LOWER BOUNDS ON THE COEFFICIENTS OF EHRHART POLYNOMIALS

MARTIN HENK AND MAKOTO TAGAMI

Abstract. We present lower bounds for the coefficients of Ehrhart polynomials of convex lattice polytopes in terms of their volume. We also introduce two formulas for calculating the Ehrhart series of a kind of a "weak" free sum of two lattice polytopes and of integral dilates of a polytope. As an application of these formulas we show that Hibi's lower bound on the coefficients of the Ehrhart series is not true for lattice polytopes without interior lattice points.

1. Introduction

Let $\mathcal{P}^d$ be the set of all convex d-dimensional lattice polytopes in the d-dimensional Euclidean space $\mathbb{R}^d$ with respect to the standard lattice $\mathbb{Z}^d$, i.e., all vertices of $P \in \mathcal{P}^d$ have integral coordinates and $\dim(P) = d$. The lattice point enumerator of a set $S \subset \mathbb{R}^d$, denoted by $\mathrm{G}(S)$, counts the number of lattice (integral) points in S, i.e., $\mathrm{G}(S) = \#(S \cap \mathbb{Z}^d)$. In 1962, Eugéne Ehrhart (see e.g. [3, Chapter 3], [7]) showed that for $k \in \mathbb{N}$ the lattice point enumerator $\mathrm{G}(k\,P)$, $P \in \mathcal{P}^d$, is a polynomial of degree d in k where the coefficients $\mathrm{g}_i(P)$, $0 \leq i \leq d$, depend only on P:

$$\mathrm{G}(k\,P) = \sum_{i=0}^{d} \mathrm{g}_i(P)\,k^i. \tag{1.1}$$

The polynomial on the right hand side is called the Ehrhart polynomial, and regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$ it is denoted by $\mathrm{G}_P(z)$. Two of the $d+1$ coefficients $\mathrm{g}_i(P)$ are almost obvious, namely, $\mathrm{g}_0(P) = 1$, the Euler characteristic of P, and $\mathrm{g}_d(P) = \mathrm{vol}(P)$, where vol() denotes the volume, i.e., the d-dimensional Lebesgue measure on $\mathbb{R}^d$. It was shown by Ehrhart (see e.g. [3, Theorem 5.6], [8]) that also the second leading coefficient admits a simple geometric interpretation as lattice surface area of P

$$\mathrm{g}_{d-1}(P) = \frac{1}{2} \sum_{F \text{ facet of P}} \frac{\mathrm{vol}_{d-1}(F)}{\det(\mathrm{aff}\,F \cap \mathbb{Z}^d)}. \tag{1.2}$$

Here $\mathrm{vol}_{d-1}(\cdot)$ denotes the $(d-1)$-dimensional volume and $\det(\mathrm{aff}\,F \cap \mathbb{Z}^d)$ denotes the determinant of the $(d-1)$-dimensional sublattice contained in the affine hull of F. All other coefficients $\mathrm{g}_i(P)$, $1 \leq i \leq d-2$, have no such known

2000 *Mathematics Subject Classification.* 52C07, 11H06.

Key words and phrases. Lattice polytopes, Ehrhart polynomial.

The second author was supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

geometric meaning, except for special classes of polytopes. For this and as a general reference on the theory of lattice polytopes we refer to the recent book of Matthias Beck and Sinai Robins [3] and the references within. For more information regarding lattices and the role of the lattice point enumerator in Convexity see [9].

In [4, Theorem 6] Ulrich Betke and Peter McMullen proved the following upper bounds on the coefficients $\mathrm{g}_i(P)$ in terms of the volume:

$$\mathrm{g}_i(P) \leq (-1)^{d-i}\mathrm{stirl}(d,i)\mathrm{vol}(P) + (-1)^{d-i-1}\frac{\mathrm{stirl}(d,i+1)}{(d-1)!}, \quad i = 1,\dots,d-1,$$

where $\mathrm{stirl}(d,i)$ denote the Stirling numbers of the first kind. In order to present our lower bounds on $\mathrm{g}_i(P)$ in terms of the volume we need some notation. For an integer i and a variable z we consider the polynomial

$$(z+i)(z+i-1)\cdot\ldots\cdot(z+i-(d-1)) = d!\binom{z+i}{i},$$

and we denote its r-th coefficient by $C^d_{r,i}$, $0 \leq r \leq d$. For instance, it is $C^d_{d,i} = 1$, and for $0 \leq i \leq d-1$ we have $C^d_{0,i} = 0$. For $d \geq 3$ we are interested in

$$M_{r,d} = \min\{C^d_{r,i} : 1 \leq i \leq d-2\}. \tag{1.3}$$

Obviously, we have $M_{d,d} = 1$ and it is also easy to see that

$$M_{d-1,d} = C^d_{d-1,1} = -\frac{d(d-3)}{2}. \tag{1.4}$$

With the help of these numbers $M_{r,d}$ we obtain the following lower bounds

Theorem 1.1. *Let $P \in \mathcal{P}^d$, $d \geq 3$. Then for $i = 1,\dots,d-1$ we have*

$$\mathrm{g}_i(P) \geq \frac{1}{d!}\left\{(-1)^{d-i}\mathrm{stirl}(d+1,i+1) + (d!\mathrm{vol}(P)-1)M_{i,d}\right\}.$$

In the case $i = d-1$, for instance, we get together with (1.4) the bound

$$\mathrm{g}_{d-1}(P) \geq \frac{1}{(d-1)!}\left\{d-1-\frac{d-3}{2}d!\mathrm{vol}(P)\right\}.$$

Since the lattice surface area of any facet is at least $1/(d-1)!$ we have the trivial inequality (cf. (1.2))

$$\mathrm{g}_{d-1}(P) \geq \frac{1}{2}\frac{d+1}{(d-1)!}. \tag{1.5}$$

Hence the lower bound on $\mathrm{g}_{d-1}(P)$ is only best possible if $\mathrm{vol}(P) = 1/d!$. In the cases $i \in \{1,2,d-2\}$, however, Theorem 1.1 gives best possible bounds for any volume

Corollary 1.2. *Let $P \in \mathcal{P}^d$. Then*

i) $\mathrm{g}_1(P) \geq 1 + \frac{1}{2} + \cdots + \frac{1}{d-2} + \frac{2}{d-1} - (d-2)!\mathrm{vol}(P)$,

ii) $\mathrm{g}_2(P) \geq \frac{(-1)^d}{d!} \left\{\mathrm{stirl}(d+1,3) + \left((d-2)! + \mathrm{stirl}(d-1,2)\right)(d!\mathrm{vol}(P) - 1)\right\}$,

iii) $\mathrm{g}_{d-2}(P) \geq \begin{cases} \frac{1}{d!}\frac{(d-1)d(d+1)}{24}\left\{3(d+1) - d!\mathrm{vol}(P)\right\} : & d \text{ odd}, \\ \frac{1}{d!}\frac{(d-1)d}{24}\left\{3d(d+2) - (d-2)\, d!\mathrm{vol}(P)\right\} : & d \text{ even}. \end{cases}$

And the bounds are best possible for any volume.

For some recent inequalities involving more coefficients of Ehrhart polynomials we refer to [2]. Next we come to another family of coefficients of a polynomial associated to lattice polytopes.

The generating function of the lattice point enumerator, i.e., the formal power series

$$\mathrm{Ehr}_P(z) = \sum_{k \geq 0} \mathrm{G}_P(k)\, z^k,$$

is called the Ehrhart series of P. It is well known that it can be expressed as a rational function of the form

$$\mathrm{Ehr}_P(z) = \frac{\mathrm{a}_0(P) + \mathrm{a}_1(P)\, z + \cdots + \mathrm{a}_d(P)\, z^d}{(1-z)^{d+1}}.$$

The polynomial in the numerator is called the $h^\star$-polynomial. Its degree is also called the degree of the polytope [1] and it is denoted by $\deg(P)$. Concerning the coefficients $\mathrm{a}_i(P)$ it is known that they are integral and that

$$\mathrm{a}_0(P) = 1, \quad \mathrm{a}_1(P) = \mathrm{G}(P) - (d+1), \quad \mathrm{a}_d(P) = \mathrm{G}(\mathrm{int}(P)),$$

where $\mathrm{int}(\cdot)$ denotes the interior. Moreover, due to Stanley's famous non-negativity theorem (see e.g. [3, Theorem 3.12], [16]) we also know that $\mathrm{a}_i(P)$ is non-negative, i.e., for these coefficients we have the lower bounds $\mathrm{a}_i(P) \geq 0$. In the case $\mathrm{G}(\mathrm{int}(P)) > 0$, i.e., $\deg(P) = d$, these bounds were improved by Takayuki Hibi [12] to

$$\mathrm{a}_i(P) \geq \mathrm{a}_1(P),\ 1 \leq i \leq \deg(P) - 1. \tag{1.6}$$

In this context it was a quite natural question whether the assumption $\deg(P) = d$ can be weaken (see e.g. [14]), i.e., whether these lower bounds (1.6) are also valid for polytopes of degree less than d. As we show in Example 1.1 the answer is already negative for polytopes having degree 3. The problem in order to study such a question is that only very few geometric constructions of polytopes are known for which we can explicitly calculate the Ehrhart series. In [3, Theorem 2.4, Theorem 2.6] the Ehrhart series of special pyramids and double pyramids over a basis Q are determined in terms of the Ehrhart series of Q. In a recent paper Braun [6] gave a very nice product formula for the Ehrhart series of the free sum of two lattice polytopes, where one of the polytopes has to be reflexive. Here we consider the following construction, which might be regarded as a "very weak" or "fake" free sum.

Lemma 1.3. *For $P \in \mathcal{P}^p$ and $Q \in \mathcal{P}^q$ let*

$$P \otimes Q = \operatorname{conv}\{(x, 0_q, 0)^\intercal, (0_p, y, 1)^\intercal : x \in P,\, y \in Q\} \in \mathcal{P}^{p+q+1},$$

where 0_p and 0_q denote the p- and q-dimensional 0-vector, respectively. Then

$$\operatorname{Ehr}_{P\otimes Q}(z) = \operatorname{Ehr}_P(z) \cdot \operatorname{Ehr}_Q(z).$$

In order to apply this Lemma we consider two families of lattice simplices . For an integer $m \in \mathbb{N}$ let

$$\begin{aligned} T_d^{(m)} &= \operatorname{conv}\{o, e_1, e_1 + e_2, e_2 + e_3, \ldots, e_{d-2} + e_{d-1}, e_{d-1} + m\, e_d\}, \\ S_d^{(m)} &= \operatorname{conv}\{o, e_1, e_2, e_3, \ldots, e_{d-1}, m\, e_d\}, \end{aligned}$$

where e_i denotes the i-th unit vector. It was shown in [4] that

$$\operatorname{Ehr}_{T_d^{(m)}}(z) = \frac{1 + (m-1)\, z^{\lceil \frac{d}{2} \rceil}}{(1-z)^{d+1}} \text{ and } \operatorname{Ehr}_{S_d^{(m)}}(z) = \frac{1 + (m-1)\, z}{(1-z)^{d+1}}. \tag{1.7}$$

Example 1.1. *For $q \in \mathbb{N}$ odd and $l, m \in \mathbb{N}$ we have*

$$\operatorname{Ehr}_{T_q^{(l+1)} \otimes S_p^{(m+1)}}(z) = \frac{1 + m\, z + l\, z^{\frac{q+1}{2}} + m\, l\, z^{\frac{q+3}{2}}}{(1-z)^{p+q+2}}.$$

In particular, for $q \geq 3$ and $l < m$ this shows that (1.6) is, in general, false for lattice polytopes without interior lattice points.

Another formula for calculating the Ehrhart Series from a given one concerns dilates. Here we have

Lemma 1.4. *Let $P \in \mathcal{P}^d$, $k \in \mathbb{N}$ and let ζ be a primitive k-th root of unity. Then*

$$\operatorname{Ehr}_{k\, P}(z) = \frac{1}{k} \sum_{i=0}^{k-1} \operatorname{Ehr}_P(\zeta^i\, z^{\frac{1}{k}}).$$

The lemma can be used, for instance, to calculate the Ehrhart series of the cube $C_d = \{x \in \mathbb{R}^d : |x_i| \leq 1,\, 1 \leq i \leq d\}$.

Example 1.2. *For two integers j, d, $0 \leq j \leq d$, let $A(d, j)$ be the Eulerian numbers (see e.g. [3, pp. 28]) and for convenience we set $A(d, j) = 0$ if $j \notin \{0, \ldots, d\}$. Then*

$$\mathrm{a}_i(C_d) = \sum_{j=0}^{d+1} \binom{d+1}{j} A(d, 2\,i + 1 - j),\; 0 \leq i \leq d.$$

Of course, the cube C_d may be also regarded as a prism over a $(d-1)$ cube, and as a counterpart to the bipyramid construction in [3] we calculate here also the Ehrhart series of some special prism.

Example 1.3. *Let $Q \in \mathcal{P}^{d-1}$, $m \in \mathbb{N}$, and let $P = \{(x, m)^\intercal : x \in Q\}$ be the prism of height m over Q. Then*

$$\mathrm{a}_i(P) = (m\,i + 1)\mathrm{a}_i(Q) + (m(d - i + 1) - 1)\,\mathrm{a}_{i-1}(Q),\ 0 \leq i \leq d,$$

where we set $\mathrm{a}_d(Q) = \mathrm{a}_{-1}(Q) = 0$.

It seems to be quite likely that for the class of 0-symmetric lattice polytopes $\mathcal{P}_o^d$ the lower bounds on $\mathrm{a}_i(P)$ can considerably be improved. In [5] it was conjectured that for $P \in \mathcal{P}_o^d$

$$\mathrm{a}_i(P) + \mathrm{a}_{n-i}(P) \geq \binom{d}{i}(\mathrm{a}_n(P) + 1)\,,$$

where equality holds for instance for the cross-polytopes $C_d^\star(2\,l-1) = \mathrm{conv}\{\pm l\,e_1, \pm e_i : 2 \leq i \leq d\}$, $l \in \mathbb{N}$, with $2l - 1$ interior lattice points. It is also conjectured that these cross-polytopes have minimal volume among all 0-symmetric lattice polytopes with a given number of interior lattice points. The maximal volume of those polytopes are known by the work of Blichfeldt and van der Corput (cf. [9, p. 51]) and, for instance, the maximum is attained by the boxes $Q_d(2\,l-1) = \{|x_1| \leq l,\ |x_i| \leq 1,\ 2 \leq i \leq d\}$ with $2\,l-1$ interior points. By the Examples 1.2 and 1.3 we can easily calculate the Ehrhart series of these boxes

Example 1.4. *Let $l \in \mathbb{N}$. Then, for $0 \leq i \leq d$,*

$$\mathrm{a}_i(Q_d(2\,l-1)) = (2\,l\,i + 1)\,a_i(C_{d-1}) + (2\,l(d - i + 1) - 1)\ a_{i-1}(C_{d-1}).$$

It is quite tempting to conjecture that these numbers form the corresponding upper bounds on $\mathrm{a}_i(P) + \mathrm{a}_{n-i}(P)$ for 0-symmetric polytope with $2\,l-1$ interior lattice points. In the 2-dimensional case this follows easily from a result of Paul Scott [15] which implies that $\mathrm{a}_1(P) \leq 6\,l = \mathrm{a}_1(Q_2(2\,l-1))$ for any 0-symmetric convex lattice polygon with $2\,l-1$ interior lattice points.

Concerning lower bounds on $\mathrm{g}_i(P)$ for 0-symmetric polytopes P we only know, except the trivial case $i = d$, a lower bound on $\mathrm{g}_{d-1}(P)$ (cf. (1.5)). Namely

$$\mathrm{g}_{d-1}(P) \geq \mathrm{g}_{d-1}(C_d^\star) = \frac{2^{d-1}}{(d-1)!},$$

where $C_d^\star = \mathrm{conv}\{\pm e_i : 1 \leq i \leq d\}$ denotes the regular cross-polytope. This follows immediately from a result of Richard P. Stanley [17, Theorem 3.1] on the h-vector of "symmetric" Cohen-Macaulay simplicial complex.

Motivated by a problem in [11] we study in the last section also the related question to bound the surface area $\mathrm{F}(P)$ of a lattice polytope P. To this end let $T_d = \mathrm{conv}\{0, e_1, \ldots, e_d\}$ be the standard simplex.

Proposition 1.5. *Let $P \in \mathcal{P}^d$, $\dim P = d$. Then*

$$\mathrm{F}(P) \geq \begin{cases} \mathrm{F}(C_d^\star) = \frac{2^d}{d!}\,d^{\frac{3}{2}} : P = -P, \\ \mathrm{F}(T_d) = \frac{d+\sqrt{d}}{(d-1)!} : \ \textit{otherwise}\,. \end{cases}$$

The paper is organized as follows. In the next section we give the proof of our main Theorem 1.1. Then, in Section 3, we prove the Lemmas 1.3 and 1.4 and show how the Ehrhart series in the Examples 1.1 and 1.2 can be deduced. Moreover, we show that some recent bounds of Jaron Treutlein [18] on the coefficients of $h^\star$-polynomials of degree 2 give indeed a complete classification of all these $h^\star$-polynomials (cf. Proposition 3.2). Finally, in the last section we provide a proof of Proposition 1.5 which in the symmetric cases is based on a isoperimetric inequality for cross-polytopes (cf. Lemma 4.1).

2. Lower bounds on $\mathrm{g}_i(P)$

In the following we denote for an integer r and a polynomial $f(x)$ the r-th coefficient of $f(x)$, i.e. the coefficient of x^r, by $f(x)|_r$. Before proving Theorem 1.1 we need some basic properties of the numbers $C^d_{r,i}$ and $M_{r,d}$ defined in the introduction (see (1.3)).

Lemma 2.1.

i) $C^d_{r,i} = (-1)^{d-r} C^d_{r,d-1-i}$ *for* $0 \leq i \leq d-1$.

ii) *Let* $d \geq 3$. *Then* $M_{r,d} \leq 0$ *for* $r < d$.

Proof. For i) we just note that $C^d_{r,l}$ is the $(d-r)$-th elementary symmetric function of $\{l, l-1, \ldots, l-(d-1)\}$. On account of i) it suffices to prove ii) when $d-r$ is even and we do that by induction on d.

For $d = 3$ and $r = 1$ we have $M_{1,3} = C^3_{1,1} = -1$. So let $d > 3$, and since $C^d_{0,i} = 0$ we may also assume $r \geq 1$. It is easy to see that

$$C^d_{r,i} = (i-d+1)\, C^{d-1}_{r,i} + C^{d-1}_{r-1,i}, \tag{2.1}$$

and by induction we may assume that there exists a $j \in \{1, \ldots, d-3\}$ with $C^{d-1}_{r-1,j} \leq 0$. Observe that $d-1-(r-1)$ is even. If $C^{d-1}_{r,j} \geq 0$ we obtain by (2.1) that $C^d_{r,j} \leq 0$ and we are done. So let $C^{d-1}_{r,j} < 0$. By part i) we know that

$$C^{d-1}_{r,j} = (-1)^{d-1-r} C^{d-1}_{r,d-2-j} \text{ and } C^{d-1}_{r-1,j} = (-1)^{d-r}\, C^{d-1}_{r-1,d-2-j}.$$

Since $d-r$ is even we conclude $C^{d-1}_{r,d-2-j} > 0$ and $C^{d-1}_{r-1,d-2-j} \leq 0$. Hence, on account of (2.1) we get $C^d_{r,d-2-j} \leq 0$ and so $M_{r,d} \leq 0$. □

Proof of Theorem 1.1. We follow the approach of Betke and McMullen used in [4, Theorem 6]. By expanding the Ehrhart series at $z = 0$ one gets (see e.g. [3, Lemma 3.14])

$$\mathrm{G}_P(z) = \sum_{i=0}^{d} \mathrm{a}_i(P) \binom{z+d-i}{d}. \tag{2.2}$$

In particular, we have

$$\frac{1}{d!} \sum_{i=0}^{d} \mathrm{a}_i(P) = \mathrm{g}_d(P) = \mathrm{vol}(P). \tag{2.3}$$

For short, we will write a_i instead of $\mathrm{a}_i(P)$ and g_i instead of $\mathrm{g}_i(P)$. With these notation we have

$$
\begin{aligned}
d!\mathrm{g}_r &= d!\mathrm{G}_P(z)|_r = d!\sum_{i=0}^{d} \mathrm{a}_i \binom{z+d-i}{d}\bigg|_r \\
&= C^d_{r,d} + (\mathrm{a}_1\, C^d_{r,d-1} + \mathrm{a}_d\, C^d_{r,0}) + \sum_{i=2}^{d-1} \mathrm{a}_i\, C^d_{r,d-i}.
\end{aligned} \tag{2.4}
$$

Since $C^d_{r,d-1} \geq 0$ we get with Lemma 2.1 i) that $C^d_{r,d-1} = |C^d_{r,0}|$. Together with $\mathrm{a}_1 \geq \mathrm{a}_d$ and $C^d_{r,d} = (-1)^{d-r}\mathrm{stirl}(d+1, r+1)$ we find

$$
\begin{aligned}
d!\mathrm{g}_r &\geq (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} \mathrm{a}_i\, C^d_{r,d-i} \\
&= (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} \mathrm{a}_i\, \left(C^d_{r,d-i} - M_{r,d}\right) + \sum_{i=1}^{d} \mathrm{a}_i\, M_{r,d} \\
&\qquad - (\mathrm{a}_1 + \mathrm{a}_d) M_{r,d} \\
&\geq (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + (d!\mathrm{vol}(P) - 1) M_{r,d},
\end{aligned} \tag{2.5}
$$

where the last inequality follows from the definition of $M_{r,d}$ and the negativity of $M_{r,d}$ (cf. Lemma 2.1 ii)). □

For $1 \leq r \leq d-1$ one can easily show that the numbers $C^d_{r,d-1}, M_{r,d}$ are negative and so the proof above (cf. (2.4) and (2.5)) gives

$$
\begin{aligned}
d!\mathrm{g}_r &\geq (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + 2\,\mathrm{a}_1(P) + (d!\mathrm{vol}(P) - 1) M_{r,d} \\
&= (-1)^{d-r}\mathrm{stirl}(d+1, r+1) - 2(d+1) + 2\mathrm{G}(P) + (d!\mathrm{vol}(P) - 1) M_{r,d}.
\end{aligned}
$$

In order to verify the inequalities in Corollary 1.2 we have to calculate the numbers $M_{r,d}$ for $r = 1, 2, d-2$.

Proposition 2.2. *Let $d \geq 3$. Then*

$$
\begin{aligned}
&\text{i) } M_{1,d} = C^d_{1,d-2} = -(d-2)!, \\
&\text{ii) } M_{2,d} = C^d_{2,d-2} = (d-2)! + (-1)^d \mathrm{stirl}(d-1, 2), \\
&\text{iii) } M_{d-2,d} = \begin{cases} C^d_{d,\frac{d-1}{2}} = -\frac{1}{4}\binom{d+1}{3}, & d \text{ odd}, \\ C^d_{d,\frac{d}{2}} = -\frac{1}{4}\binom{d}{3}, & d \text{ even}. \end{cases}
\end{aligned}
$$

Proof. $C^d_{1,i}$ is the $d-1$-st elementary symmetric function of $\{i, \dots, 0, \dots, i-(d-1)\}$. Thus $C^d_{1,i} = (-1)^{d-i-1}\, i!\,(d-i-1)!$ and

$$
M_{1,d} = \min\{C^d_{1,i} : 1 \leq i \leq d-2\} = C^d_{1,d-2} = -(d-2)!
$$

In the case $r = 2$ we obtain by elementary calculations that

$$
C^d_{2,i} = i!\mathrm{stirl}(d-i, 2) + (-1)^d\,(d-i-1)!\mathrm{stirl}(i+1, 2),
$$

from which we conclude $M_{2,d} = C^d_{2,d-2} = (d-2)! + (-1)^d\mathrm{stirl}(d-1, 2)$.

For the value of $M_{d-2,d}$ we first observe that

$$\begin{aligned}C^d_{d-2,i}-C^d_{d-2,i-1} &= (z+i)\,(z+i-1)\cdot\ldots\cdot(z+i-(d-1))\big|_{d-2}\\ &\qquad -(z+i-1)\cdot\ldots(z+i-(d-1))\,(z+i-d)\big|_{d-2}\\ &= \sum_{j=-d+i+1}^{i-1} j\,(i-(-d+i)) = d\sum_{j=-d+i+1}^{i-1} j\\ &= d\frac{(d-1)(-d+2\,i)}{2}.\end{aligned}$$

Thus the function $C^d_{d-2,i}$ is decreasing in $0\leq i\leq \lfloor d/2\rfloor$ and increasing in $\lfloor d/2\rfloor\leq i\leq d$. So it takes its minimum at $i=\lfloor d/2\rfloor$. First let us assume that d is odd. Then

$$\begin{aligned}M_{d-2,d} = C^d_{d-2,\frac{d-1}{2}} &= d!\binom{z+(d-1)/2}{d}\bigg|_{d-2}\\ &= z\,(z^2-1)\,(z^2-4)\cdot\ldots\cdot(z^2-((d-1)/2)^2)\Big|_{d-2} = -\sum_{i=0}^{(d-1)/2} i^2\\ &= -\frac{1}{4}\binom{d+1}{3}.\end{aligned}$$

The even case can be treated similarly. □

Now we are able to prove Corollary 1.2

Proof of Corollary 1.2. The inequalities just follow by inserting the value of $M_{r,d}$ given in Proposition 2.2 in the general inequality of Theorem 1.1. Here we also have used the identities

$$\operatorname{stirl}(d+1,2)=(-1)^{d+1}\,d!\sum_{i=1}^{d}\frac{1}{i}\ \text{and}\ \operatorname{stirl}(d+1,d-1)=\frac{3\,d+2}{4}\binom{d+1}{3}.$$

It remains to show that the inequalities are best possible for any volume. For $r=d-2$ we consider the simplex $T_d^{(m)}$ (cf. (1.7)) with $\mathrm{a}_0(T_d^{(m)})=1$, $\mathrm{a}_{\lceil d/2\rceil}(T_d^{(m)})=(m-1)$ and $\mathrm{a}_i(T_d^{(m)})=0$ for $i\notin\{0,\lceil d/2\rceil\}$. Then $\mathrm{vol}(T_d^{(m)})=m/d!$ and on account of Proposition 2.2 we have equality in (2.4) and (2.5).

For $r=1,2$ and $d\geq 4$ we consider the $(d-4)$-fold pyramid $\tilde{T}_d^{(m)}$ over $T_4^{(m)}$ given by $\tilde{T}_d^{(m)}=\mathrm{conv}\{T_4^{(m)},e_5,\ldots,e_d\}$. Then $\mathrm{vol}(\tilde{T}_d^{(m)})=m/d!$ and in view of (1.7) and [3, Theorem 2.4] we obtain

$$\mathrm{a}_0(\tilde{T}_d^{(m)})=1,\ \mathrm{a}_2(\tilde{T}_d^{(m)})=m-1 \text{ and } \mathrm{a}_i(\tilde{T}_d^{(m)})=0,\ i\notin\{0,2\}.$$

Again, by Proposition 2.2 we have equality in (2.4) and (2.5). In the 3-dimensional case it remains to show that the bound on $\mathrm{g}_1(P)$ is best possible. For the called Reeve simplex $T_3^{(m)}$, however, it is easy to check that $\mathrm{g}_1(R_3^{(m)})=2-\frac{m}{6}$ whereas $\mathrm{vol}(R_3^{(m)})=m/6$. □

3. Ehrhart series of some special polytopes

We start with the short proof of Lemma 1.3.

Proof of Lemma 1.3. Since

$$\mathrm{Ehr}_P(z)\,\mathrm{Ehr}_Q(z) = \sum_{k\geq 0}\left(\sum_{m+l=k} \mathrm{G}_P(m)\mathrm{G}_Q(l)\right) z^k,$$

it suffices to prove that the Ehrhart polynomial $\mathrm{G}_{P\otimes Q}(k)$ of the lattice polytope $P\otimes Q\in\mathcal{P}^{p+q+1}$ is given by

$$\mathrm{G}_{P\otimes Q}(k) = \sum_{m+l=k} \mathrm{G}_P(m)\mathrm{G}_Q(l).$$

This, however, follows immediately from the definition since

$$k\,(P\otimes Q) = \left\{\lambda\,(x,o_q,0)^\intercal + (k-\lambda)\,(o_p,y,1)^\intercal : x\in P,\ y\in Q, 0\leq\lambda\leq k\right\}.$$

□

Example 1.1 in the introduction shows an application of this construction. For example 1.2 we need Lemma 1.4.

Proof of Lemma 1.4. With $w = z^{\frac{1}{k}}$ we may write

$$\frac{1}{k}\sum_{i=0}^{k-1}\mathrm{Ehr}_P(\zeta^i\,w) = \frac{1}{k}\sum_{i=0}^{k-1}\sum_{m\geq 0}\mathrm{G}_P(m)(\zeta^i\,w)^m = \frac{1}{k}\sum_{m\geq 0}\mathrm{G}_P(m)w^m\sum_{i=0}^{k-1}\zeta^{i\,m}.$$

Since ζ is a k-th root of unity the sum $\sum_{i=0}^{k-1}\zeta^{i\,m}$ is equal to k if m is a multiple of k and otherwise it is 0. Thus we obtain

$$\frac{1}{k}\sum_{i=0}^{k-1}\mathrm{Ehr}_P(\zeta^i\,w) = \sum_{m\geq 0}\mathrm{G}_P(m\,k)w^{m\,k} = \sum_{m\geq 0}\mathrm{G}_{k\,P}(m)z^m = \mathrm{Ehr}_{k\,P}(z).$$

□

As an application of Lemma 1.4 we calculate the Ehrhart series of the cube C_d (cf. Example 1.2). Instead of C_d we consider the translated cube $2\,\tilde{C}_d$, where $\tilde{C}_d = \{x\in\mathbb{R}^d : 0\leq x_i\leq 1,\ 1\leq i\leq d\}$. In [3, Theorem 2.1] it was shown that $\mathrm{a}_i(\tilde{C}_d) = A(d,i+1)$ where $A(d,i)$ denotes the Eulerian numbers. Setting

$w=\sqrt{z}$ Lemma 1.4 leads to

$$
\begin{aligned}
\mathrm{Ehr}_{C_d}(z) &= \frac{1}{2}\left(\mathrm{Ehr}_{\tilde{C}_d}(w)+\mathrm{Ehr}_{\tilde{C}_d}(-w)\right)\\
&= \frac{1}{2}\left(\frac{\sum_{i=1}^{d} A(d,i)\,w^{i-1}}{(1-w)^{d+1}}+\frac{\sum_{i=1}^{d} A(d,i)\,(-w)^{i+1}}{(1+w)^{d+1}}\right)\\
&= \frac{1}{2}\frac{1}{(1-z)^{d+1}}\left(\sum_{i=1}^{d} A(d,i)\,w^{i-1}\,(1+w)^{d+1}\right.\\
&\qquad\qquad \left.+\sum_{i=1}^{d} A(d,i)\,(-w)^{i+1}\,(1-w)^{d+1}\right)\\
&= \frac{1}{(1-z)^{d+1}}\left(\sum_{i=1}^{d} A(d,i)\sum_{j=0,\ i+j-1 \text{ even}}^{d+1}\binom{d+1}{j}w^{i+j-1}\right)
\end{aligned}
$$

Substituting $2\,l=i+j-1$ gives

$$
\begin{aligned}
\mathrm{Ehr}_{C_d}(z) &= \frac{1}{(1-z)^{d+1}}\left(\sum_{l=0}^{d}\sum_{i=2\,l-d}^{2\,l+1}\binom{d+1}{2\,l+1-i}A(d,i)\,w^{2\,l}\right)\\
&= \frac{1}{(1-z)^{d+1}}\left(\sum_{l=0}^{d} z^{l}\sum_{j=0}^{d+1}\binom{d+1}{j}A(d,2\,l+1-j)\right),
\end{aligned}
$$

which explains the formula in Example 1.2.

In order to calculate in general the Ehrhart series of the prism $P=\{(x,m)^{\intercal}: x\in Q\}$ where $Q\in\mathcal{P}^{d-1}$, $m\in\mathbb{N}$ (cf. Example 1.3), we use the differential operator T defined by $z\frac{\mathrm{d}}{\mathrm{dz}}$. Considered as an operator on the ring of formal power series we have (cf. e.g. [3, p. 28])

$$
\sum_{k\geq 0} f(k)\,z^{k} = f(T)\frac{1}{1-z} \tag{3.1}
$$

for any polynomial f. Since $\mathrm{G}_P(k)=(m\,k+1)\,\mathrm{G}_Q(k)$ we deduce from (3.1)

$$
\mathrm{Ehr}_P(z)=(m\,T+1)\mathrm{Ehr}_Q(z)=mz\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{Ehr}_Q(z)+\mathrm{Ehr}_Q(z).
$$

Thus

$$
\begin{aligned}
\mathrm{Ehr}_P(z) &= m\,z\,\frac{\sum_{i=0}^{d-1} i\,\mathrm{a}_i(Q)z^{i-1}(1-z)+\sum_{i=0}^{d-1} d\,\mathrm{a}_i(Q)\,z^{i}}{(1-z)^{d+1}}+\frac{\sum_{i=0}^{d-1}\mathrm{a}_i(Q)\,z^{i}}{(1-z)^{d}}\\
&= \frac{\sum_{i=0}^{d-1}(m\,i+1)\mathrm{a}_i(Q)z^{i}(1-z)+\sum_{i=0}^{d-1} m\,d\,\mathrm{a}_i(Q)z^{i+1}}{(1-z)^{d+1}}\\
&= \frac{1}{(1-z)^{d+1}}\sum_{i=0}^{d}\left((m\,i+1)\mathrm{a}_i(Q)+(m(d-i+1)-1)\,\mathrm{a}_{i-1}(Q)\right)z^{i},
\end{aligned}
$$

which is the formula in Example 1.3.

In a recent paper Jaron Treutlein [18] generalized a result of Scott [15] to all degree 2 polytopes by showing

Theorem 3.1 (Treutlein). *Let $P \in \mathcal{P}^d$ of degree 2 and let $\mathrm{a}_i = \mathrm{a}_i(P)$. Then*

$$\mathrm{a}_1 \leq \begin{cases} 7, & \mathrm{a}_2 = 1, \\ 3\,\mathrm{a}_2 + 3, & \mathrm{a}_2 \geq 2. \end{cases} \tag{3.2}$$

The next proposition shows that these conditions indeed classify all $h^\star$ polynomials of degree 2.

Proposition 3.2. *Let $f(z) = \mathrm{a}_2\, z^2 + \mathrm{a}_1\, z + 1$, $\mathrm{a}_i \in \mathbb{N}$, satisfying the inequalities in* (3.2)*. Then f is the $h^\star$ polynomial of a lattice polytope.*

Proof. We recall that $\mathrm{a}_1(P) = \mathrm{G}(P) - (d+1)$ and $\mathrm{a}_d(P) = \mathrm{G}(\mathrm{int}(P))$ for $P \in \mathcal{P}^d$. In the case $\mathrm{a}_2 = 1$, $\mathrm{a}_1 = 7$ the triangle $\mathrm{conv}\{0, 3\,e_1, 3\,e_2\}$ has the desired $h^\star$-polynomial. Next we distinguish two cases:

i) $\mathrm{a}_2 < \mathrm{a}_1 \leq 3\,\mathrm{a}_2 + 3$. For integers k, l, m with $0 \leq l, k \leq m+1$ let $P \in \mathcal{P}^2$ given by $P = \mathrm{conv}\{0, l\, e_1, e_2 + (m+1)\, e_1, 2\, e_2, 2\, e_2 + k\, e_1\}$. Then it is easy to see that $\mathrm{a}_2(P) = m$ and P has $n + l + 4$ lattice points on the boundary. Thus $\mathrm{a}_1(P) = n + l + m + 1$.

ii) $\mathrm{a}_1 \leq \mathrm{a}_2$. For integers l, m with $0 \leq l \leq m$ let $P \in \mathcal{P}^3$ given by $P = \mathrm{conv}\{0, e_1, e_2, -l\, e_3, e_1 + e_2 + (m+1)\, e_3\}$. The only lattice points contained in P are the vertices and the lattice points on the edge $\mathrm{conv}\{0, -l\, e_3\}$. Thus $\mathrm{a}_3(P) = 0$ and $\mathrm{a}_1(P) = l$. On the other hand, since $(l+m)/6 = \mathrm{vol}(P) = (\sum_{i=0}^{3} \mathrm{a}_i(P))/6$ (cf. (2.3)) it is $\mathrm{a}_2(P) = m$.

□

4. 0-symmetric lattice polytopes

In order to study the surface area of 0-symmetric polytopes we first prove an isoperimetric inequality for the class of cross-polytopes.

Lemma 4.1. *Let $v_1, \ldots, v_d \in \mathbb{R}^d$ be linearly independent and let $C = \mathrm{conv}\{\pm v_i : 1 \leq i \leq d\}$. Then*

$$\frac{\mathrm{F}(C)^d}{\mathrm{vol}(C)^{d-1}} \geq \frac{2^d}{d!}\, d^{\frac{3}{2}d},$$

and equality holds if and only if C is a regular cross-polytope, i.e., $v_1, \ldots, v_d$ form an orthogonal basis of equal length.

Proof. Without loss of generality let $\mathrm{vol}(C) = 2^d/d!$. Then we have to show

$$\mathrm{F}(C) \geq \frac{2^d}{d!} d^{\frac{3}{2}}. \tag{4.1}$$

By standard arguments from convexity (see e.g. [10, Theorem 6.3]) the set of all 0-symmetric cross-polytopes with volume $2^d/d!$ contains a cross-polytope $C^\star = \mathrm{conv}\{\pm w_1, \ldots, \pm w_d\}$, say, of minimal surface area. Suppose that some of the vectors are not pairwise orthogonal, for instance, w_1 and w_2. Then we apply to $C^\star$ a Steiner-Symmetrization (cf. e.g. [10, pp. 169]) with respect to the hyperplane $H = \{x \in \mathbb{R}^d : w_i\, x = 0\}$. It is easy to check that the Steiner-symmetral of $C^\star$ is again a cross-polytope $\tilde{C}^*$, say, with $\mathrm{vol}(\tilde{C}^\star) = \mathrm{vol}(C^\star)$ (cf. [10, Proposition 9.1]). Since $C^\star$ was not symmetric with respect to the hyperplane H we also know that $\mathrm{F}(\tilde{C}^*) < \mathrm{F}(C^\star)$ which contradicts the minimality of $C^\star$ (cf. [10, p. 171]).

So we can assume that the vectors w_i are pairwise orthogonal. Next suppose that $\|w_1\| > \|w_2\|$, where $\|\cdot\|$ denotes the Euclidean norm. Then we apply Steiner-Symmetrization with respect to the hyperplane H which is orthogonal to $w_1 - w_2$ and bisecting the edge $\mathrm{conv}\{w_1, w_2\}$. As before we get a contradiction to the minimality of $C^\star$.

Thus we know that w_i are pairwise orthogonal and of same length. By our assumption on the volume we get $\|w_i\| = 1$, $1 \leq i \leq d$, and it is easy to calculate that $\mathrm{F}(C^\star) = (2^d/d!)d^{3/2}$. So we have

$$\mathrm{F}(C) \geq \mathrm{F}(C^\star) = \frac{2^d}{d!} d^{\frac{3}{2}},$$

and by the foregoing argumentation via Steiner-Symmetrizations we also see that equality holds if and only C is a regular cross-polytope generated by vectors of unit-length. □

The determination of the minimal surface area of 0-symmetric lattice polytope is an immediate consequence of the lemma above, whereas the non-symmetric case does not follow from the corresponding isoperimetric inequality for simplices.

Proof of Proposition 1.5. Let $P \in \mathcal{P}^d$ with $P = -P$ and let $\dim P = d$. Then P contains a 0-symmetric lattice cross-polytopes $C = \mathrm{conv}\{\pm v_i : 1 \leq i \leq d\}$, say, and by the monotonicity of the surface area and Lemma 4.1 we get

$$\mathrm{F}(P) \geq \mathrm{F}(C) \geq \left(\frac{2^d}{d!}\right)^{\frac{1}{d}} d^{\frac{3}{2}} \, \mathrm{vol}(C)^{\frac{d-1}{d}}. \tag{4.2}$$

Since $v_i \in \mathbb{Z}^d$, $1 \leq i \leq d$, we have $\mathrm{vol}(C) = (2^d/d!)|\det(v_1, \ldots, v_d)| \geq 2^d/d!$, which shows by (4.2) the 0-symmetric case.

In the non-symmetric case we know that P contains a lattice simplex $T = \{x \in \mathbb{R}^d : a_i\, x \leq b_i,\ 1 \leq i \leq d+1\}$, say. Here we may assume that $a_i \in \mathbb{Z}^n$ are primitive, i.e., $\mathrm{conv}\{0, a_i\} \cap \mathbb{Z}^n = \{0, a_i\}$, and that $b_i \in \mathbb{Z}$. Furthermore, we denote the facet $P \cap \{x \in \mathbb{R}^d : a_i\, x = b_i\}$ by F_i, $1 \leq i \leq d+1$. With these notations we have $\det(\mathrm{aff} F_i \cap \mathbb{Z}^n) = \|a_i\|$ (cf. [13, Proposition 1.2.9]). Hence there exist integers $k_i \geq 1$ with

$$\mathrm{vol}_{d-1}(F_i) = k_i \, \frac{\|a_i\|}{(d-1)!}, \tag{4.3}$$

and so we may write

$$\mathrm{F}(P) \geq \mathrm{F}(T) = \sum_{i=1}^{d+1} \mathrm{vol}_{d-1}(F_i) \geq \frac{1}{(d-1)!} \sum_{i=1}^{d+1} \|a_i\|.$$

We also have $\sum_{i=1}^{d+1} \mathrm{vol}_{d-1}(F_i) a_i/\|a_i\| = 0$ (cf. e.g. [10, Theorem 18.2]) and in view of (4.3) we obtain $\sum_{i=1}^{d+1} k_i\, a_i = 0$. Thus, since the $d+1$ lattice vectors a_i

are affinely independent we get

$$\sum_{i=1}^{d+1} \|a_i\|^2 \geq 2\,d. \tag{4.4}$$

Together with the restrictions $\|a_i\| \geq 1$, $1 \leq i \leq d+1$, it is easy to argue that $\sum_{i=1}^{d+1} \|a_i\|$ is minimized if and only if d norms $\|a_i\|$ are equal to 1 and one is equal to $\sqrt{d}$. For instance, the intersection of the cone $\{x \in \mathbb{R}^{d+1} : x_i \geq 1, 1 \leq i \leq d+1\}$ with the hyperplane $H_\alpha = \{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = \alpha\}$, $\alpha \geq d+1$, is the d-simplex $T(\alpha)$ with vertices given by the permutations of the vector $(1,\ldots,1,\alpha-d)^\intercal$ of length $\sqrt{d+(\alpha-d)^2}$. Therefore, a vertex of that simplex is contained in $\{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i^2 \geq 2d\}$ if $\alpha \geq d+\sqrt{d}$. In other words, we always have

$$\sum_{i=1}^{d+1} \|a_i\| \geq d+\sqrt{d},$$

which gives the desired inequality in the non-symmetric case (cf. (4.3)). □

We remark that the proof also shows that equality in Proposition 1.5 holds if and only if P is the o-symmetric cross-polytope $C_d^\star$ or the simplex T_d (up to lattice translations).

Acknowledgement. The authors would like to thank Christian Haase for valuable comments.

References

[1] V. V. Batyrev, *Lattice polytopes with a given h^* -polynomial*, Contemporary Mathematics, no. 423, AMS, 2007, pp. 1–10.
[2] M. Beck, J. De Loera, M. Develin, J. Pfeifle, and R.P. Stanley, *Coefficients and roots of Ehrhart polynomials*, Contemp. Math. **374** (2005), 15–36.
[3] M. Beck and S. Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Springer, 2007.
[4] U. Betke and P. McMullen, *Lattice points in lattice polytopes*, Monatsh. Math. **99** (1985), no. 4, 253–265.
[5] Ch. Bey, M. Henk, and J.M. Wills, *Notes on the roots of Ehrhart polynomials*, Discrete Comput. Geom. **38** (2007), 81–98.
[6] B. Braun, *An Ehrhart series formula for reflexive polytopes*, Electron. J. Combinatorics **13** (2006), no. 1, Note 15.
[7] E. Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci., Paris, Sér. A **254** (1962), 616–618.
[8] ———, *Sur un problème de géométrie diophantienne linéaire*, J. Reine Angew. Math. **227** (1967), 25–49.
[9] P. M. Gruber and C. G. Lekkerkerker, *Geometry of numbers*, second ed., vol. 37, North-Holland Publishing Co., Amsterdam, 1987.
[10] P.M. Gruber, *Convex and discrete geometry*, Grundlehren der mathematischen Wissenschaften, vol. 336, Springer-Verlag Berlin Heidelberg, 2007.
[11] M. Henk and J.M. Wills, *A Blichfeldt-type inequality for the surface area*, to appear in Mh. Math. (2007), Preprint available at `http://arxiv.org/abs/0705.2088`.
[12] T. Hibi, *A lower bound theorem for Ehrhart polynomials of convex polytopes*, Adv. Math. **105** (1994), no. 2, 162–165.

[13] J. Martinet, *Perfect lattices in Euclidean spaces*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 327, Springer-Verlag, Berlin, 2003.

[14] B. Nill, *Lattice polytopes having h^*-polynomials with given degree and linear coefficient*, (2007), Preprint available at http://arxiv.org/abs/0705.1082.

[15] P. R. Scott, *On convex lattice polygons*, Bull. Austral. Math. Soc. **15** (1976), no. 3, 395–399.

[16] R.P. Stanley, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.

[17] ———, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs and Combinatorics **3** (1987), 55–66.

[18] J. Treutlein, *Lattice polytopes of degree 2*, (2007), Preprint availabel at http://arxiv.org/abs/0706.4178.

Martin Henk, Universität Magdeburg, Institut für Algebra und Geometrie, Universitätsplatz 2, D-39106 Magdeburg, Germany

E-mail address: henk@math.uni-magdeburg.de

Makoto Tagami, Universität Magdeburg, Institut für Algebra und Geometrie, Universitätsplatz 2, D-39106 Magdeburg, Germany

E-mail address: tagami@kenroku.kanazawa-u.ac.jp