PACKINGS, SAUSAGES AND CATASTROPHES

MARTIN HENK AND JÖRG M. WILLS

Dedicated to the memory of Uli Betke (1948–2008)

ABSTRACT. In this survey we give an overview about some of the main results on parametric densities, a concept which unifies the theory of finite (free) packings and the classical theory of infinite packings.

1. INTRODUCTION

The theory of infinite packings of convex bodies, in particular, lattice packings of spheres is a fundamental and classical topic in mathematics which plays a role in various branches of mathematics as number theory, group theory, geometry of numbers, algebra, and which has numerous applications to coding theory, cryptography, crystallography and more. Here the main problem is to arrange infinitely many non-overlapping (lattice-) translative copies of a given convex body such the the whole space is covered as much as possible, i.e., packed as densely a possible.

On the other hand one may say, that all packings in real world are finite, even the atoms in crystals or sand at the beach, and in the theory of finite packings we want to arrange finitely many non-overlapping (lattice-) translative copies of a convex body as good as possible. Roughly speaking, there are two different ways to specify "as good as possible": The first one leads to so called *bin packings* where a container (bin) of a prescribed shape (ball, simplex, cube, etc.) but of minimal size (volume) is looked for containing a given number of non-overlapping (lattice-) translative copies of the given convex body. In contrast to this, we consider here so called *free (finite) packings* where the goal is to minimize the volume of the convex hull of a given number of non-overlapping (lattice-) translative copies of the convex body.

The volume–based free packing approach was introduced in 1892 by the Norwegian mathematician Axel Thue. For a given number of circles, he considered all possible packings and their convex hulls and asked for the minimal volume of these convex hulls. Thue's approach was further developed between 1940 and 1972 by many prominent mathematicians, e.g., by L. Fejes Tóth, R.P. Bambah, C.A. Rogers, H. Groemer and H. Zassenhaus. They also established a joint theory of finite and infinite packings (and coverings) in the plane. However, for higher dimensions, Thue's approach does not yield a joint theory of finite and infinite packings.

For the most interesting case of (free) finite sphere packings, L. Fejes Tóth formulated in 1975 his famous *sausage conjecture*, claiming that for dimensions ≥ 5 and any(!) number of unit balls, a linear arrangement of the balls, i.e., all midpoints are on a line and two consecutive balls touch each

other, minimizes the volume of their convex hull. Currently, the sausage conjecture has been confirmed for all dimensions ≥ 42. The sausage conjecture shows already that finite packings have nothing or only little to do with classical infinite packings. Thus there was the natural question to replace or to generalize Thue's density by a more general density, which permits a joint theory of finite and infinite packings for all dimensions and which also explains strange phenomena of finite a packings as so-called *sausage catastrophes*.

We will show in this note, that the *parametric density* found in 1993 is the right answer to this question; it contains, in particular, Thue's density as a special case. As the name indicates, the parametric density depends on a parameter $\rho > 0$, and it turns out that for small ρ sausage-like arrangements are optimal whereas for large ρ, densest infinite packing arrangements are optimal. The parametric density, in particular, allows to obtain results on infinite packings via (limit) results on finite packings. Still, these results are, with a few exceptions, weaker than their classical infinite counterparts, but it is a tempting and interesting task to improve them and we kindly invite the reader to do so.

There is also an analogous theory of parametric densities for finite and infinite covering problems of convex bodies, introduced in [7]. In this survey, however, we just consider packing problems and for a thorough treatment of finite and covering problems, including the parametric densities, we refer to the book of K. Böröczky, Jr. [10]. As a general reference to packing and covering of convex bodies see, e.g., [17, 24], and for sphere packings, e.g., [15, 16, 47].

The paper is organized as follows: After providing the necessary definitions and notations regarding infinite and finite packings in the next section, the parametric densities will be introduced in Section 3. In Section 4 we will discuss the small parameter range and in Chapter 5 the large one. Finite lattice packings with respect to parametric densities are discussed in the last section.

2. NOTATIONS AND PRELIMINARIES

We are working in the d-dimensional Euclidean space $\mathbb{R}^d$, equipped with the standard inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^\intercal \boldsymbol{y}$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ and Euclidean norm $|\boldsymbol{x}| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. $B^d = \{ \boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| \leq 1 \}$ is the Euclidean (unit) ball centered at the origin $\boldsymbol{0}$ of radius 1; its boundary bd B^d is called unit sphere and will be denoted by $\mathbb{S}^{d-1}$. The set of all convex bodies $K \subset \mathbb{R}^d$ is denoted by $\mathcal{K}^d$, i.e., $K \in \mathcal{K}^d$, if K is convex, closed, bounded and int (K), the interior of K is non-empty. The dimension of a set S is the dimension of its affine hull aff (S) and it will be denoted by dim S. For $K \in \mathcal{K}^d$ let

$$(2.1) \quad \mathcal{P}(K) = \{ C \subset \mathbb{R}^d : \text{int}\,(\boldsymbol{x}_i + K) \cap \text{int}\,(\boldsymbol{x}_j + K) = \emptyset,\, \boldsymbol{x}_i \neq \boldsymbol{x}_j \in C \}$$

be the set of all *packing sets* of K. For $C \in \mathcal{P}(K)$, the arrangement $C + K$ is called a *packing* of K. In order to define the density of such a packing we denote by vol (S) the volume, i.e., the d-dimensional Lebesgue measure of a measurable set $S \subset \mathbb{R}^d$. For a finite set $S \subset \mathbb{R}^d$ its cardinality is denoted by $\#S$, and let $W^d = [-1, 1]^d$ be the cube of edge length 2 centered at the

origin. Then for $K \in \mathcal{K}^d$, $C \in \mathcal{P}(K)$,

$$(2.2) \qquad \delta(K, C) = \limsup_{\lambda \to \infty} \frac{\#(C \cap \lambda\, W^d)\, \mathrm{vol}\,(K)}{\mathrm{vol}\,(\lambda\, W^d)}$$

is called the *density of the packing $C + K$* and

$$(2.3) \qquad \delta(K) = \sup\{\delta(K, C) : C \in \mathcal{P}(K)\}$$

is called the *density of a densest packing of K*.

Obviously, for any finite packing set C we have $\delta(K, C) = 0$. The idea of the quantity $\delta(K, C)$ is to measure how much of the space $\mathbb{R}^d$ is occupied by $C + K$, i.e., we would like to measure $\mathrm{vol}\,(C + K)/\mathrm{vol}\,(\mathbb{R}^d)$, and we do it mathematically by approximating $\mathbb{R}^d$ via the sequence $\lambda\, W^d$. In particular, $\delta(K, C)$ may depend on the gauge body (here W^d) by which we approximate $\mathbb{R}^d$. It was shown by Groemer [23], however, that the definition of $\delta(K)$ is independent of this gauge body, and that there exists an *optimal* packing set $C_K \in \mathcal{P}(K)$ such that

$$\delta(K) = \delta(K, C_K) = \lim_{\lambda \to \infty} \frac{\#(C_K \cap \lambda\, W^d)\mathrm{vol}\,(K)}{\mathrm{vol}\,(\lambda\, W^d)}.$$

Now we turn to finite (free) packings and to this end we consider for an integer $n \in \mathbb{N}$,

$$\mathcal{P}_n(K) = \{C \in \mathcal{P}(K) : \#C\}$$

the set of all packing sets of cardinality n. Here we want to find a packing set $C_{K,n} \in \mathcal{P}_n(K)$ minimizing $\mathrm{vol}\,(\mathrm{conv}\ C + K)$ among all $C \in \mathcal{P}_n(K)$, where conv denotes the convex hull. Hence, in analogy to (2.2), (2.3) we denote for $K \in \mathcal{K}^d$ and $C \in \mathcal{P}(K)$ with $\#C < \infty$ by

$$(2.4) \qquad \delta_1(K, C) = \frac{\#C\, \mathrm{vol}\,(K)}{\mathrm{vol}\,(\mathrm{conv}\ C + K)}$$

the *density of the finite packing $C + K$* and

$$\delta_1(K, n) = \sup\{\delta_1(K, C) : C \in \mathcal{P}_n(K)\}$$

is called the *density of a densest n-packing of K*. The role of the index 1 will become clear soon, and it not hard to see that for any n there exists an optimal packing set $C_{n,K}$ such that $\delta_1(K, n) = \delta_1(K, C_K(n))$.

Of particular interest are here finite packing sets $C = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} \in \mathcal{P}_n(K)$ with $\dim C = 1$, i.e., all points are collinear. Since we also want to minimize $\mathrm{vol}\,(\mathrm{conv}\ \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n\} + K)$ we may assume that for two consecutive points on this line, $\boldsymbol{x}_i, \boldsymbol{x}_j$, say, the translates $\boldsymbol{x}_i + K$ and $\boldsymbol{x}_j + K$ touch. Hence, without loss of generality the points of such a packing set can be represented as

$$S_n(K, \boldsymbol{u}) = \left\{(i-1)\frac{2}{|\boldsymbol{u}|_K}\boldsymbol{u} : 1 \le i \le n\right\},$$

where $\boldsymbol{u} \in \mathbb{S}^{d-1}$ is the direction of the line and with $|\boldsymbol{u}|_K$ we denote the norm induced by the origin symmetric body $\frac{1}{2}(K - K)$, i.e.,

$$|\boldsymbol{u}|_K = \min\left\{\mu \in \mathbb{R}_{\ge 0} : \boldsymbol{u} \in \mu\frac{1}{2}(K - K)\right\}.$$
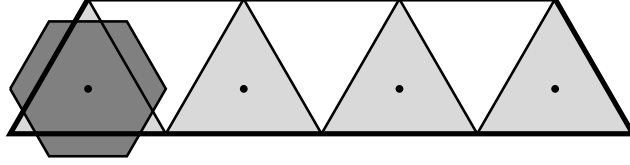
FIGURE 1. A sausage configuration of a triangle T, where $\frac{1}{2}(T - T)$ is the darker hexagon.

Packing sets $S_n(K, \boldsymbol{u})$ are called sausage configurations, where the name was coined by L. Fejes Tóth [25] in the special setting $K = B^d$. Obviously, in this case the density of such a sausage configuration is independent of the direction $\boldsymbol{u}$ and therefore, it will be only denoted by $S_n(B^d)$ and it is

$$\mathrm{vol}\left(\mathrm{conv}\left(S_n(B^d) + B^d\right) = 2(n-1)\kappa_{d-1} + \kappa_d,\right.$$

where κ_i denotes the i-dimensional volume of the i-dimensional unite ball.
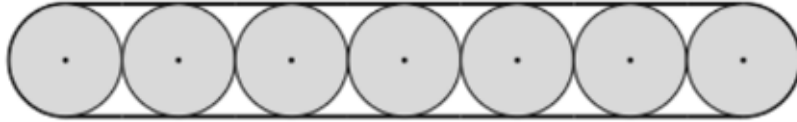

FIGURE 2. A sausage of 7 circles with density $\delta_1(B^2, S_7(B^2)) = 7\pi/(24 + \pi)$.

The famous sausage conjecture of L. Fejes Tóth [25] claims that for any number of balls, a sausage configuration is always best possible, provided $d \geq 5$.

Conjecture 2.1 (SAUSAGE CONJECTURE:). *For $d \geq 5$ and $n \in \mathbb{N}$*

$$\delta_1(B^d, n) = \delta(B^d, S_n(B^d)).$$

In the plane a sausage is never optimal for $n \geq 3$ and for "almost all" $n \in \mathbb{N}$ optimal packing configurations are known (see [30, 42, 44] and the references within).

In dimension 3 and 4 the situation is more complicated: In [3, 4] it was shown that among those finite packings sets C satisfying $\dim C \leq \min\{9, d-1\}$ or $\dim C \leq (7/12)(d-1)$ only sausages are optimal. Hence, in particular, for dimensions 3 and 4, no packings sets of intermediate dimensions are optimal, i.e., optimal packings sets are either 1-dimensional (sausages) or

d-dimensional (clusters). It is easy to see that for small n sausages are optimal while for large n clusters are optimal and so the interesting question is: when, i.e., for which 'magic" numbers n does it happen? In dimension 3, results of Wills [45, 46], Gandini and Wills [20] and Scholl[40] show that certain clusters are denser than sausage configurations when $n = 56$ or $n \geq 58$. In fact, it is conjectured that for $n < 56$ and $n = 57$ sausages are optimal. In dimension 4 it was shown by Gandini and Zucco [21], Gandini [19] that a cluster is better than a sausage configuration for $n \geq 375{,}769$. This large number of spheres motivated the name *sausage catastrophe* given in [46] referring to the abrupt change of the optimal shape of an optimal packing set. For a German popular science article about the catastrophe and conjecture see [18].

Obtaining a unified theory for finite and infinite packings covering also these phenomena of sausage conjecture and sausage catastrophe was one motivation for the parametric density which we will define in the next section.

L. Fejes Tóth's sausage conjecture was first proved via the parametric density approach in dimensions $\geq 13{,}387$ by Betke et al. [6] which was later improved to $d \geq 42$ by Betke and Henk [5].

The sausage conjecture, in particular, implies that in general

$$\delta(K) < \limsup_{n \to \infty} \delta_1(K, n)$$

and, in fact, this is known to be true for all dimensions $d \geq 3$. Thus, large optimal finite packing sets do not "approximate" optimal infinite packing sets. However, as we will see next, this will be corrected via the parametric density.

3. The parametric density

The concept of a parametric density was introduced by Betke et al. in [6], and the definitions and results presented in this section are taken from this paper.

Definition 3.1 (Parametric Density). *Let $\rho > 0$ and $K \in \mathcal{K}^d$.*

i) *Let $C \in \mathcal{P}(K)$ with $C < \infty$,*

(3.1)
$$\delta_\rho(K, C) = \frac{\#C \operatorname{vol}(K)}{\operatorname{vol}(\operatorname{conv} C + \rho K)}$$

is called the parametric density of C with respect to K and the parameter ρ.

ii)
$$\delta_\rho(K, n) = \sup\{\delta_\rho(K, C) : C \in \mathcal{P}_n(K)\}$$

is called the parametric density of a densest n-packing of K with respect to the parameter ρ.

iii)
$$\delta_\rho(K) = \limsup_{n \to \infty} \delta_\rho(K, n)$$

is called the parametric limit density of K with respect to the parameter ρ.

Apparently, for $\rho = 1$, the definitions in i) and ii) coincide with the previous given definitions of $\delta_1(K, C)$ (2.4) and $\delta_1(K, n)$(2), and again it is easy to see that the sup in ii) may be replaced by a max.

It is also easy to check, that $\delta_\rho(K, C)$, $\delta_\rho(K, n)$ and $\delta_\rho(K)$ are monotonously decreasing and continuous in ρ. Moreover, $\delta_\rho(K, n)$ and $\delta_\rho(K)$ are invariant with respect to regular affine transformations of K. By calculating the finite parametric density of large clusters of a densest infinite packing one gets for all $\rho > 0$

$$(3.2) \qquad \delta_\rho(K) \geq \delta(K).$$

In order to understand the role of the parameter ρ we briefly recall some basic facts about mixed volumes, which implicitly appear in the denominator of (3.1), and for a detailed account we refer, e.g., to [24, 38]. It is a classical fact from Convex Geometry, that the volume of conv $C + \rho K$ can be written as a polynomial in ρ of degree d, the so-called generalized Steiner-polynomial:

$$\text{vol}\,(\text{conv}\,C + \rho K) = \sum_{i=0}^{d} \binom{d}{i} \rho^i \mathrm{V}_i(\text{conv}\,C; K),$$

where the coefficients $\mathrm{V}_i(\text{conv}\,C; K)$ are called *mixed volumes*. In particular, we have $\mathrm{V}_d(\text{conv}\,C; K) = \text{vol}\,(K)$, $\mathrm{V}_0(\text{conv}\,C; K) = \text{vol}\,(\text{conv}\,C)$ and for $K \in \mathcal{K}^d$ it is $\mathrm{V}_i(\text{conv}\,C; K) = 0$ if and only if $\dim C < d - i$. So in order to determine $\delta_\rho(K, n)$ we have to minimize

$$\sum_{i=0}^{d} \binom{d}{i} \rho^i \frac{\mathrm{V}_i(\text{conv}\,C; K)}{\text{vol}\,(K)}$$

over all $C \in \mathcal{P}_n(K)$. A large parameter ρ gives a strong weight on the mixed volumes with an high index i. So it seems preferable to make mixed volumes with a small index rather large, which means that optimal packing set should be of dimensions d. Hence, we can expect that for large ρ and large n, optimal finite parametric densities converge to the density of a densest infinite packing. Therefore, we define

Definition 3.2 (Critical Parameter). *For $K \in \mathcal{K}^d$ let*

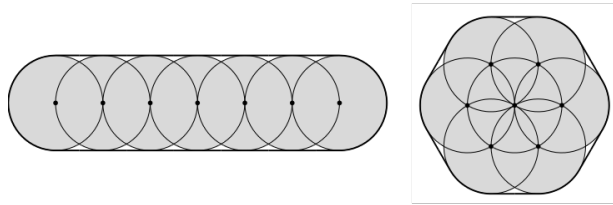$$\rho_c(K) = \inf\{\rho > 0 : \delta_\rho(K) = \delta(K)\}$$

FIGURE 3. A sausage and a hexagonal packing C of 7 unit circles with $\rho = 2$. The densities are $\delta_2(B^2, S_7(B^2)) = \frac{7\pi}{2 \cdot 24 + 2^2 \cdot \pi} \sim 0.36$ and $\delta_2(B^2, C) = \frac{7\pi}{6\sqrt{3} + 2 \cdot 12 + 2^2 \cdot \pi} \sim 0.47$.

On the other hand, a small ρ gives a strong weight to the mixed volumes with a small index, and so low-dimensional packing sets C are better, with the extreme case $\dim C = 1$. Now for such a sausage configuration $S_n(\boldsymbol{u}, K)$ we have

$$\mathrm{vol}\,(\mathrm{conv}\,S_n(\boldsymbol{u}, K) + \rho K) = 2(n-1)\frac{\mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)}{|\boldsymbol{u}|_K}\rho^{d-1} + \mathrm{vol}\,(K)\rho^d,$$

where $\mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)$ is the $(n-1)$-dimensional volume of the orthogonal porjection of K on the hyperplane orthogonal to $\boldsymbol{u}$. Hence, in order to have a sausage configuration of maximal density let $\boldsymbol{u}_k \in S^{n-1}$ such that

$$\frac{\mathrm{vol}_{n-1}(K|\boldsymbol{u}_K^\perp)}{|\boldsymbol{u}_K|_K} = \min_{\boldsymbol{u}\in S^{d-1}} \frac{\mathrm{vol}_{n-1}(K|\boldsymbol{u}^\perp)}{|\boldsymbol{u}|_K}.$$

With this notation let

$$(3.3) \qquad \delta_\rho^s(K) = \lim_{n\to\infty} \delta_\rho(K, S_n(\boldsymbol{u}_K, K)) = \rho^{1-d}\mathrm{vol}\,(K)\frac{|\boldsymbol{u}_K|_K}{2\,\mathrm{vol}_{n-1}(K|\boldsymbol{u}_K^\perp)}$$

the parametric limit density of an optimal sausage configuration of K with respect to the parameter ρ. This density may be regarded as the 1-dimensional counterpart to $\delta(K)$. In particular, for $K = B^d$ we have

$$(3.4) \qquad \delta_\rho^s(B^d) = \rho^{1-d}\frac{\kappa_d}{2\kappa_{d-1}}.$$

In analogy to the critical parameter we define

Definition 3.3 (Sausage Parameter). *For $K \in \mathcal{K}^d$ let*

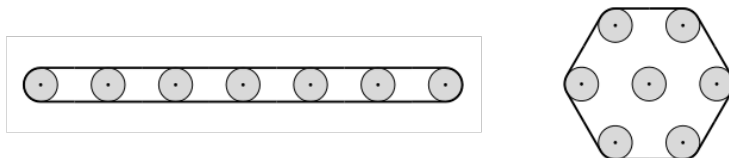$$\rho_s(K) = \sup\{\rho > 0 : \delta_\rho(K) = \delta_\rho^s(K)\}.$$

FIGURE 4. A sausage and a hexagonal packing C of 7 unit circles with $\rho = 1/2$. The densities are

$$\delta_{\frac{1}{2}}(B^2, S_7(B^2)) = \frac{7\pi}{(1/2)\cdot 24 + (1/2)^2\cdot\pi} \sim 1.72,$$

$$\delta_{\frac{1}{2}}(B^2, C) = \frac{7\pi}{6\sqrt{3} + (1/2)\cdot 12 + (1/2)^2\cdot\pi} \sim 1.28.$$

These two parameters devide the range of all parameters into three relevant areas and provide us with an simple bound on $\delta(K)$ as the next theorem shows.

Theorem 3.4. *Let $K \in \mathcal{K}^d$ and assume that $0 < \rho_s(K), \rho_c(K) < \infty$. Then*

i) $\rho_s(K) \leq \rho_c(K)$.

ii) $\delta_\rho(K) = \delta_\rho^s(K)$ *for* $\rho \in (0, \rho_s(K)]$.

iii) $\delta_\rho(K) = \delta(K)$ *for* $\rho \in [\rho_c(K), \infty)$.

iv) $\delta_{\rho_c(K)}^s(K) \leq \delta(K) \leq \delta_{\rho_s(K)}^s(K)$.

In other words, below the sausage parameter infinite sausages, i.e., 1-dimensional packings are optimal, above the critical parameter densest infinite packing arrangements are optimal, and what happens in between is (in general) rather unknown. Moreover, the parametric density of infinite sausages with respect to critical and sausage parameter yields lower and upper bounds on $\delta(K)$ (cf. (3.3))

$$(3.5) \qquad \rho_c(K)^{1-d} \frac{|\boldsymbol{u}_K|_K \operatorname{vol}(K)}{2\operatorname{vol}_{n-1}(K|\boldsymbol{u}_K^\perp)} \leq \delta(K) \leq \rho_s(K)^{1-d} \frac{|\boldsymbol{u}_K|_K \operatorname{vol}(K)}{2\operatorname{vol}_{n-1}(K|\boldsymbol{u}_K^\perp)}.$$

In particular, for the unit ball B^d, Theorem 3.4 iv) gives (cf. (3.4))

$$(3.6) \qquad \frac{\kappa_d}{2\kappa_{d-1}}(\rho_c(B^d))^{1-d} \leq \delta(B^d) \leq \frac{\kappa_d}{2\kappa_{d-1}}(\rho_s(B^d))^{1-d}.$$

Here useful estimates are (see, e.g., [22], [4])

$$(3.7) \qquad \frac{1}{d} < \frac{|\boldsymbol{u}_K|_K \operatorname{vol}(K)}{2\operatorname{vol}_{n-1}(K|\boldsymbol{u}_K^\perp)} \leq 1 \text{ and } \sqrt{\frac{2\pi}{d+1}} < \frac{\kappa_d}{\kappa_{d-1}} < \sqrt{\frac{2\pi}{d}}.$$

In general, we have $\rho_s(K) < \rho_c(K)$ for $d \geq 3$ (cf. [10, Theorem 10.7.1]), but in case of the ball it is tempting to conjecture that $\rho_c(B^d) = \rho_s(B^d)$. In fact, for the ball it is even conjectured in [6]

Conjecture 3.5 (Strong Sausage Conjecture)**.** *For* $n \in \mathbb{N}$ *and* $\rho > 0$ *it holds*

$$\delta_\rho(B^d, n) = \delta_\rho(B^d, S_n(B^d)) \text{ or } \delta_\rho(B^d, n) < \delta(B^d).$$

In particular, we have $\rho_c(B^d) = \rho_s(B^d)$.

Let us briefly point out that this conjecture also covers (essentially) L. Fejes Tóth's sausage conjecture as for $\rho = 1$ and $d \geq 5$ a (maybe large) sausage has a larger density than $\delta(B^d)$. This conjecture would also imply the equivalence of determining $\delta(B^d)$, $\rho_c(B^d)$ and $\rho_s(B^d)$. In the next sections we will briefly present what is known about these two parameters.

4. The planar case

In the planar case and for centrally symmetric convex domains we have a rather complete picture. Based on classical results of Rogers [34, 35] and Oler [32] it was shown in [6] that

Theorem 4.1. *Let* $K \in \mathcal{K}^2$, $K = -K$, *and* $n \in \mathbb{N}$.

i)

$$\frac{3}{4} \leq \rho_s(K) = \frac{\delta_1^s(K)}{\delta(K)} = \rho_c(K) \leq 1,$$

where equality on the left is attained only for an affinely regular non-degenerate hexagon and on the right only for a parallelogram.

ii) $\delta_\rho(K, n) = \delta_\rho(K, S_n(K))$ *for* $0 < \rho \leq \rho_s(K)$.

iii) $\delta_\rho(K, n) \leq \delta(K) \left(\frac{n}{n-1+\delta(K)\rho^2} \right)$, $\quad \rho_s(K) \leq \rho < \infty$.

For instance, for the circle B^2 we have $\delta_1^s(B^2) = \pi/4$ (cf. (3.4)) and by a well-known and classical result of Thue [43] we also know $\delta(B^2) = \pi/(2\sqrt{3})$. Hence

$$\rho_s(B^2) = \frac{\sqrt{3}}{2}.$$

In particular, Theorem 4.1 ii) shows that for $\rho < \rho_s(K)$ and any $n \in \mathbb{N}$ only sausage configurations are optimal. Such a strong statement is not true for arbitrary convex domains in the plane as it was shown by Böröczky, Jr. and Schnell in [12], where they also proved that Theorem 4.1 i) holds true for any convex body in the plane.

For $\rho \geq \rho_s(K)$ various optimal configurations might be possible as our running example below shows.
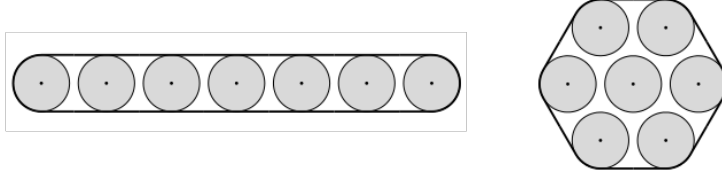

FIGURE 5. A sausage and a hexagonal packing C of 7 unit circles with $\rho = \rho_s(B^2) = \sqrt{3}/2$. Here we have $\delta_{\sqrt{3}/2}(B^2, S_7(B^2)) = \delta_{\sqrt{3}/2}(B^2, C) = 7\pi/(6\sqrt{3} + 12\sqrt{3}/2 + \pi(\sqrt{3}/2)^2) \sim 0.9503$.

5. Small parameters and sausages

First we focus on the most prominent convex body, the ball. Here the main result verifies the sausage conjecture in high dimensions in a very strong form, i.e., not only for $\rho = 1$.

Theorem 5.1 ([8]). *For every $\rho < \sqrt{2}$ there exists a dimension d_ρ such that for all $n \in \mathbb{N}$ and $d \geq d_\rho$*

$$\delta_\rho(B^d, n) = \delta(B^d, S_n(B^d)).$$

Even for parameters ≥ 1 sausages are optimal packing configurations for any $n \in \mathbb{N}$, provided the dimension is large enough. Of course, this result implies

Corollary 5.2. $\liminf_{d \to \infty} \rho_s(B^d) = \sqrt{2}$.

The proof of Theorem 5.1 is based on a local approach by comparing the volume of conv $(C + \rho B^d)$ contained in a Dirichlet-Voronoi cell of an arbitrary packing set $C \in \mathcal{P}_n(B^d)$ to the corresponding volumes of a sausage

$S_n(B^d) + \rho B^d$. The bound of $\sqrt{2}$ corresponds to the minimum distance between a vertex of a Dirichlet-Voronoi cell at c, say, and the center c. It was shown by Rogers [36, Chapter 7] that the distance between an i-dimensional face of such a cell and its center is at least $\sqrt{2(n-i)/(n-i+1)}$. In fact, in [6], Theorem 5.1 was proved for $\rho < 2/\sqrt{3}$ since only the $(n-2)$-dimensional faces of the cell were "used." It is easy to see that not all vertices of a Dirichlet-Voronoi cell of a packing set $C \in \mathcal{P}_n(B^d)$ can be as close as $\sqrt{2}$, but it seems to be hard to take advantage of this fact.

Another consequence of Theorem 5.1 is the following upper bound on $\delta(B^d)$ (cf. (3.6) and (3.7)).

Corollary 5.3. *For $\epsilon > 0$ there exists a dimension d_ϵ such that for $d \geq d_\epsilon$*

$$\delta(B^d) \leq \sqrt{\frac{\pi}{d}}(\sqrt{2} - \epsilon)^{1-d}.$$

This bound is asymptotically of the same order as the classical bounds of Blichfeldt [9] and Rogers [36, Chapter 7]. Though this is much weaker than the best known upper bound for $\delta(B^d)$ (cf. [29]), it shows that finite parameterized packings are also a tool to study infinite packings.

In view of L. Fejes Tóth's sausage conjecture the dimension d_1 of the theorem above is of particular interest.

Theorem 5.4 ([5]). *$d_1 \leq 42$, i.e., the Sausage Conjecture 2.1 is true for all dimension ≥ 42.*

The reason for 42 is given here [1]. The sausage conjecture has also been verified with respect to certain restriction on the packings sets, e.g., among those which are lower-dimensional [3, 4], or close to sausage-like arrangements [31], of whose inradius is rather large [11]. For detailed information we refer to [10, Section 8.3].

Now regarding the sausage parameter of arbitrary convex bodies the best bound is due to K. Böröczky, Jr.; he showed

Theorem 5.5 ([10, Theorem 10.1.1]). *Let $K \in \mathcal{K}^d$ and let $\rho < 1/(32\,d)$. Then for all $n \in \mathbb{N}$*

$$\delta_\rho(K,n) = \delta(B^d, S_n(K)).$$

A weaker upper bound of $1/(32\,d^2)$ was frist proved in [8]. By the theorem above we get

Corollary 5.6. *Let $K \in \mathcal{K}^d$. Then $\rho_s(K) \geq \frac{1}{32d}$.*

It is tempting to conjecture that $\rho_s(K)$ is bounded from below by an absolute constant.

6. LARGE PARAMETERS AND DENSEST INFINITE PACKINGS

Here the main result is captured by the next theorem

Theorem 6.1. *Let $K \in \mathcal{K}^d$ and let $\overline{\rho} \in \mathbb{R}_{>0}$ such that $K - K \subseteq \overline{\rho}K$. Then for each $n \in \mathbb{N}$*

$$\delta_{\overline{\rho}}(K,n) \leq \delta(K).$$

Hence, for such a $\overline{\rho}$ we have $\rho_c(K) \leq \overline{\rho}$ (cf. (3.2)). In order to estimate $\overline{\rho}$ we may assume by the translation invariance of the densities that the centroid of K is at the origin. Then, it is known that $K - K \subseteq (d+1)K$ and if K is origin symmetric then, of course, $K - K = 2K$. So we have

Corollary 6.2. *Let $K \in \mathcal{K}^d$. Then*

$$\rho_c(K) \leq \begin{cases} 2 & : K = -K \\ d+1 & : otherwise \end{cases}.$$

The general bound of $d+1$ was slightly improved to $\rho_c(K) \leq d$ in [10, Lemma 10.5.2], but also here it might be true that $\rho_c(K)$ is bounded from above by an absolute constant.

The proof of Theorem 6.1 is based on an average argument: We assume that there exists a $C \in \mathcal{P}_n(K)$ with $\delta_{\overline{\rho}}(K, n) > \delta(K)$. For each $\boldsymbol{x} \in [0, \gamma]^n$, for some large γ, a finite packing set $C_{\boldsymbol{x}}$ contained in a large cube λW^d is constructed consisting of suitable translates of C and of points of a densest infinite packing of K. For this superposition the property $K - K \subset \overline{\rho} K$ is used. In order to determine the cardinality of this new packing set $C_{\boldsymbol{x}}$ it is averaged over all $\boldsymbol{x} \in [0, \gamma]^n$. This yields the existence of a packing set $C_{\overline{\boldsymbol{x}}}$ of K such that $\#C_{\overline{\boldsymbol{x}}}\mathrm{vol}\,(K)\,\mathrm{vol}\,(\lambda W^d) > \delta(K)$, which contradicts the definition of $\delta(K)$.

Regarding infinite packings, the corollary above together with (3.5) with (3.6) gives for $K \in \mathcal{K}^d$, $K = -K$, the bound

$$\delta(K) > \frac{1}{d}\, 2^{1-d}.$$

This is up to a factor of $1/d$ of the same order than the best know lower bounds bound on $\delta(B^d)$ (cf., e.g., [36, Theorem 2.2]).

In order to present results regarding the shape of optimal finite packings, we denote for $\rho \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ by $C_{\rho,n,K}$ an optimal n-packing set of K with respect to the parameter ρ. Schnell and Wills [39] proved optimal sphere packing sets are never two-dimensional. More precisely,

Theorem 6.3. *For any $d \geq 3$, $\rho > 0$ and $n \geq 4$ we have $\dim C_{\rho,n,B^d} \neq 2$.*

A direct consequence is that for $d = 3$ and all $\rho \in (\rho_c, 1)$ sausage catastrophes occur for $n_\rho \geq 56$ balls. Exact values of these "magic" numbers n_ρ are not known, but the it grows when ρ approches ρ_c.

For large ρ and arbitrary convex bodies it was shown by Böröczky, Jr. and Schnell [12] that $\mathrm{conv}\,C_{\rho,n,K}$ resembles the shape of K; more precisely:

Theorem 6.4. *Let $K \in \mathcal{K}^d$ and*

    i) *Let $\rho > \rho_c(K)$. Then*

$$\lim_{n \to \infty} \mathrm{r}_K(\mathrm{conv}\,C_{\rho,n,K}) = \infty,$$

    where $\mathrm{r}_K(\mathrm{conv}\,C_{\rho,n,K})$ is the (relative) inradius with respect to K. i.e., $\mathrm{r}_K(\mathrm{conv}\,C_{\rho,n,K}) = \max\{r : \boldsymbol{t} + rK \subset \mathrm{conv}\,C_{\rho,n,K}$ for some $\boldsymbol{t} \in \mathbb{R}^d\}$.

ii) *Let $\rho > d + 1$. There exists a constant $\mu(\rho, d)$ such that for large n (and after possible translation)*

$$\frac{1}{\mu(\rho, d)} K \subset \sqrt[d]{\frac{\delta(K)}{n}} \, C_{n,\rho,K} \subset \mu(\rho, d) K.$$

Moreover, $\lim_{\rho \to \infty} \mu(\rho, d) = 1$, and if $K = -K$ the bound on ρ can be lowered to 2.

7. (Finite) lattice packings

In this section we will restrict the finite and infinite packings to lattice packings. Lattice packings of convex bodies have a long and famous history for which we refer to [24]. Here we just briefly recall a few facts which are relevant for finite parameterized lattice packings.

We will understand by a lattice $\Lambda \subset \mathbb{R}^d$ a regular linear image of the integral lattice $\mathbb{Z}^d$, i.e., there exists a regular matrix $B \in \mathbb{R}^{d \times d}$ such that $\Lambda = B\mathbb{Z}^d$. The determinant of the lattice, denoted by $\det \Lambda$, is the volume of the parallelepiped spanned by the columns of B, i.e., $\det \Lambda = |\det B|$.

In analogy to (2.1) the set of all packing lattice of a convex body is denoted by $\mathcal{P}(K)^*$, i.e.,

$$\mathcal{P}(K)^* = \{\Lambda \in \mathcal{P}(K) : \Lambda \text{ lattice}\}.$$

For $\Lambda \in \mathcal{P}(K)^*$ the density $\delta(K, \Lambda)$ (cf. (2.2)) can easily be calculated as $\delta(K, \Lambda) = \mathrm{vol}(K)/\det \Lambda$. In the case of (infinite) lattice packings we are interested in the determination of

$$\delta^*(K) = \sup\{\delta(K, \Lambda) : \Lambda \in \mathcal{P}(K)^*\}.$$

Confirming a conjecture of Minkowski, Hlawka [28] proved a lower bound on $\delta^*(K)$ of order to 2^{-d}, which was subsequently (slightly) improved by various authors. The current record is still due to W. Schmidt [37] with

$$\delta^*(K) \geq c \, d \, 2^{-d},$$

where c is an absolute constant. For the ball B^d there are even better bounds available and here we refer to the survey of H. Cohn [14] and the references within.

In order to deal with finite lattice packings we restrict now all given definitions in Section 3 to lattices, i.e., we set for $K \in \mathcal{K}^d$, $\rho \in \mathbb{R}_{>0}$

$$\mathcal{P}_n(K)^* = \{C \in \mathcal{P}_n(K) : \text{there exits a } \Lambda \in \mathcal{P}(K)^* \text{ with } C \subset \Lambda\}$$
$$\delta_\rho^*(K, n) = \sup\{\delta_\rho(K, C) : C \in \mathcal{P}_n(K)^*\}$$
$$\delta_\rho^*(K) = \limsup_{n \to \infty} \delta_\rho^*(K, n)$$
$$\rho_c^*(K) = \inf\{\rho \in \mathbb{R}_{>0} : \delta_\rho^*(K) = \delta^*(K)\}$$
$$\rho_s^*(K) = \sup\{\rho \in \mathbb{R}_{>0} : \delta_\rho^*(K) = \delta_\rho^s(K)\}$$

and we call all these quantities as in the general case except that we add the word "lattice".

Then Theorem 3.4 holds also true for all these lattice quantities, and we also have (cf. (3.2))

$$(7.1) \qquad\qquad \delta_\rho^*(K) \geq \delta^*(K).$$

Since $\mathcal{P}_n(K)^* \subset \mathcal{P}_n(K)$ it is $\rho_s^*(K) \geq \rho_s(K)$, and so all bounds presented in Section 5 for $\rho_s(K)$ are valid for $\rho_s^*(K)$ as well.

Regarding the critical lattice parameter $\rho_c^*(K)$ the situation is different; in particular, the superposition argument leading to Theorem 6.1 and to the lower bounds on $\rho_c(K)$ does not work in the lattice case since the superposition may destroy the lattice property.

Here the idea is to use a lattice refinement argument which goes back to Rogers [33]. It implies that for $K = -K$ and $C \subset \mathcal{P}_n(K)^*$ one can always find a lattice $\Lambda \subset \mathcal{P}(K)^*$ containing C such that all points in space are close to some lattice points; more precisely, at most at distance 3 measured with respect to the norm induced by K. Hence, the circumradius of the Dirichlet-Voronoi cells of this refined lattice Λ is at most 3. Together with some improvements for the case $K = B^d$, one gets, roughly speaking, the following theorem:

Theorem 7.1 ([26]). *Let $K \in \mathcal{K}^d$ and $n \in \mathbb{N}$. Then*

$$\delta_\rho^*(K,n) \leq \delta^*(K) \text{ for } \rho \geq \begin{cases} \sqrt{21}/2 & : K = B^d, \\ 3 & : K = -K, \\ (3/2)(d+1) & : otherwise\ . \end{cases}$$

In combination with (7.1) we obtain

Corollary 7.2. *Let $K \in \mathcal{K}^d$. Then*

$$\rho_c^*(K) \leq \begin{cases} \sqrt{21}/2 & : K = B^d, \\ 3 & : K = -K, \\ (3/2)(d+1) & : otherwise\ . \end{cases}$$

For a detailed account to Rogers lattice refinement method as well as improvements for l_p-ball packings we refer to [27] and the references within.

Regarding the structure of optimal finite lattice packings we first point out that Theorem 6.3 has also been proven by Schnell and Wills in the lattice case, i.e.,

Theorem 7.3 ([39]). *For any $d \geq 3$, $\rho > 0$ and $n \geq 4$ we have $\dim C_{\rho,n,B^d}^* \neq 2$,*

Regarding optimal packings sets for $d = 3$ and n small we refer to [41]. For large ρ and n, the asymptotic shape of optimal finite lattice packings of convex bodies is closely related to the problem to understand crystal growth and to the so called Wulff shape. We are not going to enter this subject, instead we refer to [10, Section 10.11]. However, as a strong counterpart to Theorem 6.4 i), we mention a result of Arhelger et al. on origin symmetric convex boides showing that for ρ beyond the critical lattice parameter the asymptotic shape is cluster like.

Theorem 7.4 ([2]). *Let $K \in \mathcal{K}^d$, $K = -K$, $\rho > \rho_c^*(K)$. Then the ratio of circumradius of $\operatorname{conv} C_{\rho,n}^*$ to the inradius of $\operatorname{conv} C_{\rho,n}^*$ is bounded (independent of n)*

In other words, the shape of $\operatorname{conv} C_{\rho,n}^*$ is not far from being a ball. In the same paper, it was also shown that for the 3-dimensional ball

Theorem 7.5 ([2]). $\rho_s^*(B^3) = \rho_c^*(B^3),$

indicating that also a lattice analogue of the Strong Sausage Conjecture 3.5 could be true (as well). Further evidence is contained in the paper [13].

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TECHNISCHE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMTIK, SEKR. MA4-1, STRASSE DES 17 JUNI 136, D-10623 BERLIN
Email address: `henk@math.tu-berlin.de`

UNIVERSITÄT SIEGEN, EMMY-NOETHER-CAMPUS, WALTER-FLEX-STR. 3, D-57068 SIEGEN
Email address: `wills@mathematik.uni-siegen.de`