SUCCESSIVE MINIMA AND RADII

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Abstract. In this note we present inequalities relating the successive minima of a o-symmetric convex body and the successive inner and outer radii of the body. These inequalities build a bridge between known inequalities involving only either the successive minima or the successive radii.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets with non-empty interior, in the n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathcal{K}_0^n$ be the family of all o-symmetric convex bodies, i.e., $K \in \mathcal{K}^n$ with $K = -K$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively. We denote the n-dimensional unit ball by B_n. The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$ and we set $\kappa_n = \mathrm{V}(B_n)$. If $K \subset \mathbb{R}^n$ is an i-dimensional convex body, we write $\mathrm{V}^i(K)$ to denote its i-dimensional volume.

The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement and for $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$ the orthogonal projection of K onto L is denoted by $K|L$. For $M \subset \mathbb{R}^n$, $\operatorname{lin} M$ and $\operatorname{conv} M$ denote respectively the linear and the convex hull of M.

The diameter, the minimal width, the circumradius and the inradius of a convex body K are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [3, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension of the space in which a convex body K is embedded, and if K is contained in an affine space A then we write $f(K; A)$ to denote that f has to be evaluated with respect to the space A. With this notation we define the following successive outer and inner radii.

Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1, \ldots, n$ let*

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \textit{and} \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x + L); x + L\big).$$

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So $\mathrm{R}_i(K)$ is the smallest radius of a K containing solid cylinder with i-dimensional spherical cross section, and $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. We obviously have

$$\mathrm{R}_n(K) = \mathrm{R}(K),\ \ \mathrm{R}_1(K) = \frac{\omega(K)}{2},\ \ \mathrm{r}_n(K) = \mathrm{r}(K)\ \text{ and }\ \mathrm{r}_1(K) = \frac{\mathrm{D}(K)}{2}.$$

Notice that the outer radii are increasing in i, whereas the inner radii are decreasing in i. We also have for $i \in \{1, \dots, n\}$

$$1 \leq \frac{\mathrm{R}_i(K)}{\mathrm{r}_{n-i+1}(K)} < i + 1. \tag{1.1}$$

For the lower bound, which is best possible, we refer to [2, Lemma 2.1]. To determine the optimal upper bound is still an open problem, also in the o-symmetric case. The bound presented above can be found in [15] (see also [14]). The following relation between the in- and outer radii and the volume of an arbitrary convex body $K \in \mathcal{K}^n$ can be found in [2, Corollary 2.1]:

$$\frac{2^n}{n!}\mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_n(K) \leq \mathrm{V}(K) \leq 2^n \mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_n(K). \tag{1.2}$$

In the case when K is o-symmetric, we also have (see [2, Theorem 2.1])

$$\frac{2^n}{n!}\mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_n(K) \leq \mathrm{V}(K) \leq 2^n \mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_n(K). \tag{1.3}$$

For more information on these successive radii, their size for special bodies as well as computational aspects of these radii we refer to [1, 2, 4, 5, 6, 7, 12].

Here we are mainly interested in the relations of these radii to the successive minima of a o-symmetric convex body with respect to the integer lattice, which we introduce next.

We denote by $\mathbb{Z}^n$ the integer lattice, i.e., the lattice of all points with integral coordinates in $\mathbb{R}^n$. Then any lattice Λ of $\mathbb{R}^n$ can be obtained as $\Lambda = B\mathbb{Z}^n$ with $B \in \mathrm{GL}_n(\mathbb{R})$, and the determinant of the lattice is $\det \Lambda = |\det B|$ (see [9, p. 23]).

For $K \in \mathcal{K}_0^n$ and a lattice Λ, the i-th successive minimum $\lambda_i(K, \Lambda)$ of K with respect to Λ, $i = 1, \dots, n$, is defined as

$$\lambda_i(K, \Lambda) = \min\bigl\{\lambda \in \mathbb{R} : \lambda > 0, \dim(\lambda K \cap \Lambda \geq i)\bigr\}.$$

Clearly $\lambda_1(K, \Lambda) \leq \cdots \leq \lambda_n(K, \Lambda)$. The second fundamental theorem of Minkowski (see e.g. [9, s. 9.1, 9.4], [11], [13]) relates the successive minima with the volume of a convex body $K \in \mathcal{K}_0^n$:

$$\frac{2^n}{n!}\det \Lambda \leq \lambda_1(K, \Lambda) \cdot \ldots \cdot \lambda_n(K, \Lambda)\mathrm{V}(K) \leq 2^n \det \Lambda. \tag{1.4}$$

In the case of the integer lattice $\mathbb{Z}^n$ we will just write $\lambda_i(K)$ instead of $\lambda_i(K, \mathbb{Z}^n)$. In this paper we relate the successive minima with the inner and outer radii. Of course, the most natural inequalities between these two series would be of the type $\lambda_i(K)\mathrm{r}_j(K)$ or $\lambda_i(K)\mathrm{R}_j(K)$. The next proposition shows, however, that in general we can not bound these products.

Proposition 1.1. *Let $K \in \mathcal{K}_0^n$. Then*

$$\frac{1}{\mathrm{R}(K)} \leq \lambda_i(K) \leq \frac{1}{\mathrm{r}(K)}, \quad 1 \leq i \leq n.$$

In all other cases, the products $\lambda_i(K)\mathrm{r}_j(K)$ and $\lambda_i(K)\mathrm{R}_j(K)$ can not be bounded neither from above or below by a constant depending only on the dimension.

Therefore we consider products of several radii and successive minima.

Theorem 1.1. *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdot \ldots \cdot \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_i(K), \tag{1.5}$$

$$\lambda_1(K) \cdot \ldots \cdot \lambda_i(K)\mathrm{V}(K) \geq \frac{2^n}{n!}\mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_{n-i}(K). \tag{1.6}$$

None of these inequalities can be improved.

By (1.1) we have $\mathrm{r}_{n-j+1}(K) \leq \mathrm{R}_j(K)$ and so:

Corollary 1.1. *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdot \ldots \cdot \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{R}_{n-i+1}(K) \cdot \ldots \cdot \mathrm{R}_n(K), \tag{1.7}$$

$$\lambda_1(K) \cdot \ldots \cdot \lambda_i(K)\mathrm{V}(K) \geq \frac{2^n}{n!}\mathrm{r}_{i+1}(K) \cdot \ldots \cdot \mathrm{r}_n(K). \tag{1.8}$$

None of these inequalities can be improved.

For inequality (1.5) and inequality (1.7) (inequality (1.6) and inequality (1.8)), the "limit" case $i = 0$ ($i = n$), i.e., when no radii appear in the inequalities, is Minkowski's inequality (1.4). The "limit" case $i = n$ ($i = 0$), i.e., when no successive minima appear in the formulae, gives the upper (lower) bounds for the volume in (1.2) and (1.3). Thus, these inequalities build a bridge between Minkowski's inequality and the known inequalities involving in- and outer radii.

In the next section we present the proofs of the main results, as well as some consequences for general (not necessarily o-symmetric) convex bodies.

2. Proofs of the main results

Before we start with the proof of Proposition 1.1 we have briefly to introduce the concept of polar bodies.

For a convex body $K \in \mathcal{K}^n$ containing the origin in its interior, the polar body of K is the convex body $K^* = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \text{ for all } x \in K\}$ (see e.g. [17, s. 1.6]). The in- and outer radii of a o-symmetric convex body $K \in \mathcal{K}_0^n$ and its polar are related by the following identity, for which we refer to [6, (1.2)]:

$$\mathrm{R}_i(K^*)\,\mathrm{r}_i(K) = 1 \quad \text{for } i = 1, \dots, n. \tag{2.1}$$

Proof of Proposition 1.1. Since $\mathrm{r}(K)B_n \subseteq K$ we obviously have

$$\lambda_i(K) \leq \lambda_i\big(\mathrm{r}(K)B_n\big) = \frac{1}{\mathrm{r}(K)}\lambda_i(B_n) = \frac{1}{\mathrm{r}(K)}$$

for $1 \leq i \leq n$. Analogously, from $K \subseteq \mathrm{R}(K)B_n$ we find $\lambda_i(K) \geq 1/\mathrm{R}(K)$ and so we trivially get the inequalities in the proposition.

Next we show that the inequalities above are the only possible upper and lower bounds for the products $\lambda_i(K)\mathrm{r}_j(K)$ and $\lambda_i(K)\mathrm{R}_j(K)$. In order to see that there is no upper bound on $\lambda_i(K)\mathrm{r}_j(K)$, $j = 1, \dots, n-1$, we consider the j-dimensional unit ball B_j embedded in a j-dimensional irrational plane $L \in \mathcal{L}^n_j$, i.e., $L \cap \mathbb{Z}^n = \{0\}$. Taking the convex hull of B_j and suitable points with irrational coordinates, close enough to L, we can find an n-dimensional convex body K_0 with $\mathrm{r}_j(K_0) = 1$ but arbitrarily large $\lambda_i(K_0)$.

The non-existence of lower bounds on $\lambda_i(K)\mathrm{r}_j(K)$, $j = 2, \dots, n$, is shown by the following cross-polytope $C^*_n(m)$. For $m \in \mathbb{N}$ and $i = 1, \dots, n$, let $v_i := (m^{i-1}, \dots, m, 1, 0, \dots, 0)^{\intercal} \in \mathbb{R}^n$, and $C^*_n(m) := \mathrm{conv}\{\pm v_i : i = 1, \dots, n\}$. $C^*_n(m)$ is a o-symmetric lattice cross-polytope containing the origin as the only interior lattice point. Hence

$$\lambda_i\big(C^*_n(m)\big) = 1$$

for all $i = 1, \dots, n$ and next we show the inner radii $\mathrm{r}_j\big(C^*_n(m)\big)$, $j = 2, \dots, n$, can be arbitrarily small. Since r_j are decreasing in j it suffices to verify that for r_2. Moreover, from $\mathrm{r}_2\big(C^*_n(m)\big) \leq \mathrm{R}_{n-1}\big(C^*_n(m)\big)$ (cf. (1.1)) we just have to check that for a suitable projection π, the lengths of the projected vertices $\pi(v_i)$ can be made arbitrarily small. Let π be the orthogonal projection onto the hyperplane orthogonal to v_n. The k-th coordinate of the projection $\pi(v_i) = v_i - \langle v_i, v_n\rangle / |v_n|^2 v_n$ of v_i is given by

$$\big(\pi(v_i)\big)_k = \begin{cases} m^{i-k}\dfrac{1+m^2+\cdots+m^{2(n-i-1)}}{1+m^2+\cdots+m^{2(n-1)}}, & \text{for } k = 1, \dots, i, \\[2ex] -m^{2n-i-k}\dfrac{1+m^2+\cdots+m^{2(i-1)}}{1+m^2+\cdots+m^{2(n-1)}}, & \text{for } k = i+1, \dots, n. \end{cases}$$

Hence

$$\pi(v_i) = v_i - \frac{\langle v_i, v_n\rangle}{|v_n|^2}v_n \to (0, \dots, 0)^{\intercal} \quad \text{when} \quad m \to \infty,$$

and so $\mathrm{R}_{n-1}\big(C^*_n(m)\big)$ tends to zero as m approaches infinity.

In order to deal with the outer radii we use polarity. By (2.1) we may write

$$\lambda_i(K)\mathrm{R}_j(K) = \frac{\lambda_i(K)\lambda_{n-i+1}(K^*)}{\lambda_{n-i+1}(K^*)\mathrm{r}_j(K^*)}.$$

By classical results in Geometry of Numbers we know that the numerator is bounded from above and below (cf. [8, Theorem 23.2]). Hence, by taking K as the polar body of $C^*_n(m)$ and the foregoing discussion on the inner radii we see that $\lambda_i(K)\mathrm{R}_j(K)$ is not bounded from above for $j \geq 2$; by taking $K = K_0^*$ we get that $\lambda_i(K)\mathrm{R}_j(K)$ is not bounded from below for $j \leq n-1$. □

Next we come to the proof of Theorem 1.1, providing upper and lower bounds for products of successive minima in terms of the in- and outer radii.

Proof of Theorem 1.1. We start with inequality (1.5). Let $z_1, \ldots, z_i \in K$ be i linearly independent points with $\lambda_j(K)z_j \in \lambda_j(K)K \cap \mathbb{Z}^n$. We consider a suitable $(n-i)$-dimensional coordinate plane $L_{n-i} = \{x \in \mathbb{R}^n : x_{j_1} = \cdots = x_{j_i} = 0, j_k \in \{1, \ldots, n\}\}$ such that

$$\mathrm{lin}\{z_1, \ldots, z_i\} \cap L_{n-i} = \{0\}. \tag{2.2}$$

Denoting by $\mathbb{Z}^{n-i}$ the sublattice of all points in L_{n-i} with integer coordinates, Minkowski's second fundamental theorem assures that

$$\lambda_1(K \cap L_{n-i}, \mathbb{Z}^{n-i}) \cdot \ldots \cdot \lambda_{n-i}(K \cap L_{n-i}, \mathbb{Z}^{n-i})\mathrm{V}^{n-i}(K \cap L_{n-i}) \leq 2^{n-i}.$$

From (2.2) we know that $\lambda_j(K \cap L_{n-i}, \mathbb{Z}^{n-i})K$ contains $i+j$ linearly independent points of $\mathbb{Z}^n$, for $j = 1, \ldots, n-i$. Therefore,

$$\lambda_{i+j}(K) \leq \lambda_j(K \cap L_{n-i}, \mathbb{Z}^{n-i}), \quad j = 1, \ldots, n-i,$$

and hence

$$\lambda_{i+1}(K) \cdot \ldots \cdot \lambda_n(K)\mathrm{V}^{n-i}(K \cap L_{n-i}) \leq 2^{n-i}. \tag{2.3}$$

With $L_i = L_{n-i}^{\perp}$ we get by the o-symmetry of K (cf. [10])

$$\mathrm{V}^{n-i}(K \cap L_{n-i}) \geq \frac{\mathrm{V}(K)}{\mathrm{V}^i(K|L_i)}.$$

Since $K|L_i$ is an i-dimensional o-symmetric convex body, we have (see [2, Theorem 2.1]) $\mathrm{V}^i(K|L_i) \leq 2^i \mathrm{r}_1(K|L_i) \cdot \ldots \cdot \mathrm{r}_i(K|L_i)$. Together with $\mathrm{r}_j(K|L_i) \leq \mathrm{r}_j(K)$ (see [2, Lemma 2.1]), we get

$$\mathrm{V}^i(K|L_i) \leq 2^i \mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_i(K).$$

Therefore

$$\mathrm{V}^{n-i}(K \cap L_{n-i}) \geq \frac{\mathrm{V}(K)}{2^i \mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_i(K)}$$

and using (2.3) we obtain

$$\lambda_{i+1}(K) \cdot \ldots \cdot \lambda_n(K)\mathrm{V}(K) \leq 2^n \mathrm{r}_1(K) \cdot \ldots \cdot \mathrm{r}_i(K).$$

In order to show that inequality (1.5) can not be improved it suffices to consider the tightness of inequality (1.7) in Corollary 1.1. Let $Q(\mu)$ be the orthogonal parallelepiped with edge-lengths $\mu, \mu^2, \ldots, \mu^n$, for $\mu \geq 1$. The successive minima of such a box are $\lambda_j\big(Q(\mu)\big) = 2/\mu^{n-j+1}$, $j = 1, \ldots, n$, the outer radii R_j are given by $\mathrm{R}_j\big(Q(\mu)\big) = (1/2)\big(\sum_{k=1}^{j} \mu^{2k}\big)^{1/2}$ (see [5, Theorem 4.4]) and for the volume we find $\mathrm{V}\big(Q(\mu)\big) = \mu \cdot \ldots \cdot \mu^n$. Thus

$$\frac{\prod_{j=i+1}^{n} \lambda_j\big(Q(\mu)\big)}{\prod_{j=n-i+1}^{n} \mathrm{R}_j\big(Q(\mu)\big)} \mathrm{V}\big(Q(\mu)\big) = 2^i \frac{2^{n-i}\mu^{n-i+1} \cdot \ldots \cdot \mu^n}{\prod_{j=n-i+1}^{n} \Big(\sum_{k=1}^{j} \mu^{2k}\Big)^{1/2}},$$

which tends to 2^n as μ approaches infinity.

Now we prove inequality (1.6). Again let $z_1, \dots, z_i \in K$ be i linearly independent points with $\lambda_j(K)z_j \in \lambda_j(K)K \cap \mathbb{Z}^n$. We denote by $u_j := \lambda_j(K)z_j$, and we consider the i-dimensional sublattice Λ_i of $\mathbb{Z}^n$ determined by $\{u_1, \dots, u_i\}$. Clearly, $\det \Lambda_i \geq 1$. Minkowski's lower bound in (1.4) gives

$$\frac{2^i}{i!} \leq \frac{2^i}{i!} \det \Lambda_i \leq \lambda_1(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \cdot \ldots \cdot \lambda_i(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) \mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i).$$

Since $\lambda_j(K \cap \operatorname{lin} \Lambda_i, \Lambda_i) = \lambda_j(K)$, $1 \leq j \leq i$, we can write

$$\frac{2^i}{i!} \leq \lambda_1(K) \cdot \ldots \cdot \lambda_i(K) \mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i). \tag{2.4}$$

With $L_{n-i} = (\operatorname{lin} \Lambda_i)^{\perp}$ we know that (see [16])

$$\mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i) \mathrm{V}^{n-i}(K | L_{n-i}) \leq \binom{n}{i} \mathrm{V}(K).$$

Since $K|L_{n-i}$ is an $(n-i)$-dimensional o-symmetric convex body we have (see [2, Theorem 2.1])

$$\mathrm{V}^{n-i}(K|L_{n-i}) \geq \frac{2^{n-i}}{(n-i)!} \mathrm{R}_1(K|L_{n-i}) \cdot \ldots \cdot \mathrm{R}_{n-i}(K|L_{n-i}),$$

and since $\mathrm{R}_j(K|L_{n-i}) \geq \mathrm{R}_j(K)$ (see [2, Lemma 2.1]) we arrive at

$$\mathrm{V}^{n-i}(K|L_{n-i}) \geq \frac{2^{n-i}}{(n-i)!} \mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_{n-i}(K).$$

Therefore

$$\mathrm{V}^i(K \cap \operatorname{lin} \Lambda_i) \leq \binom{n}{i} \frac{\mathrm{V}(K)}{\mathrm{V}^{n-i}(K|L_{n-i})} \leq \frac{n!}{i! 2^{n-i}} \frac{\mathrm{V}(K)}{\mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_{n-i}(K)},$$

and with (2.4) we get

$$\frac{2^n}{n!} \mathrm{R}_1(K) \cdot \ldots \cdot \mathrm{R}_{n-i}(K) \leq \lambda_1(K) \cdot \ldots \cdot \lambda_i(K) \mathrm{V}(K).$$

To show that inequality (1.6) can not be improved it suffices to consider the tightness of inequality (1.8) in Corollary 1.1. We consider for $\mu > 1$ the orthogonal cross-polytope $C_n^*(\mu) := \operatorname{conv}\{\pm \mu^i e_i : i = 1, \dots, n\}$, where e_i denotes the i-th canonical unit vector. The successive minima of such a cross-polytope are $\lambda_j\bigl(C_n^*(\mu)\bigr) = 1/\mu^{n-j+1}$, $j = 1, \dots, n$, the inner radii r_j are given by $\mathrm{r}_j\bigl(C_n^*(\mu)\bigr) = \bigl(\sum_{k=n-j+1}^{n} \mu^{-2k}\bigr)^{-1/2}$ (see [5, Theorem 4.4]) and for its volume we find $\mathrm{V}\bigl(C_n^*(\mu)\bigr) = (2^n/n!)\mu \cdot \ldots \cdot \mu^n$. Thus

$$\frac{\prod_{j=1}^{i} \lambda_j\bigl(C_n^*(\mu)\bigr)}{\prod_{j=i+1}^{n} \mathrm{r}_j\bigl(C_n^*(\mu)\bigr)} \mathrm{V}\bigl(C_n^*(\mu)\bigr) = \frac{2^n}{n!} \frac{\mu \cdot \ldots \cdot \mu^{n-i}}{\prod_{j=i+1}^{n} \bigl(\sum_{k=n-j+1}^{n} \mu^{-2k}\bigr)^{-1/2}},$$

which tends to $2^n/n!$ when $\mu \to \infty$. □

In order to present some inequalities as in Theorem 1.1 for arbitrary convex bodies, we write $DK = K+(-K)$ for the *difference body* of a convex body $K \in \mathcal{K}^n$. DK is certainly o-symmetric, and for further properties we refer for instance to [8, s. 9.5]. The *central symmetral* of K is just the convex body $\overline{K} = (1/2)DK$ (see [3, p. 79] for a study of this symmetrization).

As a consequence of Corollary 1.1 we get the following result for general convex bodies.

Corollary 2.1. *Let $K \in \mathcal{K}^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(DK) \cdot \ldots \cdot \lambda_n(DK)\mathrm{V}(K) \leq 2^i \mathrm{R}_{n-i+1}(K) \cdot \ldots \cdot \mathrm{R}_n(K). \tag{2.5}$$

$$\lambda_1(DK) \cdot \ldots \cdot \lambda_i(DK)\mathrm{V}(DK) \geq \frac{2^{2n-i}}{n!}\mathrm{r}_{i+1}(K) \cdot \ldots \cdot \mathrm{r}_n(K). \tag{2.6}$$

None of these inequalities can be improved.

Proof. Let $K \in \mathcal{K}^n$. Inequality (1.7) and inequality (1.8) applied to the central symmetral $\overline{K}$ give

$$\lambda_{i+1}(\overline{K}) \cdot \ldots \cdot \lambda_n(\overline{K})\mathrm{V}(\overline{K}) \leq 2^n \mathrm{R}_{n-i+1}(\overline{K}) \cdot \ldots \cdot \mathrm{R}_n(\overline{K})$$

and

$$\lambda_1(\overline{K}) \cdot \ldots \cdot \lambda_i(\overline{K})\mathrm{V}(\overline{K}) \geq \frac{2^n}{n!}\mathrm{r}_{i+1}(\overline{K}) \cdot \ldots \cdot \mathrm{r}_n(\overline{K}).$$

It is well known that central symmetrization does not decrease the volume (cf. e.g. [3, p. 79]) and so we have $\mathrm{V}(K) \leq \mathrm{V}(\overline{K})$. Moreover, for the outer radii R_j it holds (see [12, Lemma 2.1]) $\mathrm{R}_j(\overline{K}) \leq \mathrm{R}_j(K)$, and for the inner radii r_j we have (see [12, Remark 2.1]) $\mathrm{r}_j(\overline{K}) \geq \mathrm{r}_j(K)$, $j = 1, \dots, n$. Then writing $\overline{K} = (1/2)DK$ we obtain

$$2^{n-i}\lambda_{i+1}(DK) \cdot \ldots \cdot \lambda_n(DK)\mathrm{V}(K) \leq 2^n \mathrm{R}_{n-i+1}(K) \cdot \ldots \cdot \mathrm{R}_n(K)$$

and

$$2^i\lambda_1(DK) \cdot \ldots \cdot \lambda_i(DK)\frac{1}{2^n}\mathrm{V}(DK) \geq \frac{2^n}{n!}\mathrm{r}_{i+1}(K) \cdot \ldots \cdot \mathrm{r}_n(K),$$

which prove the result. The same orthogonal parallelepiped and the same orthogonal cross-polytope considered in the proof of Theorem 1.1 show that inequality (2.5) and inequality (2.6), respectively, can not be improved. □

Remark 2.1. *In Corollary 2.1 the well-known Rogers-Shephard inequality* $\mathrm{V}(DK) \leq \binom{2n}{n}\mathrm{V}(K)$ *(see* [17, s. 7.3]*) can be used in order to express inequality* (2.6) *in terms of the volume of K. But then the bound is not best possible.*

We finally remark that identity (2.1) allows to express the inequalities in Theorem 1.1 in terms of the in- and outer radii of the polar body.

Remark 2.2. *Let $K \in \mathcal{K}_0^n$. For $i = 1, \dots, n-1$ we have*

$$\lambda_{i+1}(K) \cdot \ldots \cdot \lambda_n(K)\, \mathrm{R}_1(K^*) \cdot \ldots \cdot \mathrm{R}_i(K^*)\mathrm{V}(K) \;\leq\; 2^n,$$

$$\lambda_1(K) \cdot \ldots \cdot \lambda_i(K)\, \mathrm{r}_1(K^*) \cdot \ldots \cdot \mathrm{r}_{n-i}(K^*)\mathrm{V}(K) \;\geq\; \frac{2^n}{n!}.$$

None of these inequalities can be improved.

In the same way we can rewrite Corollary 1.1 and Corollary 2.1 in terms of the radii of the polar body.

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