Intersection Body of n–Cube, Siegel's Lemma and Sum–Distinct Sets

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In this talk we discuss a relation between the intersection body of the n-cube and Siegel's Lemma w. r. t. the maximum norm. We show that for any non–zero vector $\mathbf{a} \in \mathbb{Z}^n$, $n \geq 5$, there exist linearly independent vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1} \in \mathbb{Z}^n$ such that $\mathbf{x}_i \mathbf{a} = 0$, $i = 1, \ldots, n-1$ and

$$0 < ||\mathbf{x}_1||_\infty \cdots ||\mathbf{x}_{n-1}||_\infty < \frac{||\mathbf{a}||_\infty}{\sigma_n}, \quad \sigma_n = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t} \right)^n dt \,. \tag{1}$$

The main tools are the Minkowski theorem on successive minima and the Busemann theorem from convex geometry.

This result can be applied to the following well known problem from additive number theory. A finite set $\{a_1, \ldots, a_n\}$ of integers is called *sum–distinct set* if any two of its 2^n subsums differ by at least 1. We shall assume w. l. o. g. that $0 < a_1 < a_2 < \ldots < a_n$. In 1955, P. Erdös and L. Moser ([2], Problem 6) asked for an estimate on the least possible a_n of such a set. They proved that $a_n > \max\{n^{-1}2^n, (1/4)n^{-1/2}2^n\}$ and Erdös conjectured that $a_n > C_0 2^n$, $C_0 > 0$. In 1986, N. D. Elkies [1] showed that $a_n > 2^{-n}\binom{2n}{n}$ and this result is cited by Guy ([3], Problem C8) as the best known lower bound for large n. Using (1), we asymptotically improve the result of Elkies with factor $\sqrt{3/2}$.

[1] , N. D. Elkies, *An Improved Lower Bound on the Greatest Element of a Sum-Distinct Set of Fixed Order*, J. Combin. Theory Ser. A **41** (1986) no. 1 89–94.
[2] , P. Erdös, *Problems and Results in Additive Number Theory*, in ,,Colloque sur la Théorie des Nombres, Bruxelles, 1955", pp. 127–137.
[3] , R. K. Guy, *Unsolved Problems in Number Theory*, Third edition. Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, I. Springer-Verlag, New York, 2004.