# SOME INEQUALITIES FOR PLANAR CONVEX FIGURES

Martin Henk and George A. Tsintsifas

Abstract. We prove the inequality $A \leq 2Dr$, between the area, diameter and inradius of a compact convex body in the 2-dimensional Euclidean space. Using this result we derive other relations of the same kind.

## 1. Introduction

Throughout this paper $E^2$ denotes the 2-dimensional Euclidean space with norm $\|\cdot\|$ and the set of plane convex figures — compact convex sets — in $E^2$ is denoted by $\mathcal{K}^2$. The area, diameter, inradius and width of $K \in \mathcal{K}^2$ is denoted by $A(K)$, $D(K)$, $r(K)$, $\Delta(K)$, respectively. For a detailed description of these functionals we refer to [BF]. For a subset $P \subset E^2$ the convex (affine) hull of $P$ is denoted by $\mathrm{conv}(P)$, $\mathrm{aff}(P)$. Further, the interior of $P$ is denoted by $\mathrm{int}(P)$.

It is not hard to see that for $K \in \mathcal{K}^2$ the area $A(K)$ is bounded from above and below by the diameter and inradius. Indeed, using the well known inequalities $D(K)\Delta(K) \leq 2A(K) \leq 2D(K)\Delta(K)$ [K] we get immediately the lower bound $A(K) \geq D(K)r(K)$ which in general can not be improved. Applying Blaschke's inequality $\Delta(K) \leq 3r(K)$ [BL] to the upper bound yields $A(K) \leq 3D(K)r(K)$. The purpose of this paper is to prove

**Theorem 1.1.** *Let $K \in \mathcal{K}^2$. Then*

$$A(K) \leq 2D(K)r(K),$$

*and equality holds iff $\mathrm{int}(K) = \emptyset$. In the case $\mathrm{int}(K) \neq \emptyset$ this bound is in general best possible.*

## 2. Proof of the Theorem

For the proof of Theorem 1.1. we need the following Lemma

**Lemma 2.1.** *Let $P = \mathrm{conv}\{x^1, x^2, x^3, x^4\} \in \mathcal{K}^2$ be a parallelogramme. With the notation of Figure 1 we have for $\|x - y\| \leq (\|x^1 - x^2\|/2)$*

$$A(\mathrm{conv}\{a, x^1, x\}) + A(\mathrm{conv}\{b, x^2, y\}) \geq A(\mathrm{conv}\{x, y, z\}).$$

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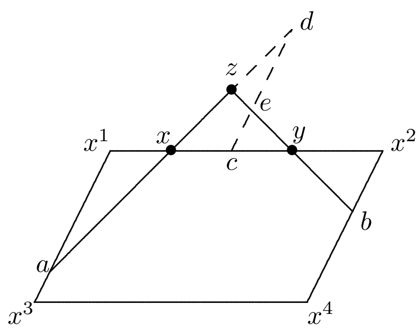


FIGURE 1

*Proof.* Let $a, b$ be arbitrary points in $\text{conv}\{x^1, x^3\}$, $\text{conv}\{x^2, x^4\}$. Without loss of generality let $\|x^1 - x\| \leq \|x^2 - y\|$ and let $c$ be the point in $\text{conv}\{x, x^2\}$ with $\|x - x^1\| = \|x - c\|$. The ray $c + \lambda(x^1 - x^3)$, $\lambda \geq 0$, intersects the ray $a + \mu(z - a)$, $\mu \geq 0$, in a point $d$ and it follows

$$A(\text{conv}\{x, c, d\}) = A(\text{conv}\{a, x^1, x\}). \tag{2.1}$$

In the case $\|c - x\| \geq \|x - y\|$ we have $\text{conv}\{x, y, z\} \subset \text{conv}\{x, c, d\}$ and we are ready. In the other case the ray $c + \lambda(x^1 - x^3)$, $\lambda \geq 0$, also intersects the ray $b + \mu(z - b)$, $\mu \geq 0$, in a point $e$. We get

$$\text{conv}\{x, y, z\} \subset \text{conv}\{x, c, d\} \cup \text{conv}\{c, y, e\}. \tag{2.2}$$

By assumption we have $\|c - y\| \leq \|x^2 - y\|$ and thus

$$A(\text{conv}\{c, y, e\}) \leq A(\text{conv}\{b, x^2, y\}).$$

On account of (2.1) and (2.2) we obtain the desired inequality. □

*Proof of Theorem 1.1.* If $\text{int}(K) = \emptyset$ then we have $A(K) = r(K) = 0$ and thus equality. So we may assume $\text{int}(K) \neq \emptyset$. Let $H = \text{conv}\{x^i, 1 \leq i \leq 6\}$ be an affine regular hexagon inscribed in $K$ with midpoint 0 (see Figure 2).

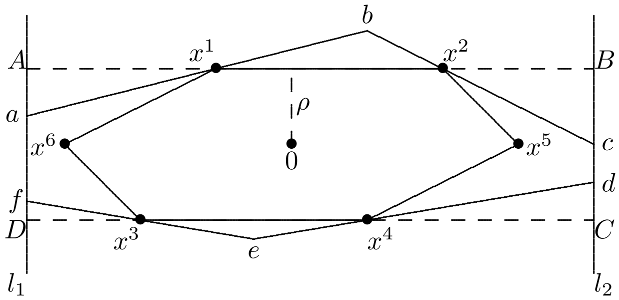


FIGURE 2

Let $D(H) = \|x^5 - x^6\|$. Then it is well known ([JB, p. 24,25, pp. 124], [E]) that $x^i$ belongs to the boundary of $K$, $1 \le i \le 6$, $\|x^1 - x^2\| = \|x^3 - x^4\| = \|x^5 - x^6\|/2$ and the edges $\mathrm{conv}\{x^1, x^2\}$, $\mathrm{conv}\{x^3, x^4\}$ have maximal length among the edges of $H$. Thus the ball with center 0 and radius $\rho$ (distance of $\mathrm{conv}\{x^1, x^2\}$ to 0) is contained in $H$. Hence

$$\rho = r(H) \le r(K). \tag{2.3}$$

Let $l_1, l_2$ be two parallel supporting lines of $K$ with normal vector $x^5$ and let $A, B, C, D$ denotes the intersection points of $\mathrm{aff}\{x^1, x^2\}$, $\mathrm{aff}\{x^3, x^4\}$ with these lines. Then we have

$$\|A - B\| = \|D - C\| \le D(K). \tag{2.4}$$

Now, let $u_i$ be supporting lines on $K$ through the points $x^i$, $1 \le i \le 4$. The intersection points with the lines $l_1, l_2$ are denoted by $a, c, d, f$, respectively. Since $x^i$, $1 \le i \le 4$, belong to the boundary of $K$ we have $l_1 \cap K \cap \mathrm{conv}\{A, D\} \ne \emptyset$ and $l_2 \cap K \cap \mathrm{conv}\{B, C\} \ne \emptyset$. Thus $a, f \in \mathrm{conv}\{A, D\}$ and $c, d \in \mathrm{conv}\{B, C\}$. The intersection point of the lines $u^1, u^2$ ($u^3, u^4$) is denoted by $b$ ($e$). Obviously,

$$\begin{aligned} K \subset \mathrm{conv}\{a, b, c, d, e, f\} &= \mathrm{conv}\{a, x^1, x^2, c, d, x^4, x^3, f\} \\ &\qquad \cup \mathrm{conv}\{x^1, x^2, b\} \cup \mathrm{conv}\{x^4, e, x^3\}. \end{aligned} \tag{2.5}$$

By Lemma 1.1. we get $A(\mathrm{conv}\{x^1, x^2, b\}) \le A(\mathrm{conv}\{a, A, x^1\}) + A(\mathrm{conv}\{x^2, B, c\})$ and $A(\mathrm{conv}\{x^4, e, x^3, \}) \le A(\mathrm{conv}\{d, C, x^4\}) + A(\mathrm{conv}\{x^3, D, f\})$ and thus by (2.5), (2.4) and (2.3)

$$A(K) \le A(\mathrm{conv}\{A, B, C, d\}) = \|A - B\| \cdot 2\rho \le 2D(K)R(K).$$

To show that this inequality is strict suppose $r(K) = \rho$. Then two parallel edges of $H$ belongs to the boundary of $K$ and every of these edges has a common point with the insphere of radius $r(K)$. Thus $K$ is contained in the parallel strip accociated to these edges and hence $A(K) < 2D(K)r(K)$. This shows $A(K) = 2D(K)r(K)$ iff $\mathrm{int}(K) = \emptyset$.

Further the quader $Q(q) = \{(x_1, x_2)^T \in E^2 \mid |x_1| \le q,\ |x_2| \le 1\}$, $q \in \mathbb{R}$, shows for $q \to \infty$ that in general this inequality can not be improved in the case $\mathrm{int}(K) \ne \emptyset$. □

## 3. Further inequalities

In this section we collect some inequalities for plane convex figures which are closely related to Theorem 1.1. To this end $R(K)$, $L(K)$ denotes the circumradius, perimeter of a convex body $K \in \mathcal{K}^2$, respectively.

(1) $(\sqrt{3}/2)\Delta(K)R(K) \le A(K) \le 2\Delta(K)R(K)$.
For the lower bound see [He, p. 29] and the upper bound can be easily deduced from $A(K) \le \Delta(K)D(K)$ [K].

(2) $L(K)r(K) \le 2A(K) \le 2r(K)(L(K) - \pi r(K))$.

The upper bound is due to Bonnesen [Bo] and the obvious lower bound can be found in [BF, p.82].

(3) $4R(K) \leq L(K) \leq 2D(K) + 4r(K)$.

For the lower bound see [N], [CK]. The upper bound follows by Theorem 1.1. and the well known Farvard's inequality $L(K)D(K) \leq 2A(K) + 2D(K)^2$ (cf. [F], [RR]).

(4) $2R(K)r(K) \leq A(K) \leq 4R(K)r(K)$.

The lower bound follows from the obvious inequality $2A(K) \geq L(K)r(K)$ [BF] and the lower bound in (3). The upper bound is an immediate consequence of Theorem 1.1. and $D(K) \leq 2R(K)$.

(5) $R(K)(L(K) - 4R(K)) \leq A(K) \leq 2R(K)(L(K) - 2R(K))$.

The lower bound is due to Favard [F]. For the upper bound let 0 center of an insphere with radius $r(K)$. Then it is easy to see that $K$ is contained in the ball with radius $D(K) - r(K)$ and center 0. Thus $D(K) \geq R(K) + r(K)$ and by $2D(K) \leq L(K)$ we get $L(K) - 2R(K) \geq 2r(K)$. Together with the upper bound in (4) we obtain the desired inequality.

(6) $D(K)(L(K) - 2D(K)) \leq 2A(K) \leq L(K)D(K)/2$.

The lower bound is also due to Favard [F] and the upper bound is due to Hayashi [H].

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## References


[Bl] W. Blaschke, *Über den größten Kreis in einer konvexen Punktmenge*, Jber. Deutsch. Math. Vereinig. **23** (1914), 369–374.

[Bo] T. Bonnesen, *Les problmes des isoprimtres et des ispiphanes*, Paris, 1929.

[BF] T. Bonnesen, W. Fenchel, *Theorie der konvexen Krper*, Springer, Berlin, 1934.

[CK] G.D. Chakerian, M.S. Klamkin, *Minimal covers for closed curves*, Math. Magazine **78(2)** (1973), 55–61.

[E] H.G. Eggleston, *Convexity*, Cambridge Univ. Press, Cambridge, 1958.

[F] J. Favard, *Problmes d'extremums relatifs aux courbes convexes I. II.*, Ann. Ecole norm. **46** (1929), 345–369.

[H] T. Hayashi, *The extremal chords of an oval*, Tôhoku Math. J. **22** (1923), 387–393.

[He] M. Henk, *Ungleichungen für sukzessive Minima und verallgemeinerte In- und Umkugelradien*, PhD thesis, University of Siegen (1991).

[JB] I.M. Jaglom, W.G. Boltjanski, *Konvexe Figuren*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

[K] T. Kubota, *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Sci. Rep. Tôhoku Univ. **12** (1923), 45–65.

[N] J.C.C. Nitsche, *The smallest sphere containing $\alpha$ rectifiable curve*, Amer. Math. Monthly **87** (1971), 881–882.

[RR] M. Rogialski, J.S. Raymond, *Inequalities about symmetric compact convex sets in the plane*, Am. Math. Monthly **92(7)** (1985), 466–480.


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Martin Henk, Mathematisches Institut, Universitt Siegen, Hlderlinstrasse 3, D-W-5900 Siegen, Federal Republic of Germany

*E-mail address*: henk@hrz.uni-siegen.dbp.de

George A. Tsintsifas, Platonos str. 23, 54631 Thessaloniki, Greece.