Kinetic Approach to Stochastic Conservation Laws

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July 30, 2015
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CHAPTER 1

What is the right notion of solution?

In this chapter, we will study the following deterministic scalar equation

\[
\begin{align*}
\frac{\partial}{\partial t} u + \text{div } B(u) &= 0, \\
u(0) &= u_0.
\end{align*}
\]  

(1.1)

Here \( t \in \mathbb{R}^+ = [0, \infty), \, x \in \mathbb{R}^N \). The coefficient \( B = (B_1, \ldots, B_N) : \mathbb{R} \to \mathbb{R}^N \) is called the flux function and the flux term \( \text{div } B(u) \) is defined as

\[
\text{div } B(u) = \sum_{j=1}^N \partial_{x_j} B_j(u) = \sum_{j=1}^N B'_j(u) \partial_{x_j} u.
\]

For notational simplicity we denote \( b = (b_1, \ldots, b_N) = (B'_1, \ldots, B'_N) \).

The solution \( u : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R} \) is a scalar function and equation (1.1) is a conservation law for \( u \), i.e. it describes conservation of the quantity \( u \). Why \( u \) is conserved can be seen easily: let \( N = 1 \) and let us calculate how the amount of \( u \) inside the interval \([a, b]\) changes in time. It holds

\[
\frac{d}{dt} \int_a^b u(t, x) \, dx = \int_a^b \partial_t u(t, x) \, dx = - \int_a^b \partial_x B(u) \, dx = B(u(t, a)) - B(u(t, b))
\]

and therefore we see that the amount of \( u \) inside \([a, b]\) changes only via the inflow and outflow across the boundary of \([a, b]\). Conservation laws represent the very basic physical phenomena such as balance of mass, energy etc.

Our main interest lies in solving (1.1), that is, we want to prove existence, uniqueness of a solution as well as continuous dependence on the initial data. In Chapter 2, we will add some stochastic perturbation on the right hand side of (1.1) which would further complicate the problem. However, as our stochastic perturbation will be rather general and the deterministic case will still be included as a special case, it is necessary to understand the difficulties in solving (1.1) first. To this end, let us start with some particular examples.

1.1. Method of characteristics

One of the basic techniques for solving PDEs is the method of characteristics. It is especially useful for first order equations. The main idea is to reduce the PDE at hand to a system of ODEs that can be solved by elementary techniques. Therefore, one has to search for curves along which the PDEs rewrites as ODEs.

1.1.1. Linear case. Let us restrict ourselves to \( N = 1 \) and let us consider (1.1) with the flux function \( B \) being linear, i.e. \( B(u) = cu \) for some \( c \in \mathbb{R} \). Then
(1.1) rewrites as
\[ \partial_t u + c \partial_x u = 0, \]
\[ u(0) = u_0. \]
In that case, the characteristic curve is governed by
\[ \frac{\dot{x}(t)}{dt} = c \]
hence the solution starting from \( x \in \mathbb{R} \) is given by \( \hat{x}(t,x) = x + ct \). We have the following result.

**Lemma 1.1.1.** Let \( u(t,x) := u_0(x - ct) \). If \( u_0 \in C^1(\mathbb{R}) \), then \( u \in C^1(\mathbb{R}^+ \times \mathbb{R}) \) and it is a classical solution to (1.2).

The proof is simple and follows immediately from the chain rule formula. It follows that in this case, (1.1) possesses an explicit solution and it is the so-called travelling wave. Indeed, the shape of the solution does not change in time as the solution is given by translation of the initial profile \( u(0) = u_0 \). This behavior is particular for the linear case and as we will see later, the main difficulties in conservation laws arise in nonlinear cases where new features appear.

**1.1.2. Nonlinear case.** Let us keep the assumption \( N = 1 \) and let \( b = B' \) be nonlinear. (1.1) rewrites as
\[ \partial_t u + b(u) \partial_x u = 0, \]
\[ u(0) = u_0. \]
Following the method of characteristics, we obtain the following formula for the characteristic curve
\[ \frac{\dot{x}(t)}{dt} = b(u(\hat{x}(t))). \]
Unlike the previous example, where the speed was constant, here it changes in time and also according to position in space. Consequently, also the shape that we see at time \( t \) differs from the one at time 0 and moreover shocks may appear. What else can we deduce?

**Lemma 1.1.2.** Let \( u \in C^1(\mathbb{R}^+ \times \mathbb{R}) \) be a solution to (1.3). Then \( u \) is constant along characteristics. Accordingly, characteristics are straight lines.

**Proof.** To show the first claim, we calculate:
\[ \frac{d}{dt} u(t, \hat{x}(t)) = \partial_t u + \partial_x u \frac{d\hat{x}(t)}{dt} = \partial_t u + \partial_x u b(u) = 0 \]
which implies that \( u(t, \hat{x}(t,x)) = u(0, \hat{x}(0,x)) = u_0(x) \) for all \( t \in \mathbb{R}^+ \). Characteristics are then given by
\[ \hat{x}(t,x) = x + \int_0^t b(u(s, \hat{x}(s,x))) \, ds = x + b(u_0(x)) t \]
and the proof is complete. \( \square \)

Let us now show some of the consequences of the above result.
1.1.3. Inviscid Burgers equation in 1D. Inviscid Burgers equation is an example of (1.1), where the flux function is given by $B(u) = \frac{1}{2}u^2$, it reads as

$$
\begin{align*}
\partial_t u + u \partial_x u &= 0, \\
u(0) &= u_0.
\end{align*}
\tag{1.4}
$$

Using Lemma 1.1.2, the characteristic lines are given by $\hat{x}(t, x) = x + u_0(x)t$ hence they are determined by the initial condition $u_0$. For certain initial conditions this gives a method of solving (1.4) but for other not.

Let $u_0 \equiv 0$. Then one can draw the characteristic lines and identify the solution $u$ at each point $(t, x)$, see Figure 1. Figure 2 shows the characteristic lines corresponding to a nondecreasing piecewise constant initial condition.

Proposition 1.1.3. If $B'' > 0$ and $u_0$ is not monotone increasing then (1.3) does not have a smooth solution.

It follows from the above considerations that we have to extend our notion of solution. Indeed, the notion of classical solution is apparently too strong as already for some simple examples such solutions do not exist. The difficulties are coming from the nonlinearity in the flux but also from absence of second order terms that could provide smoothing. Consequently, we loose regularity even for arbitrarily smooth initial conditions and flux functions and it is necessary to work within the class of discontinuous functions and to consider distributional solutions.
1.2. Weak solutions

In this part, we weaken the notion of solution and study its properties.

**Definition 1.2.1.** A measurable function \( u : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R} \) is called a weak solution to (1.1) if for every test function \( \varphi \in C^1(\mathbb{R} \times \mathbb{R}^N) \) it holds

\[
\int_0^\infty \int_{\mathbb{R}^N} \left( u \partial_t \varphi + \sum_{j=1}^N B_j(u) \partial_{x_j} \varphi \right) \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) \varphi(0, x) \, dx = 0.
\]

Clearly, weak solution does not have to be smooth or not even continuous. Let us now derive some conditions that have to be satisfied at points of jumps.

**Theorem 1.2.2 (Rankine-Hugoniot condition).** Let \( N = 1 \). Let \( N \) be an open neighborhood in \( \mathbb{R}^+ \times \mathbb{R} \). Suppose that a curve \( C : t \in (\alpha, \beta) \mapsto \hat{x}(t) \) divides \( N \) into two pieces \( N_l \) and \( N_r \). Let \( u \) be a weak solution to (1.3) such that

(i) \( u \) is a classical solution in both \( N_l \) and \( N_r \),
(ii) \( u \) has a jump discontinuity at \( C \),
(iii) the jump is continuous along \( C \).

For every \( p \in C \), let \( s = \hat{x}'(p) \) be the slope of \( C \) at \( p \). Then the following relation holds true

\[
s[u] = [B(u)],
\]

provided \([u] = u^r - u^l\) denotes the size of the jump of \( u \) and similarly \([B(u)] = B(u^r) - B(u^l)\) is the size of the jump of \( B(u) \).

**Proof.** Let \( \varphi \in C^1_c(\mathbb{R} \times \mathbb{R}) \) be a test function whose support is included in \( N \). Since \( u \) is a weak solution, we have using the Green’s identity

\[
0 = \int_{N_l} u \partial_t \varphi + B(u) \partial_{x} \varphi \, dx \, dt
\]

\[
= \int_{N_l} u \partial_t \varphi + B(u) \partial_x \varphi \, dx \, dt + \int_{N_r} u \partial_t \varphi + B(u) \partial_x \varphi \, dx \, dt
\]

\[
= -\int_{N_l} u \partial_t \varphi + B(u) \partial_x \varphi \, dx \, dt + \int_{C} u^l \varphi \, d\gamma + \int_{C} B(u^l) \varphi \, d\gamma
\]

\[
- \int_{N_r} u \partial_t \varphi + B(u) \partial_x \varphi \, dx \, dt + \int_{C} u^r \varphi \, d\gamma - \int_{C} B(u^r) \varphi \, d\gamma.
\]

The first and the fourth term on the right hand side are equal to 0 due to assumption (i). Summing the remaining terms we obtain

\[
0 = \int_{\alpha}^{\beta} ([u] \hat{x}' - [B(u)]) \varphi \, dt
\]

and the claim follows. \( \square \)

1.2.1. Nonuniqueness. In this part we will show that the concept of weak solution is in general not enough to guarantee uniqueness.

Let us continue with the example of inviscid Burgers equation in 1D. We consider a piecewise constant initial condition: let \( u_l, u_r \in \mathbb{R}, u_l \neq u_r \), and set

\[
u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0. \end{cases}
\]
The Rankine-Hugoniot condition in this case reads as
\[ s(u_+ - u_-) = \frac{(u_+)^2}{2} - \frac{(u_-)^2}{2} = \frac{1}{2}(u_+ - u_-)(u_+ + u_-) \]
hence necessarily
\[ s = \frac{u_+ + u_-}{2} \]
whenever a weak solution \( u \) jumps from \( u_- \) to \( u_+ \) over a curve with slope \( s \). Therefore, defining
\[ (1.5) \quad u(t, x) = \begin{cases} u_l, & x < \frac{u_l + u_r}{2} t, \\ u_r, & x > \frac{u_l + u_r}{2} t, \end{cases} \]
we obtain a weak solution that fulfils the Rankine-Hugoniot condition.

On the other hand, setting
\[ (1.6) \quad u(t, x) = \begin{cases} u_l, & x < s_1 t, \\ -a, & s_1 t < x < 0, \\ a, & 0 < x < s_2 t, \\ u_r, & x > s_2 t, \end{cases} \]
with
\[ s_1 = \frac{u_l - a}{2}, \quad s_2 = \frac{u_r + a}{2}, \quad a > \max\{u_l, -u_r\} \]
defines the whole family of weak solutions \( u_a \) with the same initial condition and satisfying the Rankine-Hugoniot condition. And therefore uniqueness does not hold true and some non-physical solutions might appear. Therefore, we have to look for a criterion that, on the one hand, would ensure uniqueness and, on the other hand, it would select the correct physical solution. Our first idea is to consider an additional conservation law that is satisfied by any smooth solution and that could play the role of a selector.

To be more precise, we want to find condition on functions \( \eta : \mathbb{R} \to \mathbb{R} \) and \( q = (q_1, \ldots, q_N) : \mathbb{R} \to \mathbb{R}^N \) such that any smooth solution \( u \) automatically satisfies
\[ (1.7) \quad \partial_t \eta(u) + \text{div} q(u) = 0 \]
To this end, let \( u \) be a classical solution to (1.1). To derive the new conservation law, let us multiply (1.1) by \( \eta'(u) \) and use the chain rule formula. It leads to
\[ \partial_t \eta(u) + \sum_{j=1}^N \eta'(u) B_j'(u) \partial x_j u = 0 \]
and therefore setting \( q'_j(\cdot) = \eta'(\cdot) B'_j(\cdot) \) for all \( j = 1, \ldots, N \) we obtain (1.7). Let us continue with our investigation. For instance, what are the consequences of the Rankine-Hugoniot condition for the new conservation law (1.7)? If \( u \) is a piecewise smooth solution of (1.7), that is, (1.7) is satisfied only in the corresponding subdomains, the \( N \)-dimensional version of Rankine-Hugoniot condition reads as follows
\[ \eta(u) = \sum_{j=1}^N \nu_j [q_j(u)] \]
where \( n = (-s, \nu_1, \ldots, \nu_N) \) is the outward normal that is normalized in such a way that \( |(\nu_1, \ldots, \nu_N)| = 1 \). The problem here is that, in general, Rankine-Hugoniot
condition for (1.7) is not compatible with the original Rankine-Hugoniot condition for (1.1). The set of all the conditions is just too restrictive.

Another idea (that in the end will fix the above flaw) is that any physically reasonable solution should be a limit of viscous approximation of the form

$$\partial_t u^\varepsilon + \text{div} B(u^\varepsilon) = \varepsilon \Delta u^\varepsilon,$$

$$u^\varepsilon(0) = u_0.$$  

(1.8)

In other words, we added a parabolic perturbation, i.e. a second order term that depends on a small parameter $\varepsilon$, we want to pass to the limit as $\varepsilon \to 0$ and in the limit we should see the correct physical solution of (1.1). To solve the approximation (1.8) we can use the classical theory for parabolic PDEs which is very well developed.

Let us now multiply (1.8) by $\eta'(u^\varepsilon)$, the same way as we did with (1.1) above. The chain rule now yields

$$\partial_t \eta(u^\varepsilon) + \sum_{j=1}^N \eta'(u^\varepsilon) B_j'(u^\varepsilon) \partial_{x_j} u^\varepsilon = \varepsilon \eta'(u^\varepsilon) \Delta u^\varepsilon$$

which can further rewritten as

$$\partial_t \eta(u^\varepsilon) + \text{div} q(u^\varepsilon) = \varepsilon \Delta \eta(u^\varepsilon) - \varepsilon \eta''(u^\varepsilon) |\nabla u^\varepsilon|^2$$

Assume now that $\eta \in C^2(\mathbb{R})$ is convex. Consequently,

$$-\varepsilon \eta''(u^\varepsilon) |\nabla u^\varepsilon|^2 \leq 0$$

and we obtain

$$\partial_t \eta(u^\varepsilon) + \text{div} q(u^\varepsilon) = \varepsilon \Delta \eta(u^\varepsilon) \leq 0.$$

The above inequality can be now considered as a parabolic perturbation of the inequality

$$\partial_t \eta(u^\varepsilon) + \text{div} q(u^\varepsilon) \leq 0$$

(1.9)

and this finally gives the right additional conditions that ensure uniqueness and select the correct physical solution among all weak solution.

**Definition 1.2.3.** Let $\eta \in C^1(\mathbb{R})$ be a convex function. If there exists $q_j \in C^1(\mathbb{R})$ such that for all $v \in \mathbb{R}$

$$\eta'(v) B'_j(v) = q'_j(v), \quad j = 1, \ldots, N,$$

(1.10)

then $(\eta, q)$ is called an entropy-entropy flux pair of the conservation law (1.1).

**Remark 1.2.4.** The structure condition (1.10) only becomes important when dealing with systems of conservation laws. In the case of scalar conservation, $q'_j$ can be easily defined by the left hand side of (1.10) and therefore (1.10) does not impose any further restriction on $\eta, B$.

Finally we have all in hand to define the notion of entropy solution.

**Definition 1.2.5.** A weak solution to (1.1) is called an entropy solution if for every entropy-entropy flux pair $(\eta, q)$ the so-called entropy inequality (1.9) is fulfilled in the following sense: for all $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^N)$, $\varphi \geq 0$, it holds

$$\int_0^\infty \int_{\mathbb{R}^N} \left( \eta(u) \partial_t \varphi + \sum_{j=1}^N q_j(u) \partial_{x_j} \varphi \right) \, dx \, dt + \int_{\mathbb{R}^N} \eta(u_0) \varphi(0) \, dx \geq 0.$$
Remark 1.2.6. To establish uniqueness, it is necessary to have all the entropy inequalities corresponding to all the admissible entropy-entropy flux pairs $(\eta, q)$.

Let us now go back to our example of one-dimensional inviscid Burgers equation (1.4) and verify if our selection criterion applies. Since $B(u) = \frac{u^2}{2}$, the structure condition (1.10) reads as $q'(u) = \eta'(u)u$. Recall that if a solution jumps over a straight line from $U_L$ to $U_R$ then due to Rankine-Hugoniot condition for (1.4) the slope has to satisfy $s = \frac{U_L + U_R}{2}$. The Rankine-Hugoniot condition for the entropy inequalities (1.9) is

$$s[\eta(u)] \geq [q(u)]$$

hence we obtain

$$\frac{1}{2}(U_L + U_R)(\eta(U_R) - \eta(U_L)) \geq \int_{U_L}^{U_R} \lambda \eta'\lambda d\lambda.$$ 

Let $\eta(u) = (u - c_0)^2$ where $c_0 = \frac{U_L + U_R}{2}$. Then the above left hand side vanishes and we get

$$0 \geq \frac{1}{6}(U_R - U_L)^3.$$ 

Consequently we deduce that no jumps of the form $U_L < U_R$ are allowed. Regarding the solutions constructed in (1.5) and (1.6) we obtain that the only suitable solution is the one from (1.5) with $u_l \geq u_r$.

Exercise 1.2.7. What does the above statement that no jumps of the form $U_L < U_R$ are allowed suggest about the possible definition of the solution in the region not determined by characteristics in Figure 2?

1.3. Well-posedness for entropy solutions

In this section, we present a classical well-posedness result for entropy solutions that is due to Kružkov [6].

Theorem 1.3.1. Let $b_i = B_i \in L^\infty_{\text{loc}}(\mathbb{R})$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a unique entropy solution

$$u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^N).$$

Moreover, if $u_1, u_2$ are two entropy solutions corresponding to initial conditions $u_{1,0}, u_{2,0}$, respectively, then the following $L^1$-contraction property holds true

$$\|u_1(t) - u_2(t)\|_{L^1} \leq \|u_{1,0} - u_{2,0}\|_{L^1} \quad \forall t \in \mathbb{R}^+.$$ 

This is a classical result that dates back to 1970. Concerning uniqueness, one needs all the entropy inequalities except for the case of one-dimensional conservation law with convex (or concave) fluxes and BV solutions.

We skip the proof of this result and continue with our discussion about possible concepts of solution. Later on, we prove in detail a well-posedness result in the stochastic setting that basically includes Theorem (1.3.1) as a special case.

Since the $L^1$-contraction property gives continuous dependence on the initial condition in $L^1$, it is natural to ask whether it is possible to establish a pure $L^1$-well-posedness theory. Namely, to consider unbounded initial conditions in $L^1(\mathbb{R}^N)$. Here it turns out that the concept of entropy solutions is not sufficient. To be more precise, let us for instance consider a quadratic flux function $B$, such as in the Burgers equation. If $u(0) \in L^1(\mathbb{R}^N)$ then $B(u)$ is not even locally integrable in
the space variable so div $B(u)$ is not a well-defined distribution. In other words, the equation (1.1) does not make sense as distributional equation and therefore Definition 1.2.1 is not meaningful. The same problem would of course hold true for the entropy inequalities where $\eta, q$ does not necessarily have linear growth.

To overcome this difficulties, we introduce a new (and our last one) notion of solution, the so-called kinetic solution. It was first introduced by Lions, Perthame, Tadmor \[7\] and relies on a new equation, the so-called kinetic formulation, that is derived from the conservation law at hand and that (unlike the original problem) possesses a very important feature - linearity. The two notions of solution, i.e. entropy and kinetic, are equivalent whenever both of them exist, nevertheless, kinetic solutions are more general as they are well defined even in situations when neither the original conservation law nor the corresponding entropy inequalities can be understood in the sense of distributions.

1.4. Kinetic formulation

Let us first fix some notation. For $\xi \in \mathbb{R}$ and $u \in \mathbb{R}$ we define the equilibrium function by 

$$
\chi_u(\xi) = \begin{cases} 
1, & 0 < \xi < u, \\
-1, & u < \xi < 0, \\
0, & \text{otherwise}.
\end{cases}
$$

The above can be rewritten as 

$$
\chi_u(\xi) = \mathbb{1}_{0 < \xi < u} - \mathbb{1}_{u < \xi < 0} = \mathbb{1}_{u > \xi} - \mathbb{1}_{0 > \xi}.
$$

We have the following easy lemma.

**Lemma 1.4.1.** Let $S : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and assume that $S' \in L^\infty(\mathbb{R})$. Then 

$$
\int_{\mathbb{R}} S'(\xi) \chi_u(\xi) \, d\xi = S(u) - S(0)
$$

and 

$$
\int_{\mathbb{R}} |\chi_u(\xi) - \chi_v(\xi)| \, d\xi = |u - v|.
$$

**Proof.** Both claims follow directly from the definition of the equilibrium function. \qed

**Theorem 1.4.2.** Let $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$. Then $u$ is an entropy solution to (1.1) if and only if there exists a nonnegative bounded measure $m(t, x, \xi)$ such that 

$$
\partial_t \chi_u + b(\xi) \cdot \nabla_x \chi_u = \partial_\xi m, \quad \chi_u(0) = \chi_{u_0},
$$

holds true in $D'(\mathbb{R}^+_t \times \mathbb{R}^N_x \times \mathbb{R}^\xi)$.

Equation (1.11) is called the kinetic formulation of (1.1) and it is the keystone for the notion of kinetic solution that will be introduced soon. This equation is posed in the extended space - it includes a new variable $\xi$ that basically corresponds to the values of the solution $u$. For that reason is $\xi$ usually called velocity. The measure $m$ is the kinetic defect measure and it is not known in advance, it becomes part of the solution. The main feature of the kinetic formulation is its linearity. In
fact, it is a linear transport equation for $\chi_u$ but clearly the dependence on $u$ is still nonlinear.

**Remark 1.4.3.** Considering just the left hand side of (1.11) we obtain a linear transport equation

$$\partial_t f + b(\xi) \cdot \nabla f = 0.$$  

That can be solved by the method of characteristics introduced in Section 1.1. (Remember that the problems came only with nonlinear equations!) Nevertheless, we are only interested in solutions of the form $f = \chi_u$. Therefore, posing this constraint, the kinetic measure $m$ can be regarded as the corresponding Lagrange multiplier.

In the case of existence of a classical solution to (1.1), we have the following equivalence which is proved directly by the chain rule formula.

**Proposition 1.4.4.** Let $u \in C^1(\mathbb{R}^+ \times \mathbb{R}^N).$ Then $u$ is a classical solution to (1.1) if and only if it satisfies (1.11) with $m \equiv 0$.

This result further clarifies the name kinetic defect measure. Indeed, the measure $m$ takes account of possible singularities and therefore vanishes for smooth solutions.

**Proof of Theorem 1.4.2.** In order to prepare for the proof, let us consider $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R})$ and define the function

$$\int_0^\xi \chi_u(t,x)(\zeta) \, d\zeta \in C(\mathbb{R}^+; L^1_{loc}(\mathbb{R}^{N+1}))$$

and the distribution

$$m(t,x,\xi) = \partial_t \int_0^\xi \chi_u(t,x)(\zeta) \, d\zeta + \sum_{i=1}^N \partial_{x_i} \int_0^\xi b_i(\zeta) \chi_u(t,x)(\zeta) \, d\zeta.$$  

(1.12)

Therefore, in $D'(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R})$ we have

$$\partial_t \chi_u + b(\xi) \cdot \nabla_x \chi_u = \partial_\xi m.$$  

Note that so far $u$ does not have to solve (1.1) in any sense. We can always define a distribution $m$ such that the (1.11) holds true, the nontrivial fact is that $m$ is a nonnegative and bounded measure.

Let us now multiply the above by $S' \in D(\mathbb{R})$ and integrate with respect to $\xi$. We obtain

(1.13)

$$\partial_t \int_\mathbb{R} S'(\xi) \chi_u(\xi) \, d\xi + \sum_{i=1}^N \partial_{x_i} \int_\mathbb{R} S'(\xi) b_i(\xi) \chi_u(\xi) \, d\xi = - \int_\mathbb{R} S''(\xi) m(t,x,\xi) \, d\xi.$$  

The function $S$ will eventually play the role of an entropy. Hence with Lemma 1.4.1 and the structure condition

$$q'(u) = S'(u) b(u)$$  

we deduce

(1.14)

$$\partial_t S(u) + \sum_{i=1}^N \partial_{x_i} q_i(u) = - \int_\mathbb{R} S''(\xi) m(t,x,\xi) \, d\xi.$$  

Having these preparations in hand, let us proceed with the proof.
1. WHAT IS THE RIGHT NOTION OF SOLUTION?

**Step 1:** Let \( u \) satisfy (1.11) with a nonnegative bounded measure \( m \). Then we have (1.14) for those \( S \) such that \( S' \in D(\mathbb{R}) \). By truncation, it is possible to extend the validity of (1.14) to \( S \in C^2(\mathbb{R}) \) subquadratic, i.e.

\[
|S(\xi)| \leq C(1 + |\xi|^2).
\]

Hence for such \( S \) that are in addition convex, we obtain the desired entropy inequality

\[
\partial_t S(u) + \sum_{i=1}^N \partial_{x_i} q_i(u) \leq 0.
\]

If \( S \) is not subquadratic, we make use of the boundedness of \( u \). In particular, we go back to (1.13) and observe that since \( u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^N) \), the equilibrium function is compactly supported with

\[
\text{supp } \chi_u(\cdot) \subset [-\|u\|_{L^\infty}, \|u\|_{L^\infty}].
\]

Consequently, the integration in (1.13) only runs over this interval and therefore it does not matter if we change \( S \) for large values of \( \xi \) to make it subquadratic. Finally, we conclude that \( u \) satisfies all the entropy inequalities and therefore it is an entropy solution.

**Step 2:** Conversely, let \( u \) be an entropy solution to (1.1). Then for all \( S \in C^2(\mathbb{R}) \) convex, it holds

(1.15) \[ \partial_t \int_{\mathbb{R}^N} S(u(t, x)) \, dx \leq 0. \]

Indeed, since the entropy inequalities hold true in the sense of distributions, let us test by \( \varphi_R(x) = \varphi(\frac{x}{R}) \) where \( \varphi \) is a smooth cut-off function. It leads to

\[
\partial_t \int_{\mathbb{R}^N} S(u) \varphi_R(x) \, dx - \int_{\mathbb{R}^N} q(u) \nabla \varphi_R(x) \, dx \leq 0.
\]

Since both \( S(u) \) and \( q(u) \) are integrable with respect to \( x \) we may use the dominated convergence theorem to pass to the limit as \( R \to \infty \) and obtain (1.15).

As a consequence, it follows for every \( p \in [1, \infty] \) that

\[
\|u(t)\|_{L^p_x} \leq \|u_0\|_{L^p_x}.
\]

Moreover, if \( |\xi| > \|u_0\|_{L^\infty} \) then from (1.12), the definition of the equilibrium function and the fact that \( u \) solves (1.1)

\[
m(\xi) = \partial_t \int_0^{|u_0|_{L^\infty}} \chi_u(\zeta) \, d\zeta + \sum_{i=1}^N \partial_{x_i} \int_0^{|u_0|_{L^\infty}} b_i(\zeta) \chi_u(\zeta) \, d\zeta
\]

\[
= \partial_t u + \sum_{i=1}^N \partial_{x_i} B_i(u) = 0.
\]

Hence again if \( S \in C^2(\mathbb{R}) \) is convex, it can be modified for large values of \( \xi \) to be compactly supported and (1.14) yields

\[
\int_{\mathbb{R}} S''(\xi) m(t, x, \xi) \, d\xi \geq 0
\]

and accordingly \( m \) is nonnegative.
It remains to show that $m$ is a bounded measure. We take $S(\xi) = \frac{\xi^2}{2}$ and plug it in (1.14), integrate with respect to $x$ and $t$ and obtain
\[
\int_{\mathbb{R}^+ \times \mathbb{R}^{N+1}} m \, dt \, dx \, d\xi \leq \frac{1}{2} \|u_0\|_{L^2}^2
\]
which completes the proof. \[\square\]

Finally, we have all in hand to define the notion of kinetic solution.

**Definition 1.4.5.** $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N))$ is called a kinetic solution to (1.1) if there exists a nonnegative bounded measure $m$ such that the kinetic formulation (1.11) is satisfied in the sense of distributions over $\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$.

According to Theorem 1.4.2 we know that for $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$ it is equivalent to be entropy and kinetic solution. However, the concept of kinetic solution does not require boundedness and therefore is more suitable to deal with initial conditions $u_0 \in L^1(\mathbb{R}^N)$.

The following well-posedness result can be found in [7].

**Theorem 1.4.6.** Let $u_0 \in L^1(\mathbb{R}^N)$ and $b \in L^\infty_{loc}(\mathbb{R})$. Then there exists a unique kinetic solution to (1.1) and $u \in C(\mathbb{R}^+; L^1(\mathbb{R}^N))$. Moreover, if $u_1$, $u_2$ are two entropy solutions corresponding to initial conditions $u_{1,0}$, $u_{2,0}$, respectively, then the following $L^1$-contraction property holds true
\[
\|u_1(t) - u_2(t)\|_{L^1} \leq \|u_{1,0} - u_{2,0}\|_{L^1} \quad \forall t \in \mathbb{R}^+.
\]

### 1.5. Examples of kinetic defect measure

#### 1.5.1. Burgers equation

Let us continue with our example of one-dimensional Burgers equation, in particular with the solution constructed in (1.5), and let us calculate its kinetic defect measure. Due to (1.12) we shall calculate $\partial_t \chi_u + \xi \partial_x \chi_u$ in the sense of distributions over $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. Let $\varphi \in D(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$. Then using Green’s identity and the fact that $u$ is piecewise constant, we have\(^1\)
\[
\langle \partial_t \chi_u, \varphi \rangle = -\int_0^\infty \int_{\mathbb{R}^+} \int_{\mathbb{R}^N} \chi_u \partial_t \varphi \, d\xi \, dx \, dt
\]
\[
= -\int_{\mathbb{R}^+} \int_{\{x < st\}} \chi_u \partial_t \varphi \, d\xi \, dx \, dt - \int_{\mathbb{R}^+} \int_{\{x > st\}} \chi_u \partial_t \varphi \, d\xi \, dx \, dt
\]
\[
= -\int_{\mathbb{R}^+} \int_{\{x = st\}} \chi_u \varphi(-s) \, d\xi \, dx \, dt - \int_{\mathbb{R}^+} \int_{\{x = st\}} \chi_u \varphi(s) \, d\xi \, dx \, dt
\]
\[
= \langle (-s)(\chi_{u_r} - \chi_{u_l})\delta_{x=st}, \varphi \rangle
\]
and similarly
\[
\langle \xi \partial_x \chi_u, \varphi \rangle = \langle \xi(\chi_{u_r} - \chi_{u_l})\delta_{x=st}, \varphi \rangle.
\]

So we obtain that
\[
\partial_\xi m = (\xi - s)(\chi_{u_r} - \chi_{u_l})\delta_{x=st}
\]
and
\[
m(t, x, \xi) = \delta_{x=st} \int_{-\infty}^\xi (\xi - s)(\chi_{u_r} - \chi_{u_l}) \, d\zeta.
\]

\(^1\)By $\langle \cdot, \cdot \rangle$ we denote the duality between distributions and test functions.
1. WHAT IS THE RIGHT NOTION OF SOLUTION?

In particular, we see that \( m \) is indeed supported by the shock. Besides, it can be shown that

\[
m(t, x, \xi) = \tilde{m}(\xi) \delta_{x=st} \text{sgn}(u_r - u_l)
\]

where

\[
\tilde{m}(\xi) \begin{cases} 
0, & \xi \leq \min\{u_l, u_r\} \\
0, & \xi \geq \max\{u_l, u_r\} \\
< 0, & \text{otherwise.}
\end{cases}
\]

Hence again we observe the contradiction for the case of \( u_l < u_r \) and these jumps are not allowed.

1.5.2. Linear case. Let us now consider the linear case \( B(u) = cu, c \in \mathbb{R} \), and let

\[
u(t, x) = \begin{cases} u_l, & x < st, \\
u_r, & x > st,
\end{cases}
\]

be a piecewise constant solution. From Rankine-Hugoniot condition we deduce that

\[
s(u_r - u_l) = c(u_r - u_l)
\]

so \( s = c \). The computation of the kinetic measure from Subsection 1.5.1 applies and we obtain that \( m \equiv 0 \). Therefore, in this case \( u \) and \( \chi_u \) satisfy the same equation and the kinetic measure vanishes even though the solution is not continuous.
Finally, we get to the original goal and that is the study of scalar conservation laws with stochastic forcing. Namely, we are interested in the following problem
\begin{equation}
\begin{aligned}
    du + \text{div} B(u) \, dt &= \Phi(u) \, dW, \\
    u(0) &= u_0.
\end{aligned}
\end{equation}

(2.1)

Here we restrict ourselves to a finite time interval \([0, T]\) with \(T > 0\) and consider periodic boundary conditions: \(x \in \mathbb{T}^N\) where \(\mathbb{T}^N = [0, 1]^N\) is the \(N\)-dimensional torus. As before, we have a flux function \(B = (B_1, \ldots, B_N) : \mathbb{R} \to \mathbb{R}^N\), we denote \(b = B'\) and, in addition, we assume that \(B\) is of class \(C^2\) and \(b\) has a polynomial growth.

Let us fix a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with a complete right-continuous filtration. Let \(\mathcal{P}\) denote the predictable \(\sigma\)-algebra on \(\Omega \times [0, T]\) associated to \((\mathcal{F}_t)\). That is, \(\mathcal{P}\) is generated by sets of the form \(A \times \{0\}\) where \(A \in \mathcal{F}_0\) and \(A \times (s, t]\) where \(0 \leq s < t \leq T\) and \(A \in \mathcal{F}_s\). Recall that \(\mathcal{P}\) is generated by left-continuous and adapted processes.

The initial datum may be random in general so it is \(\mathcal{F}_0\)-measurable and we assume that \(u_0 \in L^p(\Omega; L^p(\mathbb{T}^N))\) for all \(p \in [1, \infty)\). In fact, this assumption can be immediately weakened to the \(L^p\)-integrability for \(p \leq p^*\) and \(p^*\) would be determined by the growth of \(b\). However, comparing this to our discussion in Chapter 1 we would like to establish well-posedness also for initial conditions in \(L^1\) which is the natural set-up for kinetic solutions. This is indeed possible and the proof then consists of two parts. First, we assume the \(L^p\)-integrability as above and in the second step we take suitable approximation of the initial condition and pass to the limit. We will only focus on the first part, the proof of the second one can be found in [2]. Note that in the stochastic setting, due to the active white noise term, the solution is never bounded in \(\omega\) and therefore the boundedness assumption is replaced by \(L^p\)-integrability.

The driving process \(W\) is a cylindrical Wiener process in a separable Hilbert space \(\Omega\), which is formally given by
\[
W(t) = \sum_{k \geq 1} \beta_k(t)e_k
\]
where \((e_k)_{k \geq 1}\) is a complete orthonormal system in \(\Omega\) and \((\beta_k)_{k \geq 1}\) are mutually independent real-valued Wiener processes relative to \((\mathcal{F}_t)\). Recall that this sum does not converge in any good probabilistic sense hence some suitable conditions upon the coefficient \(\Phi\) are needed so that the stochastic integral in (2.1) makes sense. Regarding \(W\) itself, in order to get a process with continuous trajectories,
one introduces an auxiliary space $\mathcal{U}_0 \supset \mathcal{U}$ via

$$\mathcal{U}_0 = \left\{ v = \sum_{k \geq 1} \alpha_k e_k; \sum_{k \geq 1} \frac{\alpha_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|v\|_{\mathcal{U}_0}^2 = \sum_{k \geq 1} \frac{\alpha_k^2}{k^2}, \quad v = \sum_{k \geq 1} \alpha_k e_k.$$  

Note that the embedding $\mathcal{U} \hookrightarrow \mathcal{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of $W$ are $\mathbb{P}$-a.s. in $C([0,T];\mathcal{U}_0)$ (see [1]).

The assumptions upon the coefficient $\Phi$ are the following. If $u \in L^2(\mathbb{T}^N)$ then $\Phi(u)$ is a mapping from $\mathcal{U}$ to $L^2(\mathbb{T}^N)$ and it is defined by $\Phi(u) e_k = g_k(\cdot, u(\cdot))$ where $g_k \in C(\mathbb{T}^N \times \mathbb{R})$ satisfy

$$\begin{align*}
G^2(x,\xi) &:= \sum_{k \geq 1} |g_k(x,\xi)|^2 \leq C(1 + |\xi|^2), \\
(2.3) \sum_{k \geq 1} |g_k(x,\xi) - g_k(y,\zeta)|^2 &\leq C(|x-y|^2 + |\xi - \zeta|^{1+\alpha}),
\end{align*}$$

for some $\alpha > 0$ and for all $x,y \in \mathbb{T}^N$, $\xi,\zeta \in \mathbb{R}$.

**Exercise 2.0.1.** Check that under the above assumptions $\Phi$ maps $L^2(\mathbb{T}^N)$ to $L^2(\mathbb{T}^N)$, the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $L^2(\mathbb{T}^N)$.

Consequently, given a predictable process $u \in L^2(\Omega \times [0,T], \mathcal{P}, \mathbb{P} \otimes dt; L^2(\mathbb{T}^N))$, the stochastic integral in (2.1) is a well-defined process taking values in $L^2(\mathbb{T}^N)$ and it holds

$$\int_0^t \Phi(u) \, dW = \sum_{k \geq 1} \int_0^t \Phi(u) \, d\beta_k = \sum_{k \geq 1} \int_0^t g_k(u(s)) \, d\beta_k.$$  

Later on, we will consider our equation in the sense of distributions. In that case we have for a test function $\varphi \in C^\infty(\mathbb{T}^N)$

$$\left\langle \int_0^t \Phi(u) \, dW, \varphi \right\rangle = \sum_{k \geq 1} \int_0^t \langle g_k(u(s)), \varphi \rangle \, d\beta_k.$$  

### 2.1. Kinetic formulation

The aim of the present section is to introduce the notion of kinetic solution to (2.1). Recall, that in the deterministic case this was based on the so-called kinetic formulation of the corresponding conservation law

$$\begin{align*}
(2.4) \quad \partial_t u + \text{div} \, B(u) &= 0 \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^N.
\end{align*}$$

The kinetic formulation is an equation satisfied by the equilibrium function $\chi_u = 1_{u>\xi} - 1_{u<\xi}$ and reads as follows

$$\begin{align*}
\partial_t \chi_u + b(\xi) \cdot \nabla \chi_u &= 0 \quad \text{on} \quad \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}_\xi.
\end{align*}$$

Its includes the additional variable $\xi$ and is solved in the sense of distributions $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}_\xi)$. Regarding derivation of the kinetic formulation, if $u \in C^1(\mathbb{R}^+ \times \mathbb{R}^N)$ is a classical solution to (2.4), then it is enough to multiply (2.4) by $\delta_{u=\xi}$ and use the chain rule formula. Indeed, at least formally,

$$\partial_t 1_{u>\xi} = \delta_{u=\xi} \partial_t u$$
and the similarly for the space derivatives. In the stochastic setting, all the calculations of this form have to go through the Itô formula. To make everything rigorous, let us assume that \( u \in C([0, T]; C^1(\mathbb{T}^N)) \) a.s. is a solution to (2.1). In this case, the conservation law (2.1) is even satisfied pointwise, that is, for every \( x \in \mathbb{T}^N \) and \( t \in [0, T] \) it holds true a.s.

\[
(2.5) \quad u(t, x) = u_0(x) - \int_0^t \text{div} B(u(s, x)) \, ds + \sum_{k \geq 1} \int_0^t g_k(x, u(s, x)) \, dB_k(s).
\]

In the sequel, we denote by \( \langle \cdot, \cdot \rangle_\xi \) the duality between \( \mathcal{D}(\mathbb{R}^d) \) and \( C_c^\infty(\mathbb{R}^d) \). In order to derive the equation satisfied by \( 1_{u > \xi} \) in the sense of distributions, we will consider a test function \( \theta' \in C_c^\infty(\mathbb{R}) \) and derive an equation satisfied by \( \langle 1_{u(t, x) > \xi}, \theta' \rangle_\xi \). Since

\[
\langle 1_{u(t, x) > \xi}, \theta' \rangle_\xi = \int_{-\infty}^{u(t, x)} \theta'(\xi) \, d\xi = \theta(u(t, x)),
\]

writing down the equation for \( \langle 1_{u(t, x) > \xi}, \theta' \rangle_\xi \) is the same as applying the Itô formula to (2.5) and the function \( \theta \). Note that at this point it was important to have the regularity of \( \xi \). Since

\[
\langle 1_{u(t, x) > \xi}, \theta' \rangle_\xi = \int_{-\infty}^{u(t, x)} \theta'(\xi) \, d\xi = \theta(u(t, x)) \quad \text{and denoting} \quad f = 1_{u > \xi} \quad \text{we may finally read off the kinetic formulation of (2.1)}
\]

\[
(2.6) \quad df + b \cdot \nabla f \, dt = \delta_{u=\xi} \Phi \, dW - \frac{1}{2} \partial_x (G^2 \delta_{u=\xi}) + \partial_x m.
\]

As in the deterministic case, we added \( \xi \)-derivative of a kinetic measure \( m \) on the right hand side, although it did not pop up from our calculation as we assumed rather high regularity of \( u \).

\textbf{Remark 2.1.1.} In the stochastic setting it appears to be more convenient to write down the kinetic formulation for \( 1_{u > \xi} \) rather than \( \chi_u = 1_{u > \xi} - 1_{\theta > \xi} \). However, since \( \delta_{u=\xi} = -\partial_x 1_{u > \xi} \), (2.6) can be rewritten as

\[
df + b \cdot \nabla f \, dt = -\partial_x f \Phi \, dW + \frac{1}{2} \partial_x (G^2 \partial_x f) + \partial_x m,
\]
which is different from the same equation with \( f \) replaced by \( \chi_u \) unless \( \Phi(0) = 0 \). Kinetic solution to (2.1) is then, roughly speaking, a distributional solution to (2.6) with some kinetic measure \( m \). Let us give the precise definitions.

**Definition 2.1.2 (Kinetic measure).** A mapping \( m \) from \( \Omega \) to \( \mathcal{M}_b^+(\[0,T\] \times \mathbb{T}^N \times \mathbb{R}) \), the set of nonnegative bounded measures over \([0,T] \times \mathbb{T}^N \times \mathbb{R}\), is said to be a kinetic measure provided

(i) \( m \) is measurable in the following sense: for each \( \psi \in C_0([0,T] \times \mathbb{T}^N \times \mathbb{R}) \) the mapping \( m(\psi) : \Omega \to \mathbb{R} \) is measurable,

(ii) \( m \) vanishes for large \( \xi \): if \( B_R^\circ = \{ \xi \in \mathbb{R} ; |\xi| \geq R \} \) then

\[
\lim_{R \to \infty} \mathbb{E} m\left([0,T] \times \mathbb{T}^N \times B_R^\circ \right) = 0,
\]

(iii) for any \( \psi \in C_0(\mathbb{T}^N \times \mathbb{R}) \), the process

\[
t \mapsto \int_{\mathbb{T}^N \times [0,T] \times \mathbb{R}} \psi(x,\xi) \, dm(s,x,\xi)
\]

is predictable.

**Definition 2.1.3 (Kinetic solution).** Assume that, for all \( p \in [1,\infty) \),

\[
u \in L^p(\Omega \times [0,T], \mathcal{P}, d\mathbb{P} \otimes dt; L^p(\mathbb{T}^N)) \cap L^\infty(\Omega; L^\infty(0,T; L^p(\mathbb{T}^N))).
\]

Then \( u \) is said to be a kinetic solution to (2.1) with initial datum \( u_0 \) provided there exists a kinetic measure \( m \) such that the pair \( (f = 1_{u>\xi}, m) \) satisfies, for all \( \varphi \in C^\infty_c([0,T] \times \mathbb{T}^N \times \mathbb{R}) \), \( \mathbb{P} \)-a.s.,

\[
\begin{align*}
\int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt &+ \langle f_0(\varphi(0)) \rangle + \int_0^T \langle f(t), b \cdot \nabla \varphi(t) \rangle dt \\
&= -\sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x, u(t,x)) \varphi(t,x,u(t,x)) \, dx \, d\beta_k(t) \\
&\quad - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} G^2(x, u(t,x)) \partial_{\xi} \varphi(t,x,u(t,x)) \, dx \, dt + m(\partial_\xi \varphi).
\end{align*}
\]

Note that a kinetic solution is in fact not a stochastic process in the usual sense: it is a class of equivalence in \( L^p(\Omega \times [0,T]; L^p(\mathbb{T}^N)) \). In particular, it is a priori not defined for every time \( t \). This is connected to the fact that the kinetic formulation is solved in the sense of distributions \( D^p([0,T] \times \mathbb{T}^N \times \mathbb{R}) \), i.e. the solution that we obtain is weak also in time and consequently it is defined only almost everywhere in time. Nevertheless, it can be proved that in this class of equivalence there exists a representative with good continuity properties, namely, \( u \in C([0,T]; L^p(\mathbb{T}^N)) \) a.s. for every \( p \in [1,\infty) \).

In proof of existence of a kinetic solution to (2.1), another notion of solution will be useful and that is called generalized kinetic solution. Before we give its definition, let us present a short intermezzo about Young measures.

**2.1.1. The concept of Young measures.** Let us consider a sequence \( (v_n) \) which is bounded in \( L^\infty(\mathbb{T}^N) \). It follows from the Banach-Alaoglu theorem that there exists a subsequence, still denoted by \( (v_n) \) which converges to some limit, say \( v \) in \( L^\infty(\mathbb{T}^N) \) weak*. To be more precise, for every \( \varphi \in L^1(\mathbb{T}^N) \)

\[
\langle v_n, \varphi \rangle \to \langle v, \varphi \rangle.
\]
Since we are interested in nonlinear problems, we want to know if also $B(v_n)$ converges in some sense and what is the limit. If $B$ was continuous, then also $(B(v_n))$ is bounded in $L^\infty(\mathbb{T}^N)$. Hence we may apply the Banach-Alaoglu theorem again and get a subsequence $(B(v_n))$ and $\bar{B} \in L^\infty(\mathbb{T}^N)$ such that

$$B(v_n) \overset{w^*}{\rightharpoonup} \bar{B} \quad \text{in} \quad L^\infty(\mathbb{T}^N).$$

Is it possible to represent $\bar{B}$ in terms of $v$? In other words, under what conditions $\bar{B} = B(u)$?

To get some feeling about what the difficulties are, set $v_n(x) = \sin(nx)$ and $B(v_n) = v_n^2$. We make use of the following lemma which is based on Fourier series expansion.

**Lemma 2.1.4.** Let $v \in L^2(0, 2\pi)$ be 2\pi-periodic. Let $v_n(x) = v(nx)$. Then

$$v_n \xrightarrow{w} \frac{1}{2} a_0 \quad \text{in} \quad L^2(0, 2\pi),$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} v(x) \, dx.$$

Applying this result to our example above, in particular, to both $v_n$ and $B(v_n)$, we obtain that

$$v_n \xrightarrow{w} \frac{1}{2\pi} \int_0^{2\pi} \sin(x) \, dx = 0 \quad \text{in} \quad L^2(0, 2\pi),$$

and

$$(v_n)^2 \xrightarrow{w} \frac{1}{2\pi} \int_0^{2\pi} \sin^2(x) \, dx = \frac{1}{2} \quad \text{in} \quad L^2(0, 2\pi).$$

Hence $\bar{B} = \frac{1}{2}$ and $B(u) = 0$.

In general, the problems originate in oscillatory behavior of $(v_n)$ and one can only say that

$$B(w - \lim v_n) \leq w - \lim B(v_n).$$

Therefore, weak convergence is not enough to pass to the limit in nonlinear terms. However, limits of composite functions can be described by a family of probability measures that is called a Young measure.

**Definition 2.1.5 (Young measure).** Let $(X, \lambda)$ be a measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on $\mathbb{R}$. A mapping $\nu$ from $X$ to $\mathcal{P}_1(\mathbb{R})$ is said to be a Young measure if, for all $\psi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\psi)$ from $X$ into $\mathbb{R}$ is measurable. We say that a Young measure $\nu$ vanishes at infinity if, for all $p \geq 1$,

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) \, d\lambda(z) < \infty.$$

Let us go now go back to our problem of convergence of $B(v_n)$.

**Theorem 2.1.6.** There exists a Young measure $\nu : \mathbb{T}^N \rightarrow \mathcal{P}_1(\mathbb{R})$ which represents the sequence $(v_n)$ in the following sense: For any $B \in C(\mathbb{R})$ it holds that

$$B(v_n) \overset{w^*}{\rightharpoonup} \bar{B} \quad \text{in} \quad L^\infty(\mathbb{T}^N),$$

where

$$\bar{B}(x) = \int_{\mathbb{R}} B(\xi) \, d\nu_x(\xi) \quad \text{for a.e.} \quad x \in \mathbb{T}^N.$$
Important consequence of this result is that the Young measure \( \nu \) corresponding to the weak*-converging sequence \( (\nu_n) \) describes the limit of \( B(\nu_n) \) for every continuous \( B \) without taking any further subsequence (unlike our first naive approach using the Banach-Alaoglu theorem).

Intuitively, \( \nu_x \) gives the limit probability distribution of the values of \( v_n \) in the neighborhood of \( x \), as \( n \to \infty \).

**Theorem 2.1.7.** It holds true that \( v_n \) converges to \( v \) strongly in \( L^2(\mathbb{T}^N) \) if and only if \( \nu_x = \delta_{u(x)} \). In that case, \( B(x) = B(u(x)) \) for a.e. \( x \in \mathbb{T}^N \).

**2.1.2. Generalized kinetic solution.** Let us now go back to the kinetic formulation (2.6). The observation that the Dirac masses appearing on the right hand side can be understood as Young measures directly leads to a more general notion of kinetic solution: roughly speaking, we replace \( \delta_{u=\xi} \) by a general Young measure and define \( f \) by \( \partial_\xi f = -\delta_{u=\xi} \). The following definition makes this statement rigorous.

**Definition 2.1.8.** Let \((X, \lambda)\) be a measure space. A measurable function \( f : X \times \mathbb{R} \to [0, 1] \) is said to be a kinetic function if there exists a Young measure \( \nu \) on \( X \) that vanishes at infinity, such that for a.e. \( z \in X \) and for all \( \xi \in \mathbb{R} \)

\[
f(z, \xi) = \nu_z(\xi, \infty).
\]

Clearly, \( 1_{u>\xi} \) considered before is the kinetic function corresponding to the Young measure \( \delta_{u=\xi} \) on \( X = \Omega \times [0, T] \times \mathbb{T}^N \).

**Definition 2.1.9 (Generalized kinetic solution).** Let \( f_0 : \Omega \times \mathbb{T}^N \times \mathbb{R} \to [0, 1] \) be a kinetic function. A measurable function \( f : \Omega \times [0, T] \times \mathbb{T}^N \times \mathbb{R} \to [0, 1] \) is said to be a generalized kinetic solution to (2.1) with initial datum \( f_0 \) if \( (f(t)) \) is predictable and is a kinetic function such that the corresponding Young measure \( \nu \) satisfies for every \( p \in [1, \infty) \)

\[
\mathbb{E} \sup_{t \in [0, T]} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p \, d\nu_{t,x}(\xi) \, dx \leq C_p
\]

and there exists a kinetic measure \( m \) such that for every \( \varphi \in C^\infty_c([0, T] \times \mathbb{T}^N \times \mathbb{R}) \) it holds true

\[
\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), b \cdot \nabla \varphi(t) \rangle \, dt
\]

\[
= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, d\beta_k(t)
\]

\[
- \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} G^2(x, \xi) \partial_\xi \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, dt + m(\partial_\xi \varphi).
\]

We will see in the proof of existence that the latter notion of solution allows for a much simple proof. In particular, weak convergence of some approximate solutions is sufficient in order to pass to the limit. Recall, that according to considerations made in Subsection 2.1.1 this is not enough to pass to the limit in nonlinear terms and therefore one would expect that such a notion of solution is too weak to preserve uniqueness. Nevertheless, the key result is the so-called Reduction theorem. It states that not only it is possible to establish uniqueness for generalized kinetic solutions but, moreover, any generalized kinetic solution is actually a kinetic
solution, that is, the corresponding Young measure is Dirac. In view of Theorem 2.1.7 we thus recover the strong convergence of the approximate solutions.

**Theorem 2.1.10 (Uniqueness, Reduction).** There is at most one kinetic solution to (2.1). Besides, any generalized kinetic solution is actually a kinetic solution, i.e., if $f$ is a generalized kinetic solution with initial datum $1_{u_0} > \xi$ then there exists a kinetic solution $u$ such that $u = 1_{u > \xi}$. Moreover, if $u_1, u_2$ are kinetic solutions with initial data $u_{1,0}, u_{2,0}$ then for all $t \in [0, T]

$$E\|u_1(t) - u_2(t)\|_{L^1} \leq E\|u_{1,0} - u_{2,0}\|_{L^1}.$$  

The existence result reads as follows.

**Theorem 2.1.11 (Existence).** There exists a kinetic solution to (2.1) which is the strong limit of a parabolic approximation $(u^\varepsilon)$: for all $p \in [1, \infty)$

$$\lim_{\varepsilon \to 0} E\|u^\varepsilon - u\|_{L^p_{t,x}}^p = 0.$$  

### 2.2. Existence

This section is devoted to the proof of Theorem 2.1.11 which is based on the vanishing viscosity method and proceeds in several steps. First, the existence of a unique solution to the parabolic approximation is established, then uniform a priori estimates are derived and in the last step, the passage to the limit is performed.

To begin with, let us consider the following viscous approximation of (2.1)

$$du^\varepsilon + \text{div} B^\varepsilon(u^\varepsilon) \, dt = \varepsilon \Delta u \, dt + \Phi^\varepsilon(u^\varepsilon) \, dW,$$

(2.7)

$$u^\varepsilon(0) = u^\varepsilon_0.$$  

Here $u^\varepsilon_0$ is a suitable smooth approximation of $u_0$, the approximate flux functions $B^\varepsilon$ are smooth and compactly supported and the approximation $\Phi^\varepsilon$ is defined as follows. Recall that $\Phi(u) e_k = g_k(u)$ where $g_k$ are continuous functions. Let $g^k_\varepsilon$ be a smooth compactly supported approximation of $g_k$ if $k < \frac{1}{\varepsilon}$ and let $g^k_\varepsilon = 0$ if $k \geq \frac{1}{\varepsilon}$. Set $\Phi^\varepsilon(u) e_k = g^k_\varepsilon(u)$. In particular, we approximate the infinite dimensional noise in (2.1) by finite dimensional noises. On this level, we can consider the approximate coefficients as nice as we want so that we can not only solve (2.7) but also derive sufficient regularity properties of the solution that are needed later on in order to derive the kinetic formulation. In particular, we will show that solutions to (2.7) satisfy the corresponding kinetic formulation of (2.7) and we pass to the limit there. (Remember that the original conservation law (2.1) may not be well defined in the sense of distributions.)

Let $\varepsilon \in (0, 1)$ be fixed. Existence and uniqueness for (2.7) is classical and can be obtained by semigroup approach. Namely, let $S$ be the semigroup generated by $\varepsilon \Delta$, then we rewrite (2.7) in its mild form as

$$u^\varepsilon(t) = S(t)u^\varepsilon_0 - \int_0^t S(t-s) \text{div} B^\varepsilon(u^\varepsilon(s)) \, ds + \int_0^t S(t-s) \Phi^\varepsilon(u^\varepsilon(s)) \, dW$$

and use the Banach fixed point theorem to obtain solutions (for instance) in $L^2(\Omega \times [0, T]; L^2(\mathbb{T}^N))$. For more details we refer the reader to [1]. To establish higher regularity, we use the result of [5] which implies that the unique mild solution to (2.4) in fact belongs to $L^p(\Omega; C([0, T]; W^{1,q}(\mathbb{T}^N)))$ for all $p \in (2, \infty)$, $q \in [2, \infty)$, $l \in \mathbb{N}$. Therefore, the Sobolev embedding applies and yields that $u^\varepsilon \in C([0, T]; C^1(\mathbb{T}^N))$.
The kinetic formulation of (2.7) allows us to derive some estimates uniform in \( \epsilon \). We obtain the kinetic formulation of (2.7). To this end, we need to derive some equations and allow to prove well-posedness.

**Proposition 2.2.1 (Kinetic formulation of (2.7)).** Let \( u^\varepsilon \) be the unique solution to (2.7) and set \( f^\varepsilon = 1_{u^\varepsilon > \xi} \). Then for all \( \varphi \in C^2_c([0, T] \times \mathbb{T}^N \times \mathbb{R}) \) it holds

\[
\int_0^T \langle f^\varepsilon(t), \partial_t \varphi(t) \rangle dt + \langle f_0^\varepsilon, \varphi(0) \rangle + \int_0^T \langle \nabla \varphi(t), b \cdot \nabla \varphi(t) \rangle dt + \varepsilon \int_0^T \langle f^\varepsilon(t), \Delta \varphi(t) \rangle dt = - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x, \xi) \varphi(t, x, \xi) d\nu_{\varepsilon,x}(\xi) dx \alpha_k(t) \]

\[
- \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} G^2(x, \xi) \partial_\xi \varphi(t, x, \xi) d\nu_{\varepsilon,x}(\xi) dx dt + m^\varepsilon(\partial_\xi \varphi),
\]

where \( f_0^\varepsilon = 1_{u^\varepsilon_0 > \xi}, \nu_{\varepsilon,x}^\varepsilon = \delta_{u^\varepsilon(t,x)} \) and for \( \phi \in C_b([0, T] \times \mathbb{T}^N \times \mathbb{R}) \)

\[
m^\varepsilon(\phi) = \varepsilon \int_{[0,T] \times \mathbb{T}^N \times \mathbb{R}} \phi(t, x, u^\varepsilon(t, x)) |\nabla u^\varepsilon|^2 dx dt.
\]

**Proof.** Motivated by the calculations in Section 2.1, we take \( \theta^\prime \in C^\infty_c(\mathbb{R}) \), fix \( x \in \mathbb{T}^N \) and apply the Itô formula to

\[
\langle 1_{u^\varepsilon(t,x) > \xi}, \theta^\prime \rangle = \theta(u^\varepsilon(t, x)).
\]

Let us show what happens with the second order term, the rest being the same as before. We obtain

\[
\theta^\prime(u^\varepsilon) \varepsilon \Delta u^\varepsilon = \varepsilon \Delta \theta(u^\varepsilon) - \varepsilon |\nabla u^\varepsilon|^2 \theta''(u^\varepsilon)
\]

\[
= \varepsilon \langle \Delta 1_{u^\varepsilon > \xi}, \theta^\prime \rangle_\xi - \varepsilon \langle |\nabla u^\varepsilon|^2 \delta_{u^\varepsilon=\xi}, \theta'' \rangle_\xi
\]

and altogether

\[
d\langle 1_{u^\varepsilon > \xi}, \theta^\prime \rangle_\xi = - \langle b^\varepsilon \cdot 1_{u^\varepsilon > \xi}, \theta^\prime \rangle_\xi dt + \sum_{k \geq 1} \langle g_k^\varepsilon \delta_{u^\varepsilon=\xi}, \theta^\prime \rangle_\xi d\xi_k
\]

\[
- \frac{1}{2} \langle \partial_\xi (G^2 \delta_{u^\varepsilon=\xi}), \theta^\prime \rangle_\xi dt + \varepsilon \langle \Delta 1_{u^\varepsilon > \xi}, \theta^\prime \rangle_\xi + \langle \partial_\xi m^\varepsilon, \theta^\prime \rangle \varepsilon.
\]

Finally we set \( \theta(\xi) = \int_\xi^\infty \varphi_1(\zeta) d\zeta \) and test the above by \( \varphi_2 \in C^\infty(\mathbb{T}_x^N) \) to complete the proof.

**Remark 2.2.2.** The previous result show that the concept of kinetic solution extends to parabolic equations. In particular, it also applies to degenerate parabolic equations and allows to prove well-posedness.

As the next step, we will pass to the limit in the kinetic formulation of (2.7) to obtain the kinetic formulation of (2.1). To this end, we need to derive some estimates uniform in \( \varepsilon \).

**2.2.1. Energy estimate.**

**Lemma 2.2.3.** For all \( p \in [2, \infty) \) we have the following estimate uniform in \( \varepsilon \)

\[
\mathbb{E} \sup_{0 \leq t \leq T} ||u^\varepsilon(t)||_{L^p_x}^p \leq C(1 + \mathbb{E} ||u_0||_{L^p_x}^p).
\]
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Proof. Let us apply the infinite dimensional Itô formula to the function $F(v) = \|v\|_{L^p}^p$. A straightforward calculation yields

$$F'(v) = p|v|^{p-2}v \in L^p(\mathbb{T}^N), \quad F''(v) = p(p-1)|v|^{p-2}\text{Id} \in \mathcal{L}(L^p(\mathbb{T}^N), L^p(\mathbb{T}^N)),$$

hence

$$\|u_\varepsilon(t)\|_{L^p}^p = \|u_0\|_{L^p}^p - p \int_0^t \langle |u_\varepsilon|^{p-2}u_\varepsilon, \text{div} B_\varepsilon(u_\varepsilon) \rangle ds + \varepsilon p \int_0^t \langle |u_\varepsilon|^{p-2}u_\varepsilon, \Delta u_\varepsilon \rangle ds + p \sum_{k \geq 1} \int_0^t \langle |u_\varepsilon|^{p-2}u_\varepsilon, g_\varepsilon^{(k)}(u_\varepsilon) \rangle ds \beta_k(s)$$

$$+ \frac{1}{2} p(p-1) \int_0^t \int_{\mathbb{T}^N} |u_\varepsilon|^{p-2}G_\varepsilon^2(u_\varepsilon) dx ds = I_1 + \cdots + I_4.$$  

Let us estimate each of the terms separately. First, we set

$$H_\varepsilon(\xi) = \int_{0}^{\xi} |\zeta|^{p-2}B_\varepsilon(\zeta) d\zeta$$

and observe that

$$I_1 = p \int_0^t \int_{\mathbb{T}^N} \nabla \langle |u_\varepsilon|^{p-2}u_\varepsilon \rangle B_\varepsilon(u_\varepsilon) dx ds = p \int_0^t \int_{\mathbb{T}^N} |u_\varepsilon|^{p-2} \nabla u_\varepsilon B_\varepsilon(u_\varepsilon) dx ds$$

$$= p \int_0^t \int_{\mathbb{T}^N} \text{div} H_\varepsilon(u_\varepsilon) dx ds = 0,$$

due to periodic boundary conditions. Next, we have

$$I_2 = -\varepsilon p \int_0^t \int_{\mathbb{T}^N} |u_\varepsilon|^{p-2} |\nabla u_\varepsilon|^2 dx ds \leq 0$$

and due to (2.2)

$$I_4 \leq C \int_0^t \int_{\mathbb{T}^N} |u_\varepsilon|^{p-2}(1 + |\nabla u_\varepsilon|^2) dx ds \leq C \left(1 + \int_0^t \|u_\varepsilon(s)\|_{L^p}^p ds \right).$$

Consequently, after taking expectation (since the stochastic integral vanishes), we apply the Gronwall lemma and obtain

$$(2.8) \quad \mathbb{E}\|u_\varepsilon(t)\|_{L^p}^p \leq C\left(1 + \mathbb{E}\|u_0\|_{L^p}^p \right).$$

In order to complete the proof it is needed to first take supremum over $t \in [0,T]$ and then expectation. The only change appears in the estimation of the stochastic integral, where we make use of the Burkholder-Davis-Gundy inequality, the Schwartz
inequality, the assumption (2.2) and the weighted Young inequality:

\[
E \sup_{0 \leq t \leq T} \left| \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon|^{p-2} u^\epsilon g_k^\epsilon(u^\epsilon) dx \, d\beta_k(s) \right|
\]

\[
\leq C E \left( \int_0^T \sum_{k \geq 1} \left( \int_{\mathbb{T}^N} |u^\epsilon|^{p-1} |g_k^\epsilon(u^\epsilon)| dx \right)^2 ds \right)^{1/2}
\]

\[
\leq C E \left( \int_0^T ||u^\epsilon||_2^2 \sum_{k \geq 1} ||u^\epsilon||_2^2 |g_k^\epsilon(u^\epsilon)||_2^2 ds \right)^{1/2}
\]

\[
\leq C E \left( \sup_{0 \leq t \leq T} ||u^\epsilon||_{L^p_\epsilon}^p \left( 1 + ||u^\epsilon||_{L^p_\epsilon}^p \right) ds \right)^{1/2}
\]

\[
\leq \frac{1}{2} E \sup_{0 \leq t \leq T} ||u^\epsilon||_{L^p_\epsilon}^p + C \left( 1 + E \int_0^T \left( 1 + ||u^\epsilon||_{L^p_\epsilon}^p \right) ds \right).
\]

Finally, we put the first term on the right hand side to the left hand side, estimate the second one by (2.8) and apply the Gronwall lemma. \hfill \Box

2.2.2. Estimate of the kinetic measures. Recall that the approximate kinetic measures \( m^\epsilon \) are defined by

\[
dm^\epsilon = \epsilon |\nabla u^\epsilon|^2 \delta_{u^\epsilon}(d\xi) dx dt.
\]

Our aim is to show that \((m^\epsilon)\) is bounded in \( L^2_w(\Omega; \mathcal{M}_b([0, T] \times \mathbb{T}^N \times \mathbb{R}))\), the space of weak-star measurable mappings from \( \Omega \) to the space of bounded Borel measures with the norm given by the total variation. Since

\[
\mathcal{M}_b([0, T] \times \mathbb{T}^N \times \mathbb{R}) \cong (C_0([0, T] \times \mathbb{T}^N \times \mathbb{R}))^\ast
\]

it follows that

\[
L^2_w(\Omega; \mathcal{M}_b([0, T] \times \mathbb{T}^N \times \mathbb{R})) \cong \left( L^2(\Omega; C_0([0, T] \times \mathbb{T}^N \times \mathbb{R})) \right)^\ast
\]

and therefore we then apply the Banach-Alaoglu theorem and obtain existence of a weak-star convergent subsequence.

The proof of the necessary bound follows from the calculation in Lemma 2.2.3. Indeed,

\[
E||m^\epsilon||_{L^b_\epsilon}^2 = E[m^\epsilon([0, T] \times \mathbb{T}^N \times \mathbb{R})]^2 = E \left| \epsilon \int_0^T \int_{\mathbb{T}^N} |\nabla u^\epsilon|^2 dx dt \right|^2
\]

where the term inside the square corresponds to \( I_2 \) from Lemma 2.2.3. Hence we proceed similarly, take the square and expectation and estimate the remaining terms on the right hand side, that is, \( I_3 \) and \( I_4 \) to deduce

\[
E||m^\epsilon||_{L^b_\epsilon}^2 \leq C \left( 1 + E\|u_0\|_{L^2_\epsilon}^2 \right).
\]

The Banach-Alaoglu now gives a subsequence \((m^n)\) and \( m \in L^2_w(\Omega; \mathcal{M}_b) \) such that

\[
m^n \xrightarrow{\ast} m \quad \text{in} \quad L^2_w(\Omega; \mathcal{M}_b).
\]
2.2.3. Estimate of the Young measures. As we intend to pass to the limit in the kinetic formulation from Proposition 2.2.1, we need also to get compactness of the approximate Young measures $\nu^\varepsilon$. It is based on the following result.

**Proposition 2.2.4.** Let $(X, \lambda)$ be a finite measure space such that $L^1(X)$ is separable.\(^1\) Let $(\nu^n)$ be a sequence of Young measures such that for some $p \geq 1$

\[
\sup_n \int_X \int_\mathbb{R} |\xi|^p d\nu^n_z(\xi) d\lambda(z) < \infty.
\]

Then there exists a subsequence, still denoted by $(\nu^n)$, and a Young measure $\nu$ such that for all $h \in L^1(X)$ and for all $\phi \in C_b(\mathbb{R})$

\[
\int_X h(z) \int_\mathbb{R} \phi(\xi) d\nu^n_z(\xi) d\lambda(z) \to \int_X h(z) \int_\mathbb{R} \phi(\xi) d\nu_z(\xi) d\lambda(z).
\]

**Corollary 2.2.5.** Let $(X, \lambda)$ be a finite measure space such that $L^1(X)$ is separable. Let $(f^n)$ be a sequence of kinetic functions such that the corresponding Young measures $(\nu^n)$ satisfy (2.11). Then there exists a subsequence, still denoted by $(f^n)$, and a kinetic function $f$ such that

\[
f^n \mathrel{\rightharpoonup} f \quad \text{in} \quad L^\infty(X \times \mathbb{R}).
\]

In our case, we apply the above results to $X = \Omega \times [0,T] \times T^N$, $f^\varepsilon = 1_{u^\varepsilon > \xi}$ and $\nu^\varepsilon = \delta_{u^\varepsilon = \xi}$, hence the estimate (2.11) reads as

\[
\sup_n \mathbb{E} \int_0^T \int_{T^N} |u^\varepsilon|^p \, dz \, dt < \infty
\]

which holds true due to the energy estimate Lemma 2.2.3. Therefore, we have all in hand to pass to the limit and to obtain a generalized kinetic solution to (2.1). It only remains to verify that $m$ is a kinetic measure. The first and the third requirement of Definition 2.1.2 are technical and not difficult and hence they are left to the reader. Let us verify the point (ii). We observe that it follows, once we show, for some $p > 2$, that

\[
\mathbb{E} \int_{[0,T] \times T^N \times \mathbb{R}} |\xi|^{p-2} \, dm(t, x, \xi) \leq C.
\]

Indeed, in that case

\[
\mathbb{E} m([0,T] \times T^N \times B_R^c) = \mathbb{E} \int_{[0,T] \times T^N \times \mathbb{R}} 1_{|\xi| \geq R} \, dm(t, x, \xi)
\]

\[
\leq \frac{1}{R^{p-2}} \int_{[0,T] \times T^N \times \mathbb{R}} |\xi|^{p-2} \, dm(t, x, \xi) \leq \frac{C}{R^{p-2}} \to 0.
\]

In order to show (2.12), we consider the approximate kinetic measures $m^\varepsilon$ given by (2.9) and write

\[
\mathbb{E} \int_{[0,T] \times T^N \times \mathbb{R}} |\xi|^{p-2} \, dm^\varepsilon(t, x, \xi) = \mathbb{E} \int_0^T \int_{T^N} \varepsilon |\nabla u^\varepsilon|^2 |u^\varepsilon|^{p-2} \, dz \, dt.
\]

This has already appeared in the term $I_2$ in Lemma 2.2.3, so we know that

\[
\mathbb{E} \int_{[0,T] \times T^N \times \mathbb{R}} |\xi|^{p-2} \, dm^\varepsilon(t, x, \xi) \leq C
\]

\(^1\)It is sufficient to assume that the $\sigma$-algebra is countably generated.
uniformly in $\varepsilon$. Since $\phi(\xi) = |\xi|^{p-2}$ is not a good test function for the convergence (2.10), we have to truncate: let $h_\delta$ be a smooth cut-off function on $\mathbb{R}$. We have
\begin{align*}
E \int_{[0, T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{p-2} d\nu(t, x, \xi) &\leq \liminf_{\delta \to 0} E \int_{[0, T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{p-2} h_\delta(\xi) d\nu(t, x, \xi) \\
&\leq \liminf_{\delta \to 0} \lim_{\varepsilon \to 0} E \int_{[0, T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{p-2} h_\delta(\xi) d\nu(t, x, \xi) \\
&\leq \limsup_{\varepsilon \to 0} E \int_{[0, T] \times \mathbb{T}^N \times \mathbb{R}} |\xi|^{p-2} h_\delta(\xi) d\nu(t, x, \xi) \leq C
\end{align*}
and the claim follows.

Finally, we may apply the Reduction theorem 2.1.10 and obtain the existence of a kinetic solution with the corresponding kinetic measure $\mu$. To complete the proof of Theorem 2.1.11, it remains to verify the strong convergence of $u^\varepsilon$ to $u$ in $L^p(\Omega \times [0, T] \times \mathbb{T}^N)$.

Let $p = 2$. Recall from the theory of Young measures that, roughly speaking, the Young measure corresponding to a weakly converging sequence reduces to a Dirac mass if and only if the sequence converges weakly. Since we have already shown that the limit Young measure $\nu$ is a Dirac, the strong convergence from Theorem 2.1.11 is expected. Since $u^\varepsilon$ is bounded in $L^2(\Omega \times [0, T] \times \mathbb{T}^N \times \mathbb{R})$, due to Banach-Alaoglu, there exists a weakly converging subsequence. The limit is necessarily $u$. In order to establish the strong convergence, we need to prove the convergence of the norms, i.e.
\begin{equation}
(2.13) \quad \|u^\varepsilon\|_{L^2_{\omega,t,x}} \to \|u\|_{L^2_{\omega,t,x}}.
\end{equation}
To this end, we observe: since $f^\varepsilon$ converges to $f$ weak-star in $L^\infty(\Omega \times [0, T] \times \mathbb{T}^N \times \mathbb{R})$, taking $\psi(\xi) = \partial_\xi \varphi(\xi)$ with $\phi \in C_c^\infty(\mathbb{R})$ as a test function and integrating by parts gives
\begin{align*}
\langle f^\varepsilon, \psi \rangle &= E \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} 1_{u^\varepsilon \geq \xi} \partial_\xi \varphi d\xi dx dt = E \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \varphi d\delta_{u^\varepsilon}(\xi) dx dt \\
&= E \int_0^T \int_{\mathbb{T}^N} \varphi(u^\varepsilon) dx dt \to E \int_0^T \int_{\mathbb{T}^N} \varphi(u) dx dt.
\end{align*}
This would prove (2.13) provided we could take $\varphi(\xi) = |\xi|^2$. However, this is not possible directly, so we truncate first: let $(h_\delta)$ be a smooth cut-off function as above and set $\varphi_\delta = \varepsilon^2 h_\delta(\xi)$. Then
\begin{align*}
|E \int_0^T \int_{\mathbb{T}^N} |u^\varepsilon|^2 - |u|^2 | dx dt | &\leq |E \int_0^T \int_{\mathbb{T}^N} |u^\varepsilon|^2 (1 - h_\delta(\xi)) dx dt | \\
+ |E \int_0^T \int_{\mathbb{T}^N} |u^\varepsilon|^2 h_\delta(\xi) - |u|^2 h_\delta(\xi) dx dt | \\
+ |E \int_0^T \int_{\mathbb{T}^N} |u|^2 (1 - h_\delta(\xi)) dx dt | = I_1 + I_2 + I_3,
\end{align*}
where $I_2 \to 0$ for each $\delta$ as $\varepsilon$ vanishes. For $I_1$ and $I_3$ we make use of Lemma 2.2.3 for $p > 2$ which implies that $|u^\varepsilon|^2$ is uniformly integrable and hence $I_1 \to 0$ as $\delta \to 0$ uniformly in $\varepsilon$. Similarly, since the limit $u$ satisfies the same a priori estimates is $u^\varepsilon$, $I_3 \to 0$ as $\delta \to 0$ uniformly in $\varepsilon$. As a consequence, we obtain the convergence from Theorem 2.1.11 for $p = 2$. 

If $p > 2$, we apply the Hölder inequality and the uniform a priori estimate to deduce

$$
\mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon - u|^p \, dx \, dt = \mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon - u|^{p-1} |u^\epsilon - u| \, dx \, dt
$$

$$
\leq \left( \mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon - u|^{2(p-1)} \, dx \, dt \right)^{1/2} \left( \mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon - u|^2 \, dx \, dt \right)^{1/2}
$$

$$
\leq C \left( \mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\epsilon - u|^2 \, dx \, dt \right)^{1/2} \to 0
$$

and the proof of Theorem 2.1.11 is complete.

### 2.3. Uniqueness

The proof of uniqueness as well as the reduction theorem are very technical and lengthy. Therefore I will only give some basic ideas and explain the difficulties.

Recall that the kinetic formulation (2.6) is satisfied in the following sense: for all $\varphi \in C^\infty_c([0,T) \times \mathbb{T}^N \times \mathbb{R})$, P.-a.s.,

$$
\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), b \cdot \nabla \varphi(t) \rangle \, dt
$$

$$
= -\sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, d\beta_k(t)
$$

$$
- \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} G^2(x, \xi) \partial_\xi \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, dt + \langle m, \partial_\xi \rangle([0, t)).
$$

Roughly speaking, (2.6) holds true in the sense of distributions over $[0, T) \times \mathbb{T}^N \times \mathbb{R}$. However, it is not clear how to work with this formulation. In particular, one cannot apply the Itô formula. Is it possible to consider a stronger version of (2.6)? Namely, for a test function $\varphi \in C^\infty_c(\mathbb{T}^N \times \mathbb{R})$, we want to write

$$
\langle f(t), \varphi \rangle - \langle f_0, \varphi \rangle - \int_0^t \langle f(s), b \cdot \nabla \varphi \rangle \, ds
$$

$$
= \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, d\beta_k(t)
$$

$$
+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} G^2(x, \xi) \partial_\xi \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, dt + \langle m, \partial_\xi \rangle([0, t)).
$$

This was for instance true for the parabolic approximation as can be seen in Proposition 2.2.1. Nevertheless, in general, this is not true as the measure can have a shock at time $t$.

Let us consider the first term in (2.14). We observe that testing by $\partial_t \varphi(t, x, \xi) = \delta_{s=t} \psi(x, \xi)$ we get

$$
\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt = \langle f(s), \psi \rangle,
$$
but again, this is not a good test function. Taking a smooth approximation, so e.g. a smooth approximation of \( \varphi(t, x, \xi) = \alpha^\varepsilon(t) \psi(x, \xi) \) where

\[
\alpha^\varepsilon(t) = \begin{cases} 
1, & 0 \leq t \leq s \\
1 - \frac{t-s}{\varepsilon}, & s \leq t \leq s + \varepsilon \\
0, & s + \varepsilon \leq t,
\end{cases}
\]

we have

\[
\int_0^T \langle f(t), \psi \rangle \partial_s \alpha^\varepsilon \, dt = \int_s^{s+\varepsilon} \langle f(t), \psi \rangle \partial_s \alpha^\varepsilon \, dt \approx -\frac{1}{\varepsilon} \int_s^{s+\varepsilon} \langle f(t), \psi \rangle \, dt \to -\langle f(s), \psi \rangle
\]

for a.e. \( s \in [0, T] \) due to the Lebesgue differentiation theorem. However, this is still not enough as we would like to have such a formula for every \( s \in [0, T] \). Therefore, we are led to study the limits of \( \langle f(s \pm \varepsilon), \psi \rangle \). We obtain the following result.

**Proposition 2.3.1.** Let \( f \) be a generalized kinetic solution to (2.1). Then \( f \) admits a.s. left and right limits at all points \( t^* \in [0, T] \) in the following sense: Fix \( t^* \in [0, T] \). There exist kinetic functions \( f^{*, \pm} \) such that for all \( \psi \in C^1_c(T^N \times \mathbb{R}) \)

\[
\langle f(t^* \pm \varepsilon), \psi \rangle \to \langle f^{*, \pm}(t^*), \psi \rangle \quad \text{a.s.}
\]

Moreover, let us define \( f^{\pm}(t^*) = f^{*, \pm} \). Then \( f^\pm = t \) a.e. and \( f^- \) and \( f^+ \) are a.s. left and right-continuous, respectively, i.e. for all \( \psi \in C^1_c(T^N \times \mathbb{R}) \)

\[
\langle f(t^* \pm \varepsilon), \psi \rangle \to \langle f^{\pm}(t^*), \psi \rangle \quad \text{a.s.}
\]

In other words, recalling that \( f \in L^\infty(\Omega \times [0, T] \times T^N \times \mathbb{R}) \) is a class of equivalence, we may say that in this class of equivalence there exists a right and a left-continuous representative. As a consequence, going back to (2.15), we have

\[
-\frac{1}{\varepsilon} \int_s^{s+\varepsilon} \langle f(t), \psi \rangle \, dt = -\frac{1}{\varepsilon} \int_s^{s+\varepsilon} \langle f^{\pm}(t), \psi \rangle \, dt \to -\langle f^{\pm}(s), \psi \rangle
\]

for every \( s \in [0, T] \) and therefore we can indeed write a stronger formulation of (2.6) provided we replace \( f \) by \( f^+ \) (or similarly by \( f^- \)). To be more precise, if \( f \) is a generalized kinetic solution to (2.6), it satisfies for all \( \varphi \in C^1_c(T^N \times \mathbb{R}) \)

\[
\langle f^+(t), \varphi \rangle = \langle f_0, \varphi \rangle + \int_0^t \langle f(s), b \cdot \nabla \varphi \rangle \, ds
\]

\[
= \sum_{k \geq 1} \int_0^t \int_{T^N} g_k(x, \xi) \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, dt + \int_0^t \int_{T^N} \sum_{k \geq 1} G^2(x, \xi) \partial_{\xi} \varphi(t, x, \xi) \, d\nu_{t,x}(\xi) \, dx \, dt + \langle m, \partial_{\xi}((0, t)) \rangle.
\]

This was a very crucial step in the proof of uniqueness.

In view of the \( L^1 \)-contraction property

\[
\mathbb{E}\|u_1(t) - u_2(t)\|_{L^1} \leq \|u_{1,0} - u_{2,0}\|_{L^1},
\]

it is needed to write the above \( L^1 \)-norm of the difference \( u_1(t) - u_2(t) \) in terms of the corresponding kinetic functions \( f_1(t), f_2(t) \). It can be seen by an easy calculation that in the case of kinetic solutions, i.e. \( f_i = 1_{u_i > \xi} \), it holds true

\[
\int_{\mathbb{R}} 1_{u_1 > \xi} (1 - 1_{u_2 > \xi}) \, d\xi = (u_1 - u_2)^+.\]
Hence if we already had kinetic (not generalized) solutions, we would write down the equation for $1_{u_1 > \xi}$ and for $1 - 1_{u_2 > \xi}$ and apply the Itô formula to their product and estimate all the terms on the right hand side. According to the above considerations, we have to work with $(1_{u_1 > \xi})^+$ and $(1_{u_1 > \xi})^-$. Moreover, as we still need the test functions with respect to $x, \xi$ we have to mollify first. The following proposition is called doubling of variables.

PROPOSITION 2.3.2. Let $f_1, f_2$ be generalized kinetic solutions to (2.1). Let $(\varrho_\varepsilon)$ and $(\psi_\delta)$ be an approximation to the identity on $\mathbb{T}^N$ and $\mathbb{R}$, respectively. Then for all $t \in [0, T]$ it holds

$$
\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \varrho_\varepsilon(x - y) \psi_\delta(\xi - \zeta) f_1^+(t, x, \xi)(1 - f_2^+(t, y, \zeta)) d\xi d\zeta dx dy \\
\leq \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}^2} \varrho_\varepsilon(x - y) \psi_\delta(\xi - \zeta) f_1^+(0, x, \xi)(1 - f_2^+(0, y, \zeta)) d\xi d\zeta dx dy \\
+ I_{\varepsilon, \delta} + J_{\varepsilon, \delta},
$$

where $I_{\varepsilon, \delta}$ and $J_{\varepsilon, \delta}$ both vanish for a suitable choice of $\varepsilon, \delta \to 0$.

Accordingly, sending $\varepsilon, \delta \to 0$ in the right way leads to

$$
\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^+(t)(1 - f_2^+(t)) d\xi dx \leq \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^+(0)(1 - f_2^+(0)) d\xi dx.
$$

In the case of generalized kinetic solution $f = f_1 = f_2$ we thus obtain

$$
\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f^+(t)(1 - f^+(t)) d\xi dx \leq 0
$$

and since $f^+(t)$ is a kinetic function for every $t \in [0, T)$, we deduce that necessarily there exists $u^+ = u_1^+ > \xi$ a.e. in $\omega, x$. This implies the reduction theorem. Applying now Proposition 2.3.2 to kinetic solutions $u_1$ and $u_2$ gives the $L^1$-contraction property.
Bibliography


