Some Geometric Properties oF the Roots of the Steiner Polynomial

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(joint work with E. Saorín)

Let K be a compact convex set in the Euclidean space. It is well known that the volume of the outer parallel body $K + \rho\mathbb{B}^n$ can be expressed as a polynomial of degree the dimension n in the parameter ρ,

$$V(K + \rho\mathbb{B}^n) = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i,$$

which is known as the *Steiner polynomial* of the body K. The coefficients $W_i(K)$ so defined are called the *quermassintegrals* of the set K. In particular, $W_0 = V$ is the volume, $nW_1 = S$ is the surface area, $nW_2 = M$ is the integral mean curvature and $W_n = \kappa_n$ is the volume of the n-dimensional unit ball. Thus, it is a natural question to study the roots of this polynomial, and to try to find out wether they have any geometric meaning.

On the other hand, Blaschke (1916) considered a compact convex set K in the Euclidean 3-space $\mathbb{R}^3$, with volume $V = V(K)$, surface area $S = S(K)$, and integral of the mean curvature $M = M(K)$. He asked for a characterization of the set of all points in $\mathbb{R}^3$ of the form $\big(V(K), S(K), M(K)\big)$ as K ranges over the family of all compact convex sets in $\mathbb{R}^3$, or, equivalently, for a characterization of the set of all points (x, y) in the unit square $[0, 1]^2 \subset \mathbb{R}^2$ of the form

$$x = \frac{4\pi S}{M^2} \quad \text{and} \quad y = \frac{48\pi^2 V}{M^3}.$$

The latter set is called the *Blaschke diagram.*

In this talk we intend to analyze the relationship between these two problems: first, studying the roots of the three dimensional Steiner polynomial from a geometric point of view, in the sense of characterizing convex bodies according to the type of roots of its Steiner polynomial; then, showing that there is a close relationship with the famous Blaschke problem, for which we can obtain interesting consequences. In particular, it will provide a precise classification of the points of the unit square depending on the convex sets which are mapped into them.