_v. \end{array}$$

Differentiating the first equation with respect to v and the second one with respect to u yields

$$\begin{array}{rcl} q_{uv} & = & \eta_v \times p_u + \eta \times p_{uv}, \\ q_{vu} & = & \eta_u \times p_v + \eta \times p_{vu}. \end{array}$$

Since $q_{uv} = q_{vu}$, we can equate the right hand sides of these two equations. Then $p_{uv} = p_{vu}$ implies (48). □

Lemma 65. *For every $x \in U$, the image of the map $d\eta_x$ is contained in the image of dp_x, that is in the tangent plane $T_{p(x)}p(U)$ of the immersion p. In other words, there exist smooth functions a, b, c, d on U such that*

$$\begin{array}{rcl} \eta_u & = & ap_u + bp_v, \\ \eta_v & = & cp_u + dp_v. \end{array} \tag{49}$$

Besides, we have $a + d = 0$.

In the matrix form,

$$\begin{pmatrix}\eta_u\\ \eta_v\end{pmatrix} = A\begin{pmatrix}p_u\\ p_v\end{pmatrix} \tag{50}$$

with $\operatorname{tr} A = 0$.

Proof. Denote

$$\omega := \eta_u \times p_v = \eta_v \times p_u$$

(where the last equality holds by Lemma 64). If $\omega = 0$, then η_u is collinear to p_v, and η_v is collinear to p_u. Thus, equations (49) hold with $a = d = 0$.

If $\omega \neq 0$, then it is orthogonal to both p_u and p_v. Therefore it is a nonzero multiple of the normal vector ν to $T_{p(x)}p(U)$. As ω is also orthogonal to η_u and η_v, this implies that η_u and η_v lie in $T_{p(x)}p(U)$. Thus there exist numbers a, b, c, and d such that (49) holds.

Equation $a + d = 0$ follows by substituting (49) into (48).

Finally, a, b, c, and d depend smoothly on x because the vectors η_u, η_v, p_u, and p_v do. □

Lemma 66. *Let* I *and* II *be the matrices of the first and of the second fundamental forms of the immersion* p *in the basis* (p_u, p_v)*, and let* I_η *and* II_η *be the matrices of the first and of the second fundamental forms of* η *in the basis* (η_u, η_v)*. Then we have*

$$\mathrm{I}_\eta = A \cdot \mathrm{I} \cdot A^\top, \tag{51}$$

$$\mathrm{II}_\eta = A \cdot \mathrm{II}, \tag{52}$$

where A *is the matrix from Lemma 65.*

Proof. Due to (50), we have

$$\begin{aligned}\mathrm{I}_\eta &= \begin{pmatrix}\langle\eta_u,\eta_u\rangle & \langle\eta_u,\eta_v\rangle\\ \langle\eta_u,\eta_v\rangle & \langle\eta_v,\eta_v\rangle\end{pmatrix} = \begin{pmatrix}\eta_u\\ \eta_v\end{pmatrix}\begin{pmatrix}\eta_u & \eta_v\end{pmatrix}\\ &= A\begin{pmatrix}p_u\\ p_v\end{pmatrix}\begin{pmatrix}p_u & p_v\end{pmatrix}A^\top = A\begin{pmatrix}\langle p_u,p_u\rangle & \langle p_u,p_v\rangle\\ \langle p_u,p_v\rangle & \langle p_v,p_v\rangle\end{pmatrix}A^\top = A\cdot\mathrm{I}\cdot A^\top.\end{aligned}$$

This proves (51).

To show (52), take the partial derivatives of (50):

$$\begin{aligned}\begin{pmatrix}\eta_{uu}\\ \eta_{vu}\end{pmatrix} &= A_u\begin{pmatrix}p_u\\ p_v\end{pmatrix} + A\begin{pmatrix}p_{uu}\\ p_{vu}\end{pmatrix},\\ \begin{pmatrix}\eta_{uv}\\ \eta_{vv}\end{pmatrix} &= A_v\begin{pmatrix}p_u\\ p_v\end{pmatrix} + A\begin{pmatrix}p_{uv}\\ p_{vv}\end{pmatrix}.\end{aligned}$$

By Lemma 65, the unit normal ν to the surface $p(U)$ is also normal to $\eta(U)$. Therefore by multiplying scalarly with ν, we obtain

$$\mathrm{II}_\eta = \begin{pmatrix}\langle\eta_{uu},\nu\rangle & \langle\eta_{uv},\nu\rangle\\ \langle\eta_{uv},\nu\rangle & \langle\eta_{vv},\nu\rangle\end{pmatrix} = A\begin{pmatrix}\langle p_{uu},\nu\rangle & \langle p_{uv},\nu\rangle\\ \langle p_{uv},\nu\rangle & \langle p_{vv},\nu\rangle\end{pmatrix} = A\cdot\mathrm{II}.$$

□

Remark 67. In Lemma 66 we tacitly assume that the vectors η_u and η_v are linearly independent, so that the second fundamental form of η is well-defined. Even without this assumption, the argument in the second half of the proof of Lemma 66 shows that the matrix $A \cdot \mathrm{II}$ is symmetric. By using the standard notation

$$\mathrm{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

we can write this down as

$$aM + bN = cL + dM. \tag{53}$$

Lemma 68. *If the immersion p has a positive Gauss curvature at x, then* $\det A \le 0$ *at x. Besides, if* $\det A = 0$, *then* $A = 0$.

Proof. On the space of symmetric 2×2 matrices consider the following bilinear form:

$$D\left(\begin{pmatrix} x_1 & y_1 \\ y_1 & z_1 \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ y_2 & z_2 \end{pmatrix}\right) = \frac{1}{2}(x_1 z_2 + z_1 x_2) - y_1 y_2.$$

The associated quadratic form is just the determinant:

$$D(X, X) = \det X.$$

Consider two symmetric matrices

$$\mathrm{II} = \begin{pmatrix} L & M \\ M & N \end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

(The matrix A' is symmetric because we have $a + d = 0$ by Lemma 65.) The positivity of the Gauss curvature of p means

$$D(\mathrm{II}, \mathrm{II}) > 0. \tag{54}$$

And the equation (53) can be rewritten as

$$D(A', \mathrm{II}) = 0. \tag{55}$$

It is easy to see that the signature of the form D is $(+, -, -)$. Therefore equations (54) and (55) imply

$$D(A', A') \le 0$$

with an equality if and only if the matrix vanishes. Since $\det A = \det A' = D(A', A')$, the lemma follows. □

In the next two theorems we assume that, at the point x, the vectors η_u and η_v are linearly independent.

Theorem 17. *Assume that the immersion p has a positive Gauss curvature at x. Then either* $\mathrm{rk}\, d\eta_x = 2$ *and η has a negative Gauss curvature at x, or* $d\eta_x \equiv 0$.

Proof. Equation (50) implies

$$\operatorname{rk} d\eta_x = \operatorname{rk} A.$$

Thus, if $\operatorname{rk} d\eta_x = 2$, then the matrix A is non-degenerate. Then, by Lemma 68, we have $\det A < 0$. From equation (52) it follows that

$$\det \mathrm{II}_\eta = \det A \cdot \det \mathrm{II} < 0.$$

Similarly, $\operatorname{rk} d\eta_x < 2$ implies $\operatorname{rk} A < 2$ and hence by Lemma 68 we have $A = 0$, that is $d\eta_x = 0$. □

Theorem 18. *If x is an umbilic point of p, then η has zero mean curvature at x. In particular, if $p(U)$ is a subset of a sphere, then the surface $\eta(U)$ is minimal at its regular points.*

An *umbilic point* is a point where principal curvatures are equal.

Proof. If the principal curvatures of p are equal, then the equation $\det(\mathrm{II} - \lambda \mathrm{I}) = 0$ has a double root. Since the matrices II and I are symmetric, it follows that

$$\mathrm{II} = k\mathrm{I},$$

where k is the principal curvature of p at x. By applying Lemma 66, we obtain

$$\begin{aligned} \mathrm{II}_\eta - \lambda \mathrm{I}_\eta &= A \cdot (\mathrm{II} - \lambda \mathrm{I} \cdot A^\top) = A \cdot (k\mathrm{I} - \lambda \mathrm{I} \cdot A^\top) \\ &= A \cdot \mathrm{I} \cdot (kE - \lambda A^\top) = -\lambda A \cdot \mathrm{I} \cdot (A - \frac{k}{\lambda} E)^\top. \end{aligned}$$

(If $k = 0$, then $\lambda = 0$ is a double root of $\det(\mathrm{II}_\eta - \lambda \mathrm{I}_\eta) = 0$, and both principal curvatures of η at x vanish. If $k \neq 0$, then $\lambda = 0$ is not a root.) Equation (50) and assumption that η_u and η_v are linearly independent imply that the matrix A is non-degenerate. Therefore

$$\det(\mathrm{II}_\eta - \lambda \mathrm{I}_\eta) = 0 \quad \Longleftrightarrow \quad \det(A - \frac{k}{\lambda} E) = 0.$$

Thus the principal curvatures λ_1, λ_2 of η are related to eigenvalues μ_1, μ_2 of the matrix A through

$$\lambda_i = \frac{k}{\mu_i}.$$

By Lemma 65, the matrix A is traceless, therefore we have $\mu_1 + \mu_2 = 0$. It follows that $\lambda_1 + \lambda_2 = 0$, that is the mean curvature of η vanishes. □

Exercise 16. Assume that at a point x the vectors q_u and q_v are linearly independent. Show that if p has a positive Gauss curvature at x, then q has a negative Gauss curvature at x.

3. **Infinitesimal rigidity of smooth surfaces of positive Gauss curvature.**

§3.1. *Main theorem.*

Theorem 19. *Let $p : \mathbb{S}^2 \to \mathbb{R}^3$ be an immersion with everywhere positive Gauss curvature:*

$$K(x) > 0 \text{ for all } x.$$

Then p is infinitesimally rigid.

Remark 69. The condition $K(x) > 0$ for all x implies that the surface $p(\mathbb{S}^2)$ is convex, that is lies to one side from each of its tangent planes. Moreover, $p(\mathbb{S}^2)$ is convex if and only if $K(x) \geq 0$ for all x.

However, the condition $K(x) \geq 0$ is not sufficient for the immersion p to be infinitesimally rigid. As we have seen in Example 7, every surface that contains a planar piece is infinitesimally flexible, and it is possible to construct a convex smooth surface with a planar piece. But if a convex surface contains no planar piece, then it is infinitesimally rigid. With a bit of extra work, the argument we will use to prove Theorem 19 can be extended to cover this case.

Let us discuss one possible approach to Theorem 19.

By Theorem 17, the rotation field η viewed as a smooth map $\mathbb{S}^2 \to \mathbb{R}^3$ has Gauss curvature $K_\eta(x) < 0$ for all x except those where $d\eta_x \equiv 0$. If we assume for the moment that η is an immersion at all points x, then we arrive to a contradiction. Indeed, a smooth surface homeomorphic to the sphere cannot have negative Gauss curvature everywhere, because by the Gauss-Bonnet formula the integral of the curvature is 4π. Even more, no *compact* surface can have everywhere negative Gauss curvature, because a compact surface always has a tangent plane that has the surface on one side (take the plane $z = z_{\max}$ through the point with the largest z-coordinate), in which case the Gauss curvature at a point of tangency is non-negative. But what to do if η has singular points?

In his original proof of Theorem 19 in 1900, Liebmann showed that if the map η is not constant, then at its singular points it has the “character of a surface of negative curvature”, which means that, for a singular point x, no plane through $\eta(x)$ has the surface $\eta(\mathbb{S}^2)$ on one side. By the argument in the previous paragraph, this cannot happen with a compact surface, thus $\eta(\mathbb{S}^2)$ must degenerate to a point, which means that the corresponding deformation q is trivial.

It should be noted that Liebmann assumed the analyticity of the immersion p and of the deformation q. Without the analyticity assumption, the study of singularities of the map η is more complicated and was accomplished by Sabitov in 1965.

The proof that we will present here is due to Blaschke (about 1915). It also relies on Theorem 17 (either $K_\eta(x) < 0$ or $d\eta_x \equiv 0$). Before proceeding to the proof, we need to introduce an additional tool.

§3.2. *Support function.*

Definition 70. *Let* $p : \mathbb{S}^2 \to \mathbb{R}^3$ *be a smooth convex embedding. The* support function *of* p *is a map*

$$h : \mathbb{S}^2 \to \mathbb{R}$$

that assigns to x *the distance from the coordinate origin to the tangent plane at* $p(x)$ *to the surface* $p(\mathbb{S}^2)$*:*

$$h(x) = \operatorname{dist}\big(0, T_{p(x)}p(\mathbb{S}^2)\big).$$

The distance is signed, so that it is positive if the origin and the surface lie on the same side of the tangent plane.

Lemma 71. *In any coordinate chart, the following equation holds:*

$$h \cdot \mathrm{d\,area} = -\langle p, p_u \times p_v\rangle \cdot du \wedge dv. \tag{56}$$

Here $\mathrm{d\,area}$ *is the volume form of the Riemannian metric induced by the immersion* p*.*

Proof. Let $\nu(x)$ be the inward pointing unit normal to the surface $p(\mathbb{S}^2)$ at the point $p(x)$. It is easy to see that

$$h = -\langle p, \nu\rangle.$$

On the other hand, we have

$$\nu \cdot \mathrm{d\,area} = p_u \times p_v \cdot du \wedge dv.$$

Thus we have

$$h \cdot \mathrm{d\,area} = -\langle p, \nu \cdot \mathrm{d\,area}\rangle = -\langle p, p_u \times p_v\rangle du \wedge dv.$$

□

Note that both sides of equation (56) represent thrice the volume of a pyramid spanned by the coordinate origin and by the area element of the surface. In particular, we have

$$\operatorname{vol}(B) = \frac{1}{3}\int_{\mathbb{S}^2} h \cdot \mathrm{d\,area},$$

where B is the body enclosed by the surface $p(\mathbb{S}^2)$.

§3.3. *Proof of the main theorem.* Consider the 2–form

$$2 \det A \cdot h \cdot \mathrm{d\,area}$$

on $\mathbb{S}^2$. By Lemma 73, this form is exact: there is a 1–form ω on $\mathbb{S}^2$ such that

$$d\omega = 2 \det A \cdot h \cdot \mathrm{d\,area}\,.$$

It follows that

$$\int_{\mathbb{S}^2} \det A \cdot h \cdot \mathrm{d\,area} = 0. \tag{57}$$

Without loss of generality we can assume that the coordinate origin lies in the region bounded by the surface $p(\mathbb{S}^2)$. Then we have

$$h(x) > 0 \text{ for all } x \in \mathbb{S}^2.$$

Since, by Lemma 18, we have

$$\det A \leq 0 \text{ for all } x \in \mathbb{S}^2,$$

equation (57) implies that $\det A = 0$ for all x. This, by Lemma 18, implies that $A = 0$, and thus

$$d\eta = 0 \text{ for all } x \in \mathbb{S}^2.$$

Therefore η is constant, which means that the corresponding infinitesimal isometric deformation is trivial.

Remark 72. The matrix A was defined as a transition matrix from (p_u, p_v) to (η_u, η_v), therefore it depends on the choice of local coordinates. However, it can be shown that $\det A$ does not depend on that choice. This also follows from Lemma 73.

Lemma 73. *Put* $\omega = \langle \eta \times p, d\eta \rangle$. *Then we have* $d\omega = 2 \det A \cdot h \cdot \mathrm{d\,area}$.

Proof. In every coordinate chart, $d\omega$ can be written as $f du \wedge dv$, where f is a function in variables u, v. The function f can be computed via

$$f = d\omega(\partial_u, \partial_v),$$

where ∂_u, ∂_v are the coordinate vector fields. For arbitrary vector fields X, Y, we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Since coordinate vector fields commute, we have

$$\begin{array}{rcl} f & = & \partial_u(\omega(\partial_v)) - \partial_v(\omega(\partial_u)) \\ & = & \partial_u \langle \eta \times p, \eta_v \rangle - \partial_v \langle \eta \times p, \eta_u \rangle \\ & = & (\langle \eta_u \times p, \eta_v \rangle - \langle \eta_v \times p, \eta_u \rangle) \\ & + & (\langle \eta \times p_u, \eta_v \rangle - \langle \eta \times p_v, \eta_u \rangle) \\ & + & (\langle \eta \times p, \eta_{vu} \rangle - \langle \eta \times p, \eta_{uv} \rangle) = \text{(i)} + \text{(ii)} + \text{(iii)}. \end{array}$$

The summand (iii) vanishes because of $\eta_{uv} = \eta_{vu}$. The summand (ii) also vanishes, because

$$\begin{array}{rcl} \langle \eta \times p_u, \eta_v \rangle - \langle \eta \times p_v, \eta_u \rangle & = & \langle p_u \times \eta_v, \eta \rangle - \langle p_v \times \eta_u, \eta \rangle \\ & = & \langle p_u \times \eta_v - p_v \times \eta_u, \eta \rangle, \end{array}$$

and we have $p_u \times \eta_v - p_v \times \eta_u = 0$ by Lemma 14. The summand (i) can be rewritten as

$$\langle \eta_u \times p, \eta_v \rangle - \langle \eta_v \times p, \eta_u \rangle = \langle \eta_v \times \eta_u, p \rangle - \langle \eta_u \times \eta_v, p \rangle = -2 \langle \eta_u \times \eta_v, p \rangle.$$

Thus we have

$$d\omega = -2 \langle \eta_u \times \eta_v, p \rangle du \wedge dv.$$

Equation

$$\begin{pmatrix} \eta_u \\ \eta_v \end{pmatrix} = A \begin{pmatrix} p_u \\ p_v \end{pmatrix}$$

easily implies

$$\eta_u \times \eta_v = \det A \cdot p_u \times p_v.$$

Hence

$$d\omega = -2 \det A \cdot \langle p, p_u \times p_v \rangle du \wedge dv = 2 \det A \cdot h \cdot \mathrm{d}\,\mathrm{area},$$

where the last equation follows from (56). □

The following exercise is not related to the infinitesimal rigidity, but its proof is similar to the proof of Lemma 73, and the result is a very nice geometric identity.

Exercise 17. Let $p : \mathbb{S}^2 \to \mathbb{R}^3$ be a smooth convex embedding, and let $\nu : \mathbb{S}^2 \to \mathbb{R}^3$ be the corresponding unit normal. Consider a 1–form

$$\beta = \langle p \times \nu, d\nu \rangle.$$

Show that

$$d\beta = 2(H - hK)\mathrm{d}\,\mathrm{area},$$

where H is the mean curvature, K is the Gauss curvature, and h is the support function of p.

Derive from this the equation

$$\int_{\mathbb{S}^2} H \mathrm{d}\,\mathrm{area} = \int_{\mathbb{S}^2} hK \mathrm{d}\,\mathrm{area}\,. \tag{58}$$

The left hand side of (58) is called the *total mean curvature* of the surface $p(\mathbb{S}^2)$. The right hand side has the following interpretation. Since $K\mathrm{d}\,\mathrm{area} = d\nu$, we are integrating the support function with respect to the measure of the Gauss image. We have

$$\int_{\mathbb{S}^2} h d\nu = \frac{1}{2} \int_{\mathbb{S}^2} (h(\nu) - h(-\nu)) d\nu,$$

and the difference $h(\nu) - h(-\nu)$ is the distance between two parallel tangent planes to the surface. This is called the *width* in the direction of ν. Thus equation (58) says that the total mean curvature is proportional to the average width.

4. **Affine and projective invariance of the infinitesimal rigidity of surfaces.**

§4.1. *Affine invariance.* Let M be a smooth n-dimensional manifold.

Theorem 20. *Let $p : M \to \mathbb{R}^d$ be an immersion, and let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be an affine transformation:*

$$\Phi(x) = Cx + b,$$

$C \in \mathrm{GL}(d, \mathbb{R}), b \in \mathbb{R}^d$. *Then the immersion $p' = \Phi \circ p$ is infinitesimally rigid if and only if p is infinitesimally rigid.*

Moreover, if $q : M \to \mathbb{R}^d$ is an infinitesimal isometric deformation of p, then

$$q' = (C^{-1})^* \circ q$$

is an infinitesimal isometric deformation of p'. Here $*$ denotes the adjoint with respect to the scalar product in $\mathbb{R}^d$. The deformation q' is trivial if and only if q is trivial.

Proof. Since the infinitesimal rigidity is, by definition, triviality of all infinitesimal isometric deformations, the first part of the theorem follows from the second part. So let us prove the second part.

We will use Lemma 6(a): q is an infinitesimal isometric deformation of p if and only if

$$\langle dp, dq \rangle = 0$$

(by which we mean $\langle dp_x(X), dq_x(X) \rangle = 0$ for all $x \in M$, $X \in T_xM$). We have

$$\begin{aligned} dp' &= d\Phi \circ dp = C \circ dp, \\ dq' &= (C^{-1})^* \circ dq. \end{aligned}$$

Therefore

$$\langle dp', dq' \rangle = \langle C \circ dp, (C^{-1})^* \circ dp \rangle = \langle C^{-1} \circ C \circ dp, dq \rangle = \langle dp, dq \rangle.$$

Thus $\langle dp', dq' \rangle = 0$ if and only if $\langle dp, dq \rangle = 0$.

Let us show that q' is trivial if and only if q is. If q is trivial, then we have

$$q = Zp + d,$$

where Z is a skew-symmetric $d \times d$ matrix, and $d \in \mathbb{R}^d$. Denote for brevity $A = C^{-1}$. Then we have $p' = A(p - b)$, and thus

$$q' = A^*(Zp + d) = A^*(Z(Ap' - Ab) + d) = A^*ZAp' + d',$$

where $d' = A^*ZAb + A^*d \in \mathbb{R}^d$. Since $A^* = A^\top$ and Z is skew-symmetric, the matrix A^*ZA is also skew-symmetric, that is q' is a trivial deformation of p. Similarly, triviality of q' implies triviality of q. □

§4.2. *Projective interpretation of infinitesimal isometric deformations.* Let us recall our strategy in proving the projective invariance of the infinitesimal rigidity of frameworks. We introduced the notion of static rigidity that deals with forces applied at the vertices of frameworks rather than with infinitesimal motions of the vertices. The forces were shown to be in some sense dual to the motions. Then we interpreted forces as bivectors in $\mathbb{R}^{d+1}$ where $\mathbb{R}^d$ is embedded as a hyperplane $x_0 = 1$. This led us to a formulation of static rigidity in terms of linear algebra of $\mathbb{R}^{d+1}$. As projective transformations of $\mathbb{R}^d$ are represented by linear transformations of $\mathbb{R}^{d+1}$, such a formulation immediately yields projective invariance of the static rigidity, and hence of the infinitesimal rigidity.

In the case of smooth manifolds, the theory of static rigidity seems not to be developed. But we can make a shortcut and try to find an interpretation of infinitesimal isometric deformations in terms of linear algebra of $\mathbb{R}^{d+1}$.

What the discrete case does teach us is that the motions should be interpreted dually to the forces, that is they should be viewed as antisymmetric bilinear forms in $\mathbb{R}^{d+1}$.

Recall that we put $\mathbb{R}^d = \{x \in \mathbb{R}^{d+1} \mid x_0 = 0\}$. This leads to a double notation $v = (0, v)$ for every $v \in \mathbb{R}^d$.

Definition 74. *Let $p : M \to \mathbb{R}^d$ be an immersion. A smooth map $\bar{p} : M \to \mathbb{R}^{d+1}$ is called a* lift *of p if*

$$\bar{p}(x) = \lambda(x) \cdot (1, p(x)),$$

for some smooth function $\lambda : M \to \mathbb{R} \setminus \{0\}$.

The map $x \mapsto (1, p(x))$ will be called the *standard lift* of p.

Definition 75. *Let $q : M \to \mathbb{R}^d$ be a smooth map viewed as a vector field along p. A smooth map $\bar{q} : M \to \Lambda^2(\mathbb{R}^{d+1})^*$ is called a* lift *of q if*

$$\bar{q}((1, p) \wedge f) = \langle f, q \rangle \quad \textit{for all } f \in \mathbb{R}^d. \tag{59}$$

Remark 76. As so often, in (59) we suppressed a variable. In an expanded form, the equation reads

$$\bar{q}(x)((1, p(x)) \wedge f) = \langle f, q(x) \rangle \quad \text{for all } x \in M \text{ and all } f \in \mathbb{R}^d.$$

Definition 75 is motivated by our study of static rigidity. The bivector $(1, p(x)) \wedge f$ corresponds to a force f applied at the point $p(x)$. The scalar product $\langle f, q(x) \rangle$ is the virtual work of the force f on the infinitesimal motion $q(x)$.

The equation (59) prescribes the value of $\bar{q}(x)$ only on bivectors divisible through $(1, p)$. This determines $\bar{q}(x)$ modulo $\Lambda^2((1, p(x))^\perp)$. It is not hard to show that every smooth map q has a lift.

Lemma 77. *The map q is an infinitesimal isometric deformation of p if and only if*

$$d\bar{q}\,(\bar{p} \wedge d\bar{p}) = 0 \tag{60}$$

for some (and then for all) lifts $\bar{p}$, $\bar{q}$ of p, q.

Proof. In an expanded form, equation (60) reads

$$d\bar{q}_x(X)(\bar{p}(x) \wedge d\bar{p}_x(X)) = 0,$$

for all $x \in M$, $X \in T_xM$. Consider a path $\gamma : [0, 1] \to M$ such that $\gamma(0) = x$, $\dot{\gamma}(0) = X$. Denote

$$\begin{aligned} v(t) &:= p(\gamma(t)), \\ w(t) &:= q(\gamma(t)), \\ W(t) &:= \bar{q}(\gamma(t)). \end{aligned}$$

Substitute in (59) $x = \gamma(t)$, $f = \dot{v}(t)$. Then we have

$$W(t)\big((1, v(t)) \wedge \dot{v}(t)\big) = \big\langle \dot{v}(t), w(t) \big\rangle.$$

By differentiating this with respect to t, we obtain

$$\dot{W}(t)\big((1, v(t)) \wedge \dot{v}(t)\big) + W(t)\big(\dot{v}(t) \wedge \dot{v}(t)\big) + W(t)\big((1, v(t)) \wedge \ddot{v}(t)\big) \\ = \big\langle \ddot{v}(t), w(t) \big\rangle + \big\langle \dot{v}(t), \dot{w}(t) \big\rangle.$$

The second term on the left hand side vanishes. Also, because of (59), we have

$$W(t)\big((1, v(t)) \wedge \ddot{v}(t)\big) = \big\langle \ddot{v}(t), w(t) \big\rangle.$$

Hence

$$\dot{W}(t)\big((1, v(t)) \wedge \dot{v}(t)\big) = \big\langle \dot{v}(t), \dot{w}(t) \big\rangle,$$

which at $t = 0$ can be rewritten as

$$d\bar{q}_x(X)\big((1, p(x)) \wedge dp_x(X)\big) = \big\langle dp_x(X), dq_x(X) \big\rangle. \tag{61}$$

The map q is an infinitesimal isometric deformation of p if and only if the right hand side of (61) vanishes for all x and X. This proves Lemma 77 in the case when $\bar{q}$ is an arbitrary lift of q, and $\bar{p}$ is the standard lift $(1, p)$ of p.

For an arbitrary lift $\bar{p} = \lambda \cdot (1, p)$ of p, we have

$$\bar{p} \wedge d\bar{p} = \lambda \cdot (1, p) \wedge (d\lambda \cdot (1, p) + \lambda \cdot dp) = \lambda^2 \cdot (1, p) \wedge dp.$$

Thus, if the left hand side of (60) vanishes for some lift of p, then it vanishes for all lifts of p. □

Remark 78. Recall that due to Lemma 6 the relation between p and q is symmetric: if q is an immersion, then the map p can be viewed as an infinitesimal isometric deformation of q. Definitions of lifts $\bar{p}$ and $\bar{q}$ destroy this symmetry, so that we arrive to an "asymmetric" criterion (60).

Lemma 79. *A map q is a trivial infinitesimal isometric deformation of p if and only if it posesses a constant lift $\bar{q}$.*

Proof. Assume that q has a constant lift

$$\bar{q}(x) = \bar{Z} \in \Lambda^2(\mathbb{R}^{d+1})^*$$

for all $x \in M$. The standard Euclidean structure of $\mathbb{R}^{d+1}$ identifies $(\mathbb{R}^{d+1})^*$ with $\mathbb{R}^{d+1}$, which allows to represent $\bar{Z}$ as

$$\bar{Z} = \begin{pmatrix} 0 & b^\top \\ -b & Z \end{pmatrix} \tag{62}$$

with $b \in \mathbb{R}^d$ and a $d \times d$ antisymmetric matrix Z. Equation (59) implies then

$$q = Zp + b,$$

that is q is an infinitesimal isometric deformation of p.

In the opposite direction, if $q = Zp + b$ with an antisymmetric matrix Z, then equation (62) defines a constant lift of q. □

§4.3. *Projective invariance.*

Theorem 21. *Let $p : M \to \mathbb{R}^d$ be an immersion, and let $\mathbb{R}\mathrm{P}^d = \mathbb{R}^d \cup l_\infty$. Let*

$$\Phi : \mathbb{R}\mathrm{P}^d \to \mathbb{R}\mathrm{P}^d$$

be a projective transformation such that $\Phi(p(x)) \notin l_\infty$, for all $x \in M$. Then the immersion $p' = \Phi \circ p$ is infinitesimally rigid if and only if p is infinitesimally rigid.

Proof. Let $C \in \mathrm{GL}(d+1, \mathbb{R})$ be a linear map representing Φ. There is an isomorphism

$$C^\sharp : \Lambda^2(\mathbb{R}^{d+1})^* \to \Lambda^2(\mathbb{R}^{d+1})^*$$

such that

$$C^\sharp(\omega)(x \wedge y) = \omega(C^{-1}x \wedge C^{-1}y) \tag{63}$$

for all $x \wedge y \in \Lambda^2\mathbb{R}^{d+1}$. For a map $q : M \to \Lambda^2(\mathbb{R}^{d+1})^*$, put

$$\bar{q}' := C^\sharp(\bar{q}),$$

where $\bar{q}$ is an arbitrary lift of q. Since $\bar{p}' = C\bar{p}$ is a lift of p', equation (63) implies

$$d\bar{q}'\,(\bar{p}' \wedge d\bar{p}') = d\bar{q}\,(\bar{p} \wedge d\bar{p}).$$

Equation (59) allows to reconstruct from $\bar{q}'$ a map $q' : M \to \mathbb{R}^d$. (Consider the covector $i_{(1,p'(x))}\bar{q}'|_{\mathbb{R}^d}$; the scalar product on $\mathbb{R}^d$ associates to it a vector; this vector is $q'(x)$.) Thus, q' is an infinitesimal isometric deformation of p' if and only if q is an infinitesimal isometric deformation of p.

If q has a constant lift $\bar{q}$, then $\bar{q}'$ is also constant. Thus, if q is trivial, then q' is also trivial. Since the above construction can be inverted, triviality of q' implies triviality of q as well. □

Lemma 80. *Let p, Φ, and p' be as in Theorem 21, and let q be an infinitesimal isometric deformation of p. Then*

$$q'(x) = h_l^{-2}(p(x)) \cdot (d\Phi^{-1}_{p(x)})^* q(x)$$

is an infinitesimal isometric deformation of p'. Here $l \subset \mathbb{R}^d$ is a hyperplane such that $\Phi(l) = l_\infty$, and h_l is the distance to l.

Also, q' is trivial if and only if q is trivial.

Proof. This can be derived from the formula for $\bar{q}'$ in the proof of Theorem 21 by arguments similar to those in the proof of Theorem 14. □

5. **Translation field of an infinitesimal isometric deformation.** Now we come back to the case of a surface M and an immersion $p : M \to \mathbb{R}^3$. Recall that with an infinitesimal isometric deformation q of p we associate a *rotation field* η such that

$$dq = \eta \times dp.$$

Vector $\eta(x)$ is the common angular velocity vector for the infinitesimal motions of the tangent vectors to the immersion p at the point $p(x)$. Aside from a rotation, the tangent space $T_{p(x)}p(M)$ undergoes also a translation so that we have

$$q(x) = \eta(x) \times p(x) + \tau(x)$$

for all $x \in M$.

Definition 81. *Let η be the rotation field of an infinitesimal isometric deformation q of an immersion p. The map*

$$\tau = q - \eta \times p$$

is called the translation field *of deformation q.*

The next lemma shows that the relation between pairs (p, q) and (η, τ) is symmetric.

Lemma 82. *Assume that η is an immersion. Then τ is an infinitesimal isometric deformation of η, with rotation field p and translation field q.*

Proof. Exercise. □

Lemma 83. *Let $d\eta = A \cdot dp$. Then $d\tau = A \cdot (p \times dp)$.*

Proof. Exercise. □

6. **Examples of infinitesimal isometric deformations.**

§6.1. *Formulas in the case of graphs of smooth functions.* Let $U \subset \mathbb{R}^2$ be a simply-connected region, and let $f : U \to \mathbb{R}$ be a smooth function. Consider an immersion $p : U \to \mathbb{R}^3$ which is the graph of f:

$$p(u, v) = (u, v, f(u, v)). \tag{64}$$

We are going to describe the infinitesimal isometric deformations of p.

Assume that $q = (x, y, z)$ is an infinitesimal isometric deformation of p with the associated rotation field $\eta = (\alpha, \beta, \gamma)$. By definition of the rotation field we have $dq = \eta \times dp$, which is equivalent to

$$\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ f_u \end{pmatrix}, \qquad \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_v \end{pmatrix}.$$

(We will switch between row and column notations for vectors, choosing one which is more appropriate for computations or layout.) Calculation of cross-products yields

$$\begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix} = \begin{pmatrix} \beta f_u \\ \gamma - \alpha f_u \\ -\beta \end{pmatrix}, \qquad \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix} = \begin{pmatrix} \beta f_v - \gamma \\ -\alpha f_v \\ \alpha \end{pmatrix}.$$

It follows that $\alpha = z_v$ and $\beta = -z_u$ can be eliminated. As a result, we obtain the following expressions for η, q_u, and q_v in terms of a known function f and two unknown functions γ and z:

$$\eta = (z_v, -z_u, \gamma), \tag{65}$$

$$q_u = \begin{pmatrix} -f_u z_u \\ \gamma - f_u z_v \\ z_u \end{pmatrix}, \qquad q_v = \begin{pmatrix} -f_v z_u - \gamma \\ -f_v z_v \\ z_v \end{pmatrix}. \tag{66}$$

Since $q_{uv} = q_{vu}$, we have

$$\frac{d}{dv} \begin{pmatrix} -f_u z_u \\ \gamma - f_u z_v \\ z_u \end{pmatrix} = \frac{d}{du} \begin{pmatrix} -f_v z_u - \gamma \\ -f_v z_v \\ z_v \end{pmatrix},$$

which is equivalent to

$$\left\{ \begin{array}{rcl} \gamma_u & = & f_u z_{uv} - f_v z_{uu} \\ \gamma_v & = & f_u z_{vv} - f_v z_{uv} \end{array} \right. \tag{67}$$

Conversely, if we have a pair of functions γ, z satisfying system (67), then system (66) can be solved for q, because the domain U is simply-connected. Thus finding all infinitesimal isometric deformations of p is equivalent to finding all solutions of (67).

The system (67) can be solved for γ if and only if

$$(f_u z_{uv} - f_v z_{uu})_v = (f_u z_{vv} - f_v z_{uv})_u,$$

which is equivalent to

$$f_{vv} z_{uu} - 2 f_{uv} z_{uv} + f_{uu} z_{vv} = 0. \tag{68}$$

Thus we have reduced the problem to a second-order partial differential equation with variable coefficients.

Theorem 22. *Functions z solving system* (68) *are in one-to-one correspondence with infinitesimal isometric deformations of the immersion* (64)*, up to composition with an infinitesimal rotation along a vertical axis.*

Moreover, trivial infinitesimal isometric deformations correspond to linear functions z.

Proof. The above arguments show that the vertical component of an infinitesimal isometric deformation of p always satisfies equation (68), and conversely, every solution of (68) leads to an infinitesimal isometric deformation.

By Lemma 13, trivial infinitesimal isometric deformations are those with constant rotation fields. By equation (65), if η is constant, then z is linear. Conversely, if z is linear, then due to (67) γ is constant, and thus rotation field η is constant.

Function z determines the first two coordinates of the rotation field η uniquely, and the third coordinate up to a constant summand. This implies the first part of the theorem. □

§6.2. *Infinitesimal isometric deformations of the paraboloid of revolution.* In the equation (64), put

$$f(u,v) = \frac{u^2+v^2}{2}.$$

Then the image of the immersion p is a paraboloid of revolution. Since we have

$$f_{uu} = 1, \quad f_{uv} = 0, \quad f_{vv} = 1,$$

the equation (68) becomes

$$z_{uu} + z_{vv} = 0.$$

Thus, infinitesimal isometric deformations of a paraboloid of revolution are described by harmonic functions.

If $U \subset \mathbb{R}^2$ is a closed subset with a smooth boundary, then any smooth function on ∂U has a unique extension to a harmonic function inside U. This means that the vertical component z of an infinitesimal isometric deformation can be chosen arbitrarily, and determines the deformation uniquely.

Let us compute a particular infinitesimal isometric deformation of the whole paraboloid of revolution. Harmonic functions on $\mathbb{R}^2$ are just real parts of holomorphic functions, so let us take

$$z = \mathrm{Re}\frac{1}{2}(u+iv)^2 = \frac{u^2-v^2}{2}.$$

From (67) we have

$$\begin{cases} \gamma_u &= -v \\ \gamma_v &= -u. \end{cases}$$

Thus we can put $\gamma = -uv$. For the map q, system (66) becomes

$$q_u = \begin{pmatrix} -u^2 \\ 0 \\ u \end{pmatrix}, \qquad q_v = \begin{pmatrix} 0 \\ v^2 \\ -v \end{pmatrix},$$

so that we can put

$$q = \left(-\frac{u^3}{3}, \frac{v^3}{3}, \frac{u^2-v^2}{2} \right).$$

Also, from (65) we have

$$\eta = (-v, -u, -uv),$$

and we can compute the translation field

$$\tau = q - \eta \times p = \left(\frac{u^3}{6} - \frac{uv^2}{2}, \frac{-v^3}{6} + \frac{u^2 v}{2}, \frac{-u^2+v^2}{2} \right).$$

Note that by Lemma 32, the map τ is an infinitesimal isometric deformation of immersion η.

§6.3. *Infinitesimal isometric deformations of a punctured sphere.* The paraboloid of revolution is projectively equivalent to the punctured sphere (sphere without a point). Therefore, by results of §6.2 and by Theorem 21, the punctured sphere is infinitesimally flexible. Moreover, Lemma 30 provides a formula that associate to every infinitesimal isometric deformation of the paraboloid an infinitesimal isometric deformation of the punctured sphere. Let us write down that formula more explicitely.

Let u, v be coordinates in $\mathbb{R}^2$, and x, y, z coordinates in $\mathbb{R}^3$. We have an immersion $p : \mathbb{R}^2 \to \mathbb{R}^3$:

$$p(u, v) = (u, v, u^2 + v^2)$$

and a projective transformation $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$:

$$\Phi(x, y, z) = \left(\frac{2x}{z+1}, \frac{2y}{z+1}, \frac{z-1}{z+1} \right).$$

The map Φ is not defined on the plane $z + 1 = 0$; when Φ is extended to a map from $\mathbb{R}\mathrm{P}^3$ to $\mathbb{R}\mathrm{P}^3$, then this plane is sent to the plane at infinity. It is easy to see that Φ maps the paraboloid $p(\mathbb{R}^2)$ to a unit sphere centered at $(0, 0, 0)$ with the point $(0, 0, 1)$ removed. In fact, the immersion $p' = \Phi \circ p$ is the inverse of the stereographic projection:

$$p'(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

By Lemma 30, the family of maps

$$h_l^{-2}(p(u, v)) \cdot (\mathrm{d}\Phi^{-1}_{p(u,v)})^* : T_{p(u,v)}\mathbb{R}^3 \to T_{p'(u,v)}\mathbb{R}^3$$

transform infinitesimal isometric deformations of p into infinitesimal isometric deformations of p'. We have already identified the plane l sent by Φ to infinity: this is the plane $z + 1 = 0$. Thus we have

$$h_l(x, y, z) = z + 1.$$

Further, we have

$$\mathrm{d}\Phi_{(x,y,z)} = \frac{2}{(z+1)^2} \begin{pmatrix} z+1 & 0 & -x \\ 0 & z+1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$(\mathrm{d}\Phi^{-1}_{(x,y,z)})^* = \frac{z+1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & z+1 \end{pmatrix}.$$

As a result we have

$$q'(u, v) = \frac{1}{2(u^2 + v^2 + 1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & u^2 + v^2 + 1 \end{pmatrix} q(u, v). \tag{69}$$

In order to compute some particular infinitesimal isometric deformation of the punctured sphere, let us use the deformation of the paraboloid computed in §6.2:

$$q(u,v) = \left(-\frac{u^3}{3}, \frac{v^3}{3}, \frac{u^2-v^2}{4}\right).$$

(This differs from the formula in §6.2 by factor $\frac{1}{2}$ in the third coordinate because the paraboloid we consider here is scaled by factor 2 in the third coordinate.) By substituting this in (69), we obtain

$$q'(u,v) = \frac{1}{6(u^2+v^2+1)}\left(-u^3, v^3, \frac{(v^2-u^2)(u^2+v^2-3)}{4}\right).$$