INTRINSIC VOLUMES AND SUCCESSIVE RADII

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ABSTRACT. Motivated by a problem of Teissier to bound the intrinsic volumes of a convex body in terms of the inradius and the circumradius of the body, we give upper and lower bounds for the intrinsic volumes of a convex body in terms of the elementary symmetric functions of the so called successive inner and outer radii. These results improve on former bounds and, in particular, they also provide bounds for the elementary symmetric functions of the roots of Steiner polynomials in terms of the elementary symmetric functions of these radii.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ be the standard inner product and Euclidean norm in $\mathbb{R}^n$, respectively. We denote the n-dimensional unit ball by B_n and its boundary, i.e., the $(n-1)$-dimensional unit sphere, by $\mathbb{S}^{n-1}$.

The volume of a set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\mathrm{V}(M)$ and we write $\kappa_n = \mathrm{V}(B_n)$. The set of all i-dimensional linear subspaces of $\mathbb{R}^n$ is denoted by $\mathcal{L}_i^n$. For $L \in \mathcal{L}_i^n$, $L^\perp$ denotes its orthogonal complement and for $K \in \mathcal{K}^n$ and $L \in \mathcal{L}_i^n$ the orthogonal projection of K onto L is denoted by $K|L$.

The diameter, the minimal width, the circumradius and the inradius of a convex body K are denoted by $\mathrm{D}(K)$, $\omega(K)$, $\mathrm{R}(K)$ and $\mathrm{r}(K)$, respectively. For more information on these functionals and their properties we refer to [5, pp. 56–59]. If f is a functional on $\mathcal{K}^n$ depending on the dimension in which a convex body K is embedded, and if K is contained in an affine space A then we write $f(K;A)$ to denote that f has to be evaluated with respect to the space A. With this notation we define the following successive outer and inner radii.

Definition 1.1. *For $K \in \mathcal{K}^n$ and $i = 1, \dots, n$ let*

$$\mathrm{R}_i(K) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) \quad \textit{and} \quad \mathrm{r}_i(K) = \max_{L \in \mathcal{L}_i^n} \max_{x \in L^\perp} \mathrm{r}\big(K \cap (x+L); x+L\big).$$

2000 *Mathematics Subject Classification.* Primary 52A20, 52A39; Secondary 52A40.

Key words and phrases. Intrinsic volumes, inner and outer radii, roots of Steiner polynomials.

Second author is supported in part by Dirección General de Investigación (MEC) MTM2004-04934-C04-02 and by Fundación Séneca (C.A.R.M.) 00625/PI/04.

So $\mathrm{R}_i(K)$ is the smallest radius of a K containing solid cylinder with i-dimensional spherical cross section, and $\mathrm{r}_i(K)$ is the radius of the greatest i-dimensional ball contained in K. We obviously have

$$\mathrm{R}_n(K)=\mathrm{R}(K),\ \ \mathrm{R}_1(K)=\frac{\omega(K)}{2},\ \ \mathrm{r}_n(K)=\mathrm{r}(K)\ \ \text{and}\ \ \mathrm{r}_1(K)=\frac{\mathrm{D}(K)}{2}.$$

If we replace in the definition of R_i the min-condition by a max-condition and in the definition of r_i the first max-condition by a min-condition, we obtain another series of successive outer and inner radii.

Definition 1.2. *For $K\in\mathcal{K}^n$ and $i=1,\dots,n$ let*

$$\overline{\mathrm{R}}_i(K)=\max_{L\in\mathcal{L}_i^n}\mathrm{R}(K|L)\quad \textit{and}\quad \overline{\mathrm{r}}_i(K)=\min_{L\in\mathcal{L}_i^n}\max_{x\in L^\perp}\mathrm{r}\big(K\cap(x+L);x+L\big).$$

The outer (inner) radii now start with half of the diameter (half of the minimal width) and end with the circumradius (inradius). It is clear that both types of outer radii are increasing in i, whereas the inner radii are decreasing in i.

For more information on these successive radii, their size for special bodies as well as computational aspects of these radii we refer to [1, 2, 4, 6, 7, 9, 10, 12]. In particular, we want to mention an open problem concerning the ratio $\mathrm{R}_{n-i+1}(K)/\mathrm{r}_i(K)$. In [18] (see also [17]) it was shown that

$$\frac{\mathrm{R}_{n-i+1}(K)}{\mathrm{r}_i(K)}\le i+1,$$

but the optimal bound is still not known. Here, however, we are mainly interested in the relations of these radii to the intrinsic volumes, which we introduce next.

For two convex bodies $K,E\in\mathcal{K}^n$ and a non-negative real number ρ, the *mixed volumes of K and E*, $\mathrm{V}_i(K,E)$, are defined as the coefficients of the following polynomial describing the volume of the Minkowski sum $K+\rho\,E$,

$$\mathrm{V}(K+\rho E)=\sum_{i=0}^{n}\binom{n}{i}\mathrm{V}_i(K,E)\rho^i. \tag{1.1}$$

For characterizations and properties of the mixed volumes of convex bodies we refer to [19, s. 5.1]. If $E=B_n$ the polynomial (1.1) becomes the classical *Steiner polynomial* [19, p. 210], which can be written via the normalization $\mathrm{V}_i(K)=\binom{n}{i}\mathrm{V}_{n-i}(K,B_n)/\kappa_{n-i}$ as

$$\sum_{i=0}^{n}\kappa_{n-i}\mathrm{V}_i(K)\rho^{n-i}. \tag{1.2}$$

$\mathrm{V}_i(K)$ is called the *i-th intrinsic volume* of K since, if K is i-dimensional, then $\mathrm{V}_i(K)$ is the ordinary i-dimensional volume of K. In particular, we have that $\mathrm{V}_n(K)$ is the volume of K, $2\mathrm{V}_{n-1}(K)$ is the *surface area* of K, $2\kappa_{n-1}/(n\kappa_n)\mathrm{V}_1(K)$ is the *mean width* of K (see [19, p. 42]) and $\mathrm{V}_0(K)=1$ is the Euler characteristic.

Further the *relative inradius* $\mathrm{r}(K;E)$ and *relative circumradius* $\mathrm{R}(K;E)$ of K with respect to E are defined, respectively, by $\mathrm{r}(K,E)=\sup\{r:\exists\,x\in\mathbb{R}^n \text{ with } x+r\,E\subset K\}$ and $\mathrm{R}(K,E)=\inf\{R:\exists\,x\in\mathbb{R}^n \text{ with } K\subset x+R\,E\}$. When $E=B_n$ the classical values $\mathrm{r}(K)$ and $\mathrm{R}(K)$ are obtained. In [20] Teissier posed the problem to give bounds of the mixed volumes $\mathrm{V}_i(K,E)$ in terms of the inradius $\mathrm{r}(K;E)$ and circumradius $\mathrm{R}(K;E)$, as well as to bound this in- and circumradius in terms of the roots of the polynomial (1.1) regarded as a formal polynomial in a complex variable.

In [2] bounds for the volume of a convex body are given in terms of the product of the successive inner and outer radii. In this paper we will give more general bounds for the intrinsic volumes in terms of the elementary symmetric functions of the inner and outer radii, which in particular relate the intrinsic volumes with the circumradius and the inradius of the set. To this end we denote by

$$\mathrm{s}_i\bigl(x_1,\dots,x_m\bigr)=\sum_{1\le j_1<\dots<j_i\le m} x_{j_1}\cdot\ldots\cdot x_{j_i}$$

the i-th elementary symmetric function of $x_1,\dots,x_m\in\mathbb{R}$, $1\le i\le m$, and we set $\mathrm{s}_0\bigl(x_1,\dots,x_m\bigr)=1$.

Theorem 1.1. *Let $K\in\mathcal{K}^n$. Then for $i=0,\dots,n$*

$$\frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\bigl(\bar{\mathrm{r}}_1(K),\dots,\bar{\mathrm{r}}_n(K)\bigr)\le \mathrm{V}_i(K)\le \frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\bigl(\overline{\mathrm{R}}_1(K),\dots,\overline{\mathrm{R}}_n(K)\bigr). \tag{1.3}$$

For $K\in\mathcal{K}^n$ with nonempty interior, equality holds in both inequalities if and only if K is a ball.

For the successive radii of Definition 1.1 we obtain the following upper bound.

Theorem 1.2. *Let $K\in\mathcal{K}^n$. Then for $i=0,\dots,n$*

$$\mathrm{V}_i(K)\le 2^i\,\mathrm{s}_i\bigl(\mathrm{R}_1(K),\dots,\mathrm{R}_n(K)\bigr). \tag{1.4}$$

The bound is best possible.

Unfortunately, we are not aware of a best possible lower bound on $\mathrm{V}_i(K)$ in terms of $\mathrm{s}_i\bigl(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\bigr)$. It can easily be shown (see Remark 3.2) that

$$\mathrm{V}_i(K)\ge\frac{2^i}{i!\binom{n}{i}}\,\mathrm{s}_i\bigl(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\bigr), \tag{1.5}$$

but in general the result is not tight for $1\le i\le n-1$ (see Remark 3.3). It is quite tempting to conjecture a lower bound of $(2^i/i!)\,\mathrm{s}_i\bigl(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\bigr)$, but such a bound does not exist (cf. Remark 3.3). In fact, we believe that $(2^i/i!)\,\mathrm{s}_i\bigl(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\bigr)^{1/2}$ are the right candidates for obtaining sharp lower bounds, and here we get the following partial results.

Theorem 1.3. *Let* $K\in\mathcal{K}^n$. *Then*

$$\mathrm{V}_{n-1}(K)\geq\frac{2^{n-1}}{(n-1)!}\sqrt{\mathrm{s}_{n-1}\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)},\tag{1.6}$$

$$\mathrm{V}_{n-2}(K)\geq\frac{2\sqrt{2}}{\pi}\frac{2^{n-2}}{(n-2)!}\sqrt{\mathrm{s}_{n-2}\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)}.\tag{1.7}$$

The bound in (1.6) *is best possible.*

Finally, we remark that in the case $i=n$, i.e., with respect to the volume, the bounds in (1.3), (1.4) and (1.5) were already proved in [2].

The paper is organized as follows. In Section 2 we give some preliminary results on these radii and related functionals which are needed for the proof of the theorems. Then, in Section 3 we present the proofs of the main theorems, as well as some consequences, in particular for the roots of the Steiner polynomial (cf. Corollary 3.2). Finally, in Section 4 we prove a formula for the external angles of orthogonal cross-polytopes used in the proof of Theorem 1.3.

2. Some preliminary results

First we introduce some additional notation. $K\in\mathcal{K}^n$ is 0-*symmetric* if it is symmetric with respect to the origin, i.e., if $K=-K$. For $K\in\mathcal{K}^n$ we denote by $K^0=\big(K+(-K)\big)/2$ its *central symmetral* (see [5, p. 79]). Analogously to the definition of outer radii R_i we introduce a series of successive diameters, whose relation to the intrinsic volumes was studied in [2].

Definition 2.1. *For* $K\in\mathcal{K}^n$ *let* $\mathrm{D}_i(K)=\min_{L\in\mathcal{L}_i^n}\mathrm{D}(K|L)$, $i=1,\dots,n$.

Clearly $\mathrm{D}_n(K)=\mathrm{D}(K)$ and $\mathrm{D}_1(K)=\omega(K)$. The next lemma studies the behavior of $\mathrm{R}_i(K)$ and $\mathrm{D}_i(K)$ with respect to central symmetrization.

Lemma 2.1. *Let* $K\in\mathcal{K}^n$. *Then* $\mathrm{D}_i(K^0)=\mathrm{D}_i(K)$ *and* $\mathrm{R}_i(K^0)\leq\mathrm{R}_i(K)$, *for* $i=1,\dots,n$.

Proof. Let $\omega(K,u)$ be the width of the body K in the direction $u\in\mathbb{S}^{n-1}$, which can be expressed in terms of the support function $h(K,\cdot)$ of K as $\omega(K,u)=h(K,u)+h(K,-u)$. Clearly $\mathrm{D}(K)=\max_{u\in\mathbb{S}^{n-1}}\omega(K,u)$ (see e.g. [19, p. 42]). Furthermore, for $L\in\mathcal{L}_i^n$ let $\mathbb{S}_L^{i-1}=\mathbb{S}^{n-1}\cap L$. Using the well known facts that central symmetrization preserves the width in any direction (cf. e.g. [5, p. 79]) and that $h(K|L,u)=h(K,u)$ for all $u\in L$ (cf. e.g. [19, pp. 37–38]), we get

$$\begin{aligned}\mathrm{D}(K^0|L)&=\max_{u\in\mathbb{S}_L^{i-1}}\omega(K^0|L,u)=\max_{u\in\mathbb{S}_L^{i-1}}\big\{h(K^0|L,u)+h(K^0|L,-u)\big\}\\&=\max_{u\in\mathbb{S}_L^{i-1}}\big\{h(K^0,u)+h(K^0,-u)\big\}=\max_{u\in\mathbb{S}_L^{i-1}}\omega(K^0,u)=\max_{u\in\mathbb{S}_L^{i-1}}\omega(K,u)\\&=\max_{u\in\mathbb{S}_L^{i-1}}\big\{h(K,u)+h(K,-u)\big\}=\max_{u\in\mathbb{S}_L^{i-1}}\big\{h(K|L,u)+h(K|L,-u)\big\}\\&=\max_{u\in\mathbb{S}_L^{i-1}}\omega(K|L,u)=\mathrm{D}(K|L).\end{aligned}$$

Hence, for $i = 1, \dots, n$, we get

$$\mathrm{D}_i(K^0) = \min_{L \in \mathcal{L}_i^n} \mathrm{D}(K^0|L) = \min_{L \in \mathcal{L}_i^n} \mathrm{D}(K|L) = \mathrm{D}_i(K).$$

Since $K^0|L$ is 0-symmetric it is $2\mathrm{R}(K^0|L) = \mathrm{D}(K^0|L)$ and thus $\mathrm{R}(K^0|L) = \mathrm{D}(K^0|L)/2 = \mathrm{D}(K|L)/2 \leq \mathrm{R}(K|L)$. Hence we also obtain

$$\mathrm{R}_i(K^0) = \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K^0|L) \leq \min_{L \in \mathcal{L}_i^n} \mathrm{R}(K|L) = \mathrm{R}_i(K). \qquad \square$$

Remark 2.1. *By the same reasoning one can show that* $\overline{\mathrm{R}}_i(K^0) \leq \overline{\mathrm{R}}_i(K)$ *whereas in the case of the inner radii it is easy to see that* $\mathrm{r}_i(K^0) \geq \mathrm{r}_i(K)$ *and* $\bar{\mathrm{r}}_i(K^0) \geq \bar{\mathrm{r}}_i(K)$.

In order to state the next lemma, we need some basic definitions from the theory of polytopes. For an arbitrary polytope $P \in \mathcal{K}^n$ let $\mathcal{F}_i(P)$ denote the set of all i-dimensional faces of P, and for $F \in \mathcal{F}_i(P)$ let $\gamma(F, P)$ denote the external angle of F. For a definition of $\gamma(F, P)$ we refer to Section 4. Then the i-th intrinsic volume of P can be computed by the formula (see e.g. [19, p. 210])

$$\mathrm{V}_i(P) = \sum_{F \in \mathcal{F}_i(P)} \gamma(F, P)\mathrm{V}_i(F). \tag{2.1}$$

For $0 < \lambda_1 \leq \dots \leq \lambda_n$ we denote by $C_n^*(\lambda_1, \dots, \lambda_n)$ the orthogonal cross-polytope given by $C_n^*(\lambda_1, \dots, \lambda_n) = \mathrm{conv}\{\pm\lambda_i e_i : i = 1, \dots, n\}$, where e_i denotes the i-th canonical unit vector. We will write just C_n^* for the regular cross-polytope $C_n^*(1, \dots, 1)$.

Following the approach used in Lemma 2.1 of [3] for computing the external angles of a regular cross-polytope we obtain the following generalized formula for the external angles of an orthogonal cross-polytope. For the sake of completeness, a proof of this lemma will be given in the last section.

Lemma 2.2. *Let* $F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}}) = \mathrm{conv}\{\lambda_{l_1} e_{l_1}, \dots, \lambda_{l_{i+1}} e_{l_{i+1}}\}$, $0 \leq i \leq n-1$, *be an* i*-dimensional face of* $C_n^*(\lambda_1, \dots, \lambda_n)$, $1 \leq l_1 < \dots < l_{i+1} \leq n$. *The external angle of* $F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}})$ *is given by*

$$\frac{2^{n-i-1}}{\pi^{(n-i)/2}} \int_0^\infty \mathrm{e}^{-x^2} \left(\prod_{\substack{j=1 \\ j \notin \{l_1, \dots, l_{i+1}\}}}^{n} \int_0^{\frac{x}{\lambda_j \sqrt{\sum_{k=1}^{i+1} \frac{1}{\lambda_{l_k}^2}}}} \mathrm{e}^{-y^2}\, dy \right) dx.$$

The i-face $F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}})$ is the i-simplex $\mathrm{conv}\{\lambda_{l_1} e_{l_1}, \dots, \lambda_{l_{i+1}} e_{l_{i+1}}\}$ with i-dimensional volume

$$\mathrm{V}_i\big(F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}})\big) = \frac{1}{i!}\sqrt{\mathrm{s}_i\big(\lambda_{l_1}^2, \dots, \lambda_{l_{i+1}}^2\big)}.$$

Since $C_n^*(\lambda_1, \dots, \lambda_n)$ has 2^{i+1} congruent i-faces of the type $F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}})$ we get by (2.1) the following formulae for the intrinsic volumes.

Corollary 2.1. *The intrinsic volumes of* $C_n^*(\lambda_1,\dots,\lambda_n)$ *are given by*

$$\mathrm{V}_n\big(C_n^*(\lambda_1,\dots,\lambda_n)\big)=\frac{2^n}{n!}\lambda_1\cdot\ldots\cdot\lambda_n,\quad \textit{and for } 0\leq i\leq n-1,$$

$$\mathrm{V}_i\big(C_n^*(\lambda_1,\dots,\lambda_n)\big)=\frac{2^n}{i!\pi^{(n-i)/2}}\sum_{1\leq l_1<\cdots<l_{i+1}\leq n}\left[\sqrt{\mathrm{s}_i\big(\lambda_{l_1}^2,\dots,\lambda_{l_{i+1}}^2\big)}\right.$$

$$\left.\int_0^\infty \mathrm{e}^{-x^2}\left(\prod_{j\notin\{l_1,\dots,l_{i+1}\}}\int_0^{\frac{x}{\lambda_j\sqrt{\sum_{k=1}^{i+1}\frac{1}{\lambda_{l_k}^2}}}}\mathrm{e}^{-y^2}dy\right)dx\right].$$

In particular, we have $\mathrm{V}_{n-1}\big(C_n^*(\lambda_1,\dots,\lambda_n)\big)=\dfrac{2^{n-1}}{(n-1)!}\sqrt{\mathrm{s}_{n-1}\big(\lambda_1^2,\dots,\lambda_n^2\big)}$.

3. Proofs of the main results

We start with proving upper and lower bounds on the intrinsic volumes $\mathrm{V}_i(K)$ in terms of the i-th elementary symmetric functions of the radii given by Definition 1.2.

Proof of Theorem 1.1. It is well known that the i-th mixed volume $\mathrm{V}_i(K,B_n)$ can be expressed as

$$\mathrm{V}_i(K,B_n)=\frac{\kappa_n}{\kappa_i}\int_{\mathcal{L}_i^n}\mathrm{V}_i(K|L)\,d\sigma(L),\tag{3.1}$$

where $\sigma(L)$ is the Haar measure on the set $\mathcal{L}_i^n$ such that $\sigma(\mathcal{L}_i^n)=1$ (see e.g. [8, Theorem 19.3.2]). As mentioned in the introduction the case $i=n$ of Theorem 1.1 was shown in [2, Theorem 2.2]. So we can conclude that for any $L\in\mathcal{L}_i^n$, since $K|L$ is an i-dimensional convex body it holds

$$\kappa_i\,\bar{\mathrm{r}}_1(K|L;L)\cdot\ldots\cdot\bar{\mathrm{r}}_i(K|L;L)\leq\mathrm{V}_i(K|L)\leq\kappa_i\,\overline{\mathrm{R}}_1(K|L)\cdot\ldots\cdot\overline{\mathrm{R}}_i(K|L),$$

with equality if and only if $K|L$ is an i-ball. By the definition of the radii $\overline{\mathrm{R}}_j(K)$ and $\bar{\mathrm{r}}_j(K)$ we have $\overline{\mathrm{R}}_j(K|L)\leq\overline{\mathrm{R}}_j(K)$ and $\bar{\mathrm{r}}_j(K|L;L)\geq\bar{\mathrm{r}}_j(K)$. Hence, in view of (3.1) we conclude

$$\kappa_n\bar{\mathrm{r}}_1(K)\cdot\ldots\cdot\bar{\mathrm{r}}_i(K)\leq\mathrm{V}_i(K,B_n)\leq\kappa_n\overline{\mathrm{R}}_1(K)\cdot\ldots\cdot\overline{\mathrm{R}}_i(K).$$

Since $\mathrm{V}_i(K)=\binom{n}{i}/\kappa_{n-i}\mathrm{V}_i(K,B_n)$ and on account of $\overline{\mathrm{R}}_j(K)\leq\overline{\mathrm{R}}_{j+1}(K)$, $\bar{\mathrm{r}}_j(K)\geq\bar{\mathrm{r}}_{j+1}(K)$ for $j=1,\dots,n-1$, we finally get

$$\mathrm{V}_i(K)\leq\frac{\kappa_n}{\kappa_{n-i}}\binom{n}{i}\overline{\mathrm{R}}_1(K)\cdot\ldots\cdot\overline{\mathrm{R}}_i(K)\leq\frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\big(\overline{\mathrm{R}}_1(K),\dots,\overline{\mathrm{R}}_n(K)\big),$$

$$\mathrm{V}_i(K)\geq\frac{\kappa_n}{\kappa_{n-i}}\binom{n}{i}\bar{\mathrm{r}}_1(K)\cdot\ldots\cdot\bar{\mathrm{r}}_i(K)\geq\frac{\kappa_n}{\kappa_{n-i}}\,\mathrm{s}_i\big(\bar{\mathrm{r}}_1(K),\dots,\bar{\mathrm{r}}_n(K)\big).$$

Obviously, equality holds in both inequalities if and only if K is a ball. □

Remark 3.1.

i) *In general there is no lower (upper) bound on* $\mathrm{V}_i(K)$ *in terms of the elementary symmetric functions of the outer radii* $\overline{\mathrm{R}}_i(K)$ *(inner radii* $\bar{\mathrm{r}}_i(K)$*), since for* $i = 2, \ldots, n$ $(i = 1, \ldots, n)$ $\mathrm{V}_i(K)$ *can be arbitrarily small (large) in comparison to the diameter (minimal width) of* K.

ii) *The lower bound in inequality* (1.3) *can be improved by replacing* $\bar{\mathrm{r}}_i(K)$ *by inner radii defined via projections, i.e.,* $\min_{L\in\mathcal{L}_i^n} \mathrm{r}(K|L;L)$. *The proof is the same, but for sake of simplicity we omit this series of inner radii.*

We also want to remark that concerning the so called *dual mixed volumes* $\widetilde{\mathrm{V}}_i(K)$ of $K \in \mathcal{K}^n$ (cf. e.g. [16], [8, §24]) one can get in the same way lower and upper bounds. Now the lower bound is given in terms of the inner radii $\bar{\mathrm{r}}_i(K)$, but for the upper bound one has to consider outer radii defined via sections: $\max_{L\in\mathcal{L}_i^n} \max_{x\in L^\perp} \mathrm{R}\big(K \cap (x+L)\big)$. Instead of (3.1) one has to use for the dual mixed volumes the integral representation (cf. e.g. [8, p. 158])

$$\widetilde{\mathrm{V}}_i(K) = \frac{\kappa_n}{\kappa_i} \int\int_{\mathcal{L}_i^n} \mathrm{V}_i(K \cap L)\, d\sigma(L).$$

Next we prove Theorem 1.2 providing an upper bound on $\mathrm{V}_i(K)$ in terms of the i-th elementary symmetric function of the outer radii given by Definition 1.1.

Proof of Theorem 1.2. It is well known that central symmetrization does not decrease the intrinsic volumes (cf. e.g. [5, p. 79]) and so we have $\mathrm{V}_i(K) \leq \mathrm{V}_i(K^0)$ for $0 \leq i \leq n$. By Lemma 2.1 we also know that $\mathrm{R}_i(K^0) \leq \mathrm{R}_i(K)$ and therefore, it suffices to prove the inequality for a 0-symmetric convex body $K \in \mathcal{K}^n$.

We now construct iteratively n pairwise orthogonal unit vectors $u_i \in \mathbb{S}^{n-1}$ such that for $1 \leq i \leq n$

$$K \subset \big\{x \in \mathbb{R}^n : \big|\langle u_j, x\rangle\big| \leq \mathrm{R}_j(K),\ 1 \leq j \leq i\big\}. \tag{3.2}$$

For $i = 1$ let $u_1 \in \mathbb{S}^{n-1}$ be the direction which determines the minimal width of K, i.e., $\omega(K) = \omega(K, u_1)$. Then $\mathrm{R}_1(K) = \omega(K)/2 = \omega(K, u_1)/2$ and obviously (3.2) is satisfied. In the i-th step, $i \geq 2$, let $L = \mathrm{lin}\{u_1, \ldots, u_{i-1}\}^\perp$, i.e., the orthogonal complement of the linear hull of the vectors $u_1, \ldots, u_{i-1}$. Moreover, let $L_i \in \mathcal{L}_i^n$ be such that $\mathrm{R}_i(K) = \mathrm{R}(K|L_i)$. Since $\dim L = n-i+1$ there exists a $u_i \in (L \cap L_i) \cap \mathbb{S}^{n-1}$. Thus the vectors $u_1, \ldots, u_i$ are pairwise orthogonal and by the definition of $\mathrm{R}_i(K)$ we have $K \subset \big\{x \in \mathbb{R}^n : \big|\langle u_j, x\rangle\big| \leq \mathrm{R}_j(K),\ 1 \leq j \leq i\big\}$ which shows (3.2).

Hence, after n steps K is contained in the orthogonal parallelepiped

$$P = \big\{x \in \mathbb{R}^n : \big|\langle u_j, x\rangle\big| \leq \mathrm{R}_j(K),\ 1 \leq j \leq n\big\}.$$

By the monotonicity of the intrinsic volumes (cf. e.g. [19, p. 277]) and since $\mathrm{V}_i(P)$ can easily be computed via formula (2.1), we finally get

$$\mathrm{V}_i(K) \leq \mathrm{V}_i(P) = 2^i\, \mathrm{s}_i\big(\mathrm{R}_1(K), \ldots, \mathrm{R}_n(K)\big),$$

which proves inequality (1.4).

To show that the bounds are in general best possible let $Q(\mu)$ be the orthogonal parallelepiped with edge-lengths $\mu, \mu^2, \ldots, \mu^n$, for $\mu \geq 1$. The outer radii R_i of such a box are given by $\mathrm{R}_i\big(Q(\mu)\big) = (1/2)\big(\sum_{j=1}^{i} \mu^{2j}\big)^{1/2}$ (see [6, Theorem 4.4]) and it follows

$$\frac{\mathrm{V}_i\big(Q(\mu)\big)}{\mathrm{s}_i\big(\mathrm{R}_1(Q(\mu)), \ldots, \mathrm{R}_n(Q(\mu))\big)} = 2^i \frac{\sum_{1\leq j_1<\cdots<j_i\leq n} \mu^{j_1} \cdot \ldots \cdot \mu^{j_i}}{\sum_{1\leq j_1<\cdots<j_i\leq n} \left(\prod_{k=1}^{i} \left(\sum_{l=1}^{j_k} \mu^{2l}\right)^{1/2}\right)}.$$

When $\mu \to \infty$, the right hand side tends to 2^i. □

As pointed out in the proof above it is sufficient to prove (1.4) for 0-symmetric convex bodies. Hence, on account of Lemma 2.1, Theorem 1.2 is equivalent (cf. Definition 2.1) to:

Corollary 3.1. *Let $K \in \mathcal{K}^n$. Then for $i = 0, \ldots, n$*

$$\mathrm{V}_i(K) \leq \mathrm{s}_i\big(\mathrm{D}_1(K), \ldots, \mathrm{D}_n(K)\big). \tag{3.3}$$

The bound is best possible.

Next we deal with lower bounds on the intrinsic volumes in terms of the inner radii $\mathrm{r}_i(K)$. In [2, Theorem 3.1] it is shown that for $i = 0, \ldots, n$,

$$\mathrm{V}_i(K) \geq \frac{1}{i!}\mathrm{D}_n(K) \cdot \ldots \cdot \mathrm{D}_{n-i+1}(K).$$

Since $\mathrm{D}_i(K) \leq \mathrm{D}_{i+1}(K)$ and $\mathrm{D}_j(K) \geq 2\,\mathrm{r}_{n-j+1}(K)$ (cf. [2, Lemma 2.1]) we find:

Remark 3.2. *Let $K \in \mathcal{K}^n$. Then for $i = 0, \ldots, n$*

$$\mathrm{V}_i(K) \geq \frac{2^i}{i!\binom{n}{i}}\,\mathrm{s}_i\big(\mathrm{r}_1(K), \ldots, \mathrm{r}_n(K)\big).$$

For $1 \leq i \leq n-1$, however, this bound is in general not best possible as already the 2-dimensional case and $i = 1$ shows. From the known inequality for $K \in \mathcal{K}^2$ giving the minimum value of $\mathrm{V}_1(K)$ for fixed $\mathrm{D}(K)$ and $\omega(K)$ [5, p. 87], it can be easily obtained that $\mathrm{V}_1(K) \geq \sqrt{\mathrm{D}(K)^2 - 2\mathrm{r}(K)^2} + 2\mathrm{r}(K)\arcsin\big(2\mathrm{r}(K)/\mathrm{D}(K)\big)$. Here the 0-symmetric cap-bodies given by the convex hull of a circle of radius $\mathrm{r}(K)$ and two diametrically opposite points exterior to it at distance apart $\mathrm{D}(K)$ give the equality. Then it is a simple computation to check that:

Remark 3.3. *Let $K \in \mathcal{K}^2$. Then*

$$\mathrm{V}_1(K) \geq c\left(\frac{\mathrm{D}(K)}{2} + \mathrm{r}(K)\right) = c\,\mathrm{s}_1\big(\mathrm{r}_1(K), \mathrm{r}_2(K)\big),$$

where $c = 2\arcsin t_0 = 1.478\ldots$ and t_0 is the unique solution of the equation $\arcsin t = \sqrt{1-t^2}$. Equality holds if and only if K is the 0-symmetric cap-body K^c with $2\mathrm{r}(K^c)/\mathrm{D}(K^c) = t_0$.

The cap-bodies of Remark 3.3 also show that a lower bound of the form $\mathrm{V}_i(K) \geq (2^i/i!)\,\mathrm{s}_i\big(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\big)$ does not exit in general. However, as it will be shown next, we can get an optimal lower bound at least on $\mathrm{V}_{n-1}(K)$ if we replace the i-th elementary symmetric function $\mathrm{s}_i\big(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\big)$ by $\sqrt{\mathrm{s}_i\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)}$.

Proof of Theorem 1.3. Without loss of generality we may assume $\dim K = n$. First we construct iteratively n pairs of points x_i, y_i, $1 \leq i \leq n$, such that

$$\text{(3.4)}\qquad \begin{aligned} &\text{i) } \|x_i - y_i\| = 2\,\mathrm{r}_i(K) \quad \text{and} \\ &\text{ii) } \{x_i - y_i : i = 1,\dots,n\} \text{ are pairwise orthogonal.}\end{aligned}$$

For $i = 1$ let $x_1, y_1 \in K$ with $\|x_1 - y_1\| = \mathrm{D}(K) = 2\mathrm{r}_1(K)$. In the i-th step, $i \geq 2$, let $L = \mathrm{lin}\{x_1 - y_1, \dots, x_{i-1} - y_{i-1}\}^{\perp}$ and let $L_i \in \mathcal{L}_i^n$, $z_i \in \mathbb{R}^n$, such that $\mathrm{r}_i(K) = \mathrm{r}\big(K \cap (z_i + L_i); z_i + L_i\big)$. Obviously, we have $\dim(L \cap L_i) \geq 1$ and let u be a non-trivial vector in $L \cap L_i$. By the definition of $\mathrm{r}_i(K)$ we can find two points $x_i, y_i \in K$ contained in the line $z_i + \mathrm{lin}\{u\}$ with $\|x_i - y_i\| = 2\,\mathrm{r}_i(K)$. Hence we have verified (3.4).

Now let $P = \mathrm{conv}\{x_i, y_i : i = 1,\dots,n\} \subset K$ and without loss of generality we may assume that $x_i - y_i = 2\,\mathrm{r}_i(K)\,e_i$. Applying successive Steiner symmetrizations (cf. [11, pp. 168]) with respect to the coordinate hyperplanes $\mathrm{lin}\{e_i\}^{\perp}$, $1 \leq i \leq n$, we transform P into a polytope P^s which is symmetric with respect to all coordinate hyperplanes and such that $\big\|P^s \cap \mathrm{lin}\{e_i\}\big\| \geq 2\,\mathrm{r}_i(K)$. Hence P^s contains the orthogonal cross-polytope $C_n^*\big(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\big)$. Since Steiner symmetrizations do not increase the intrinsic volumes (see e.g. [11, p. 171]) we get

$$\mathrm{V}_i(K) \geq \mathrm{V}_i\Big(C_n^*\big(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\big)\Big). \tag{3.5}$$

In the particular case $i = n-1$ the required lower bound in (1.6)

$$\mathrm{V}_{n-1}(K) \geq \frac{2^{n-1}}{(n-1)!}\sqrt{\mathrm{s}_{n-1}\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)}$$

is a direct consequence of (3.5) and of Corollary 2.1.

In order to show that this bound can not be improved in general we consider the orthogonal cross-polytope $C_n^*(\mu) := C_n^*(\mu, \mu^2, \dots, \mu^n)$, for $\mu > 1$. The inner radii r_i of such a cross-polytope are given by $\mathrm{r}_i\big(C_n^*(\mu)\big) = \big(\sum_{j=n-i+1}^{n} \mu^{-2j}\big)^{-1/2}$ (see [6, Theorem 4.4]) and we get

$$\frac{\mathrm{V}_{n-1}\big(C_n^*(\mu)\big)}{\sqrt{\mathrm{s}_{n-1}\big(\mathrm{r}_1(C_n^*(\mu))^2,\dots,\mathrm{r}_n(C_n^*(\mu))^2\big)}} = \frac{\frac{2^{n-1}}{(n-1)!}\sqrt{\sum_{i=1}^{n}\prod_{j\neq i}\mu^{2j}}}{\sqrt{\sum_{i=1}^{n}\prod_{j\neq i}\big(\sum_{k=n-j+1}^{n}\mu^{-2k}\big)^{-1}}}.$$

When $\mu \to \infty$, the right hand side tends to $2^{n-1}/(n-1)!$.

In the case $i = n-2$ the formula for $\mathrm{V}_{n-2}\big(C_n^*(\lambda_1,\dots,\lambda_n)\big)$ given in Corollary 2.1 can be rewritten as

$$\mathrm{V}_{n-2}\big(C_n^*(\lambda_1,\dots,\lambda_n)\big) = \frac{2^{n-2}}{\pi(n-2)!}\, g(\lambda_1,\dots,\lambda_n), \tag{3.6}$$

where

$$g(\lambda_1,\dots,\lambda_n) = \sum_{i=1}^{n} \sqrt{\mathrm{s}_{n-2}\big(\lambda_1^2,\dots,\widehat{\lambda_i^2},\dots,\lambda_n^2\big)}\, \arccos \frac{\left(\sum_{j=1}^n \frac{1}{\lambda_j^2}\right) - 2\frac{1}{\lambda_i^2}}{\sum_{j=1}^n \frac{1}{\lambda_j^2}}.$$

Here $\widehat{\lambda}$ means that we omit the value λ. By elementary but tedious calculations one can show that $f(\lambda_1,\dots,\lambda_n) = g(\lambda_1,\dots,\lambda_n)/\sqrt{\mathrm{s}_{n-2}\big(\lambda_1^2,\dots,\lambda_n^2\big)}$ attains its minimum when $\lambda_1 = \cdots = \lambda_n$, i.e., when $C_n^*(\lambda_1,\dots,\lambda_n)$ is a regular cross-polytope. Hence

$$f(\lambda_1,\dots,\lambda_n) \geq \sqrt{2n}\, \arccos \frac{n-2}{n},$$

and together with (3.6), (3.5) we obtain

$$\begin{aligned}\mathrm{V}_{n-2}(K) &\geq \mathrm{V}_{n-2}\Big(C_n^*\big(\mathrm{r}_1(K),\dots,\mathrm{r}_n(K)\big)\Big)\\ &\geq \frac{2^{n-2}}{(n-2)!}\frac{\sqrt{2n}}{\pi}\arccos\frac{n-2}{n}\sqrt{\mathrm{s}_{n-2}\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)}.\end{aligned}$$

A direct computation shows that $\sqrt{n}\,\arccos\big((n-2)/n\big) \geq 2$, which finally verifies inequality (1.7). □

We want to remark that it seems to be quite likely that

$$\mathrm{V}_i\big(C_n^*(\lambda_1,\dots,\lambda_n)\big)/\sqrt{\mathrm{s}_i\big(\lambda_1^2,\dots,\lambda_n^2\big)}$$

is minimized when all λ_i coincide, i.e., for a regular cross-polytope. This would immediately lead to an extension of the bounds given in Theorem 1.3 to all other intrinsic volumes. We think, however, that the right lower bound on $\mathrm{V}_i(K)$ is given by $(2^i/i!)\sqrt{\mathrm{s}_i\big(\mathrm{r}_1(K)^2,\dots,\mathrm{r}_n(K)^2\big)}$. The orthogonal cross-polytopes $C_n^*(\mu,\mu^2,\dots,\mu^n)$, μ large, show that this bound would be best possible (cf. Corollary 2.1).

Finally, at the end of this section we apply our bounds on the intrinsic volumes to the elementary symmetric functions of the roots of the Steiner polynomial. The problem to bound the roots in terms of the in- circumradius or more generally in terms of successive inner and outer radii was the starting point of our investigations. For more information on the roots of Steiner polynomials, their locations and their sizes we refer to [13, 14, 15].

Let γ_i, $i = 1,\dots,n$, be the roots of the Steiner polynomial

$$f(K,s) = \sum_{i=0}^{n} \kappa_{n-i}\mathrm{V}_i(K)s^{n-i}$$

regarded as a formal polynomial in a complex variable $s \in \mathbb{C}$. From the identity $\sum_{i=0}^{n} \kappa_{n-i} \mathrm{V}_i(K) s^{n-i} = \kappa_n \prod_{i=1}^{n} (s - \gamma_i)$ we get

$$(-1)^i \frac{\kappa_{n-i}}{\kappa_n} \mathrm{V}_i(K) = \mathrm{s}_i\big(\gamma_1, \dots, \gamma_n\big),$$

and so the inequalities of Theorem 1.1 and Theorem 1.2 imply:

Corollary 3.2. *Let $K \in \mathcal{K}^n$ and γ_j, $j = 1, \dots, n$ be the roots of $f(K, s)$. Then for $i = 0, \dots, n$*

$$\mathrm{s}_i\big(\gamma_1, \dots, \gamma_n\big) \le (-1)^i \, \mathrm{s}_i\big(\overline{\mathrm{R}}_1(K), \dots, \overline{\mathrm{R}}_n(K)\big),$$

$$\mathrm{s}_i\big(\gamma_1, \dots, \gamma_n\big) \ge (-1)^i \, \mathrm{s}_i\big(\overline{\mathrm{r}}_1(K), \dots, \overline{\mathrm{r}}_n(K)\big),$$

$$\mathrm{s}_i\big(\gamma_1, \dots, \gamma_n\big) \le (-2)^i \frac{\kappa_{n-i}}{\kappa_n} \, \mathrm{s}_i\big(\mathrm{R}_1(K), \dots, \mathrm{R}_n(K)\big).$$

In the cases $i = n-1, n-2$ lower bounds for the elementary symmetric functions of the roots γ_j in terms of the inner radii $\mathrm{r}_j(K)$ can be obtained using Theorem 1.3.

4. Intrinsic volumes of orthogonal cross-polytopes

In order to give the proof of Lemma 2.2 we need some more notation. For a polytope $P \in \mathcal{K}^n$ and an i-face $F \in \mathcal{F}_i(P)$ let $N(F, P)$ be the *normal cone* of P in F, i.e., the positive hull of all outer unit normal vectors of the supporting hyperplanes of F, embedded in the Euclidean space $\mathbb{R}^{n-i}$. The *external angle* of F, denoted by $\gamma(F, P)$, is the $(n-i-1)$-dimensional spherical measure of $N(F, P) \cap \mathbb{S}^{n-i-1}$ divided by $(n-i)\kappa_{n-i}$, i.e., the total spherical measure of the $(n-i-1)$-dimensional unit sphere (cf. e.g. [19, pp. 98–100]).

Proof of Lemma 2.2. Let $i \in \{0, \dots, n-1\}$ and let $F^i(\lambda_{l_1}, \dots, \lambda_{l_{i+1}}) = \mathrm{conv}\{\lambda_{l_1} e_{l_1}, \dots, \lambda_{l_{i+1}} e_{l_{i+1}}\}$ be an i-face of $C_n^*(\lambda_1, \dots, \lambda_n)$, $1 \le l_1 < \dots < l_{i+1} \le n$. For the sake of brevity we will denote that i-face just by F^i, its normal cone by $N(F^i)$ and the external angle by $\gamma(F^i)$.

The 2^{n-i-1} outer normal vectors of the supporting hyperplanes of the facets containing F^i are given by

$$\left\{ \sum_{j \notin \{l_1, \dots, l_{i+1}\}} \frac{\varepsilon_j}{\lambda_j} \, e_j + \sum_{k=1}^{i+1} \frac{1}{\lambda_{l_k}} \, e_{l_k} : \varepsilon_j \in \{-1, 1\} \right\},$$

and the normal cone $N(F^i)$ is the positive hull of these vectors. Using polar coordinates we find (cf. [3])

$$\begin{aligned} \int_{N(F^i)} \mathrm{e}^{-\|\mathrm{x}\|^2} \, d\mathrm{x} &= \gamma(F^i)(n-i)\kappa_{n-i} \int_0^\infty \mathrm{e}^{-r^2} r^{n-i-1} \, dr \\ &= \gamma(F^i)(n-i)\kappa_{n-i} \, \frac{1}{2} \Gamma\left(\frac{n-i}{2}\right) = \gamma(F^i) \pi^{(n-i)/2}. \end{aligned} \tag{4.1}$$

In order to evaluate the integral on the left hand side let $U=\big\{\mathrm{x}\in\mathbb{R}^{n-i}: x_{l_1}\geq 0,\ |x_j|\leq x_{l_1},\ j\notin\{l_1,\dots,l_{i+1}\}\big\}$ and let $f:U\longrightarrow N(F^i)$ be the linear and bijective map defined as

$$f(\mathrm{x})=\sum_{j\notin\{l_1,\dots,l_{i+1}\}}\frac{x_j}{\lambda_j}e_j+x_{l_1}\sum_{k=1}^{i+1}\frac{1}{\lambda_{l_k}}e_{l_k}.$$

By this parametrization of the normal cone $N(F^i)$ we get

$$\int_{N(F^i)}\mathrm{e}^{-\|\mathrm{x}\|^2}d\mathrm{x}=\frac{\sqrt{\sum_{k=1}^{i+1}\frac{1}{\lambda_{l_k}^2}}}{\prod_{j\notin\{l_1,\dots,l_{i+1}\}}\lambda_j}\int_U\mathrm{e}^{-\|f(\mathrm{x})\|^2}d\mathrm{x}.$$

Setting $\alpha_i=\sqrt{\sum_{k=1}^{i+1}1/\lambda_{l_k}^2}$ and denoting for short by $\prod_{*j}$ the product $\prod_{j\notin\{l_1,\dots,l_{i+1}\}}$ we obtain

$$\begin{aligned}\int_{N(F^i)}\mathrm{e}^{-\|\mathrm{x}\|^2}d\mathrm{x}&=\frac{\alpha_i}{\prod_{*j}\lambda_j}\int_U\mathrm{e}^{-\|f(\mathrm{x})\|^2}d\mathrm{x}\\&=\frac{\alpha_i}{\prod_{*j}\lambda_j}\int_U\mathrm{e}^{-\sum_{j\notin\{l_1,\dots,l_{i+1}\}}\frac{x_j^2}{\lambda_j^2}-\left(\sum_{k=1}^{i+1}\frac{1}{\lambda_{l_k}^2}\right)x_{l_1}^2}d\mathrm{x}\\&=\frac{\alpha_i}{\prod_{*j}\lambda_j}\int_0^\infty\mathrm{e}^{-\alpha_i^2t^2}\left(\prod_{*j}\int_{-t}^t\mathrm{e}^{-\frac{x_j^2}{\lambda_j^2}}dx_j\right)dt\\&=\frac{2^{n-i-1}\alpha_i}{\prod_{*j}\lambda_j}\int_0^\infty\mathrm{e}^{-\alpha_i^2t^2}\left(\prod_{*j}\int_0^t\mathrm{e}^{-\frac{x_j^2}{\lambda_j^2}}dx_j\right)dt.\end{aligned}$$

Making the changes of variable $x=\alpha_i\,t$ and $y=x_j/\lambda_j$ for $j\notin\{l_1,\dots,l_{i+1}\}$, yields by (4.1) the desired formula. □

Acknowledgements. This work has been developed during a research stay of the second author at the Otto-von-Guericke Universität Magdeburg, Germany, supported by Programa Nacional de ayudas para la movilidad de profesores de universidad e investigadores españoles y extranjeros (MEC), Ref. PR2006-0351.

References

[1] K. Ball, Ellipsoids of maximal volume in convex bodies, *Geom. Dedicata* **41** (1992), 241-250.
[2] U. Betke, M. Henk, Estimating sizes of a convex body by successive diameters and widths, *Mathematika* **39** (2) (1992), 247–257.
[3] U. Betke and M. Henk, Intrinsic volumes and lattice points of crosspolytopes, *Monatsh. Math.* **115** (1–2) (1993), 27–33.

[4] U. Betke, M. Henk, A generalization of Steinhagen's theorem, *Abh. Math. Sem. Univ. Hamburg* **63** (1993), 165-176.
[5] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*. Springer, Berlin, 1934, 1974. English translation: *Theory of convex bodies*. Edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, ID, 1987.
[6] R. Brandenberg, Radii of regular polytopes, *Discrete Comput. Geom.* **33** (1) (2005), 43–55.
[7] R. Brandenberg, T. Theobald, Radii minimal projections of polytopes and constrained optimization of symmetric polynomials, *Adv. Geom.* **6** (1) (2006), 71–83.
[8] J. D. Burago, V. A. Zalgaller, *Geometric Inequalities*. Springer, Berlin, 1988; (Russian original: 1980).
[9] P. Gritzmann, V. Klee, Inner and outer j-radii of convex bodies in finite-dimensional normed spaces, *Discrete Comput. Geom.* **7** (1992), 255-280.
[10] P. Gritzmann, V. Klee, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces, *Math. Program.* **59** (1993) 163–213.
[11] P. M. Gruber, *Convex and Discrete Geometry*. Springer, Berlin Heidelberg, 2007.
[12] M. Henk, A generalization of Jung's theorem, *Geom. Dedicata* **42** (1992), 235–240.
[13] M. Henk, M. A. Hernández Cifre, Notes on the roots of Steiner polynomials, *Preprint*, `arXiv:math/0703373`.
[14] M. A. Hernández Cifre, E. Saorín, On the roots of the Steiner polynomial of a 3-dimensional convex body, *Adv. Geom.* **7** (2007), 275–294.
[15] V. Katsnelson, On H. Weyl and H. Minkowski polynomials, *Preprint*, `arXiv:math/0702139v2`.
[16] E. Lutwak, Dual mixed volumes, *Pacific J. Math.* **58** (2) (1975), 531–538.
[17] G. Ya. Perel'man, On the k-radii of a convex body, (Russian) *Sibirsk. Mat. Zh.* **28** (4) (1987), 185–186. English translation: *Siberian Math. J.* **28** (4) (1987), 665–666.
[18] S. V. Puhov, Inequalities for the Kolmogorov and Bernšteĭn widths in Hilbert space, (Russian) *Mat. Zametki* **25** (4) (1979), 619–628, 637. English translation: *Math. Notes* **25** (4) (1979), 320–326.
[19] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge, 1993.
[20] B. Teissier, Bonnesen-type inequalities in algebraic geometry I. Introduction to the problem. *Seminar on Differential Geometry*, Princeton Univ. Press, Princeton, N. J., 1982, 85–105.

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