Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Károly Böröczky, Jr. and Martin Henk

Random Projections of Regular Polytopes

Preprint SC 97-21 (May 1997)

RANDOM PROJECTIONS OF REGULAR POLYTOPES

KÁROLY BÖRÖCZKY, JR. AND MARTIN HENK

Abstract. Based on an approach of Affentranger&Schneider we give an asymptotic formula for the expected number of k-faces of the orthogonal projection of a regular n-crosspolytope onto a randomly chosen isotopic subspace of fixed dimension, as n tends to infinity. In particular, we present a precise asymptotic formula for the (spherical) volume of spherical regular simplices, which generalizes Daniel's formula.

1. Introduction

Throughout the paper the n-dimensional Euclidean space equipped with inner product $\langle \cdot, \cdot \rangle$ is denoted by $\mathbb{R}^n$. For a polytope $P \subset \mathbb{R}^n$ the set of all k-faces is denoted by $\mathcal{F}_k(P)$ and its cardinality by $f_k(P)$, i.e., $f_k(P) = \#\mathcal{F}_k(P)$. The expected value of k-faces of an orthogonal projection of an n-dimensional polytope P onto a randomly chosen d-dimensional linear subspace with isotropic distribution is denoted by $E(f_k(\Pi_d P))$, $1 \leq d \leq n-1$, $0 \leq k \leq d-1$. It was proved by Affentranger&Schneider [AS92] that

$$E(f_k(\Pi_d P)) = 2\sum_{s\geq 0}\sum_{F\in\mathcal{F}_k(P)}\sum_{G\in\mathcal{F}_{d-1-2s}(P)} \beta(F,G)\gamma(G,P), \tag{1.1}$$

or, equivalently,

$$E(f_k(\Pi_d P)) = f_k(P) - 2\sum_{s\geq 0}\sum_{F\in\mathcal{F}_k(P)}\sum_{G\in\mathcal{F}_{d+1+2s}(P)} \beta(F,G)\gamma(G,P), \tag{1.2}$$

where $\beta(F,G)$ and $\gamma(G,F)$ denote the internal and external angle of G at its face F, respectively (cf. [Grü67]). By definition the internal or external angles are spherical volumes and therefore, in general, it is impossible to give an explicit formula of them. However, it was shown by Ruben [Rub60] (see also [Had79]) that for a regular n-simplex $T^n \subset E^n$

$$\gamma(T^k, T^n) = \sqrt{\frac{k+1}{\pi}} \int_{-\infty}^{\infty} e^{-(k+1)x^2} \left(\frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} dy \right)^{n-k} dx.$$

The research of the first author has been supported by the Magyary Zoltán Fund, by OTKA 14220, and by the Volkswagen Stiftung.

The work of the second author is supported by a "Leibniz Preis" of the German Science Foundation (DFG) awarded to M. Grötschel.

Using this representation Affentranger&Schneider established the asymptotic formula (cf. [AS92])

$$E(f_k(\Pi_d T^n)) \sim \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta(T^k,T^{d-1})(\pi\ln n)^{(d-1)/2}, \quad n\to\infty. \tag{1.3}$$

This formula still involves the "unknown" internal angles $\beta(T^k,T^{d-1})$. For $k=0$ an asymptotic formula of $\beta(T^0,T^{d-1})$, $d\to\infty$, was given by Daniel in the context of densest sphere packings (cf. [Rog64]). In Section 2 we generalize the approach of Daniel and prove (see Corollary 2.1)

$$\beta(T^k,T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-3}{2}}}{\sqrt{2}^{d-k}\sqrt{\pi}^{d-k-1} d^{\frac{d-k-2}{2}}}\cdot\left(1+O\left(\frac{k^2+1}{d}\right)\right).$$

Beside the regular simplex there are two more regular polytopes in arbitrary dimensions, namely the n-cube W^n and the regular n-crosspolytope C^n. It is easy to see that for $F\in\mathcal{F}_k(W^n)$, $G\in\mathcal{F}_l(W^n)$ with $F\subset G$ we have $\gamma(F,W^n)=(1/2)^{n-k}$ and $\beta(F,G)=\beta(F,W^l)=(1/2)^{l-k}$. Furthermore, $f_k(W^n)=2^{n-k}\binom{n}{k}$ and the number of l-faces containing a fixed k-face is equal to $\binom{n-k}{l-k}$. Thus we get by (1.1)

$$E(f_k(\Pi_d W^n)) = 2\binom{n}{k}\sum_{s\geq 0}\binom{n-k}{d-1-2s-k} \sim 2\frac{n^{d-1}}{(d-1-k)!k!}.$$

In Section 3 we complete the determination of $E(f_k(\Pi_d P))$ for regular polytopes by proving for the regular n-crosspolytope C^n

Theorem 1.1. *For any given integers* $0\leq k<d\leq n-1$,

$$E(f_k(\Pi_d C^n)) \sim \frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta(T^k,T^{d-1})(\pi\ln n)^{(d-1)/2}.$$

as n tends to infinity.

Observe that the asymptotic value of the expected number of k–faces is the same for the cross polytope C^n as for T^n in (1.3). At the moment, we are not aware of any direct argument leading to this coincidence. We note the following consequence of the proof of the estimates:

Corollary 1.1. *For fixed $k\in\mathbb{N}$, if $d/k^2\to\infty$ and $n/d\to\infty$ then*

$$E(f_k(\Pi_d T^n)) \sim E(f_k(\Pi_d C^n)) \sim \frac{\sqrt{2}^{d+k}\sqrt{\pi}^k (k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-3}{2}}}{(k+1)!\,d^{\frac{d-3k-3}{2}}}\cdot(\ln n)^{\frac{d-1}{2}}.$$

We remark that by a result of Baryshnikov&Vitale, $E(f_k(\Pi_d T^n))$ coincides with the expected number of k-faces of a standard Gaussian sample in d-space (cf. [BV94]). They prove actually more: Let $v^1,\dots,v^m$ be vectors in $\mathbb{R}^n$ with the same positive length such that each $\langle v^i,v^j\rangle$, $i\neq j$, equals to the same non-positive value. Then the orthogonal projection of a random rotation of $v^1,\dots,v^m$ onto $\mathbb{R}^d$, up to an independent affine transformation coincides in distribution with a standard Gaussian sample of m points in

$\mathbb{R}^d$. If $\langle v^i, v^j\rangle = 0$ then the affine transformation can be chosen to be linear. We deduce choosing $m = n$ that

Remark. *$E(f_k(\Pi_d C^n))$ coincides with the expected number of k-faces of a standard Gaussian sample of n pairs $x^1, -x^1, \dots, x^n, -x^n$ of points in $\mathbb{R}^d$.*

Finally, in Section 4 we give a list of some numerical computations of $E(f_k(\Pi_d C^n))$.

2. Spherical volumes of regular simplices

Let $B^n, S^{n-1} \subset \mathbb{R}^n$ be denote the n-dimensional unit ball and n-dimensional unit sphere, respectively. For $\alpha \in (0,1)$ a regular n-cone of angle α is defined as the positive hull of n unit vectors $a^1, \dots, a^n \in S^{n-1}$ satisfying

$$\langle a^i, a^j\rangle = \alpha, \quad i \neq j.$$

It is denoted by $\sigma(\alpha, n)$ and $T_*^{n-1}(\alpha) = \sigma(\alpha, n) \cap S^{n-1}$ is a regular spherical $(n-1)$-simplex of angle α.

By definition, the internal angle $\beta(F, G)$ is the "fraction" of the linear hull of $G - x^F$ taken up by the cone (positive hull) $\text{pos}\{G - x^F\}$, where x^F is a relative interior point of the face F. Now, it is easy to check that for $F = T^k$ and $G = T^{d-1}$, $k < d$, the cone $\text{pos}\{G - x^F\}$ can be written as a direct sum of $\text{lin}\{F - x^F\}$ and a regular $(d-k)$-dimensional cone $\sigma(1/(k+2), d-k)$ of angle $1/(k+2)$. Thus

$$\beta(T^k, T^{d-1}) = \frac{V_*^{d-k-1}\left(T_*^{d-k-1}\left(\frac{1}{k+2}\right)\right)}{V_*^{d-k-1}(S^{d-k-1})}, \tag{2.1}$$

where $V_*^{d-k-1}(\cdot)$ denotes the spherical volume w.r.t. S^{d-k-1}, and the internal angle $\beta(T^k, T^{d-1})$ may be regarded as the normalized spherical volume of a regular $(d-k-1)$-simplex of angle $1/(k+2)$. In the following we will study the asymptotic behavior of the normalized volume of an arbitrary regular spherical simplex. To this end, for $n \in \mathbb{N}$ and $\alpha \in (0,1)$ let

$$\tau(\alpha, n) = \frac{V_*^{n-1}\left(T_*^{n-1}(\alpha)\right)}{V_*^{n-1}(S^{n-1})}.$$

The asymptotic behavior $(n \to \infty)$ of $\tau(\alpha, n)$ for the special case $\alpha = 1/2$ has been investigated by Daniel (see [Rog64]). He proved

$$\tau(1/2, n) \sim \frac{e^{n/2-1}}{\sqrt{2}^{n+1}\sqrt{n}^{n-1}\sqrt{\pi}^{n}}.$$

Here we show the generalization

Lemma 2.1. *Let $0 < \alpha < 1$. If n tends to infinity then*

$$\tau(\alpha, n) = \sqrt{\frac{1-\alpha}{\alpha}}^{\,n-1} \frac{e^{\frac{n+2}{2} - \frac{1}{\alpha}}}{\sqrt{2}^{n+1}\sqrt{\pi}^{n}\sqrt{n}^{n-1}} \left(1 + O\left(\frac{1}{\alpha^2 n}\right)\right).$$

Proof. It is well known that $V_*^{n-1}(S^{n-1}) = n\pi^{n/2}/\Gamma(n/2+1)$, where $\Gamma(\cdot)$ denotes the Γ-function. Hence (cf. [Had79])

$$\int_{\sigma(\alpha,n)} e^{-\langle x,x\rangle}dx = \tau(\alpha,n)V_*^{n-1}(S^{n-1})\int_0^\infty e^{-r^2}r^{n-1}dr = \tau(\alpha,n)\pi^{n/2}. \tag{2.2}$$

Now let $a^1,\dots,a^n \in S^{n-1}$ such that $\sigma(\alpha,n) = \mathrm{pos}\{a^1,\dots,a^n\}$ and let A be the $n\times n$-matrix with columns a^i. Calculating the volume of the simplex $\mathrm{conv}\{0,a^1,\dots,a^n\}$ yields that $\det A = \sqrt{1-\alpha+\alpha n}\sqrt{1-\alpha}^{\,n-1}$. Applying the linear transformation $x = Ay$ to the integral on the left hand side of (2.2) gives

$$\tau(\alpha,n) = \frac{\sqrt{(1-\alpha+\alpha n)(1-\alpha)^{n-1}}}{\sqrt{\pi}^n}\int_0^\infty\cdots\int_0^\infty e^{-\langle Ay,Ay\rangle}\,dy_1\dots dy_n.$$

As $\langle Ay,Ay\rangle = \sum_{i=1}^n y_i^2 - 2\alpha\sum_{1\le i<j\le n} y_iy_j$ the substitution $z=\sqrt{\alpha}y$ leads to

$$\begin{aligned}\tau(\alpha,n) = {}&\frac{\sqrt{(1-\alpha+\alpha n)(1-\alpha)^{n-1}}}{\sqrt{\alpha\pi}^n}\times\\ &\int_0^\infty\cdots\int_0^\infty e^{-\theta\sum_{i=1}^n z_i^2-\left(\sum_{i=1}^n z_i\right)^2}\,dz_1\dots dz_n\end{aligned} \tag{2.3}$$

with $\theta = (1-\alpha)/\alpha$. Let $\Phi(n)$ be denote the integral on the right hand side. In the following we give an asymptotic formula for $\Phi(n)$ as n tends to infinity. To this end we fix an $s\in\mathbb{R}$. Since $\int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{\pi}$ integrating along the line $t-si$, $t\in(-\infty,\infty)$, shows

$$\sqrt{\pi}e^{-s^2} = \int_{-\infty}^\infty e^{-t^2+2its}\,dt.$$

We deduce

$$\begin{aligned}\Phi(n) &= \frac{1}{\sqrt{\pi}}\int_0^\infty\cdots\int_0^\infty\int_{-\infty}^\infty e^{-\theta\sum_{i=1}^n z_i^2-t^2+2it\sum_{i=1}^n z_i}\,dtdz_1\dots dz_n\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-t^2}\left(\int_0^\infty e^{-\theta s^2+2its}\,ds\right)^n\,dt.\end{aligned} \tag{2.4}$$

Next we observe that the function $g(t) = e^{-t^2}(\int_0^\infty e^{-\theta s^2+2its}\,ds)^n$ regarded as complex function in the variable $t=\nu+i\chi$ is an entire function. Since

$$|g(\nu+i\chi)| \le e^{-\nu^2+\chi^2}\left(\int_0^\infty e^{-\theta s^2}ds\right)^n = (2\sqrt{\pi})^{-n}e^{-\nu^2+\chi^2}$$

for $\chi\ge 0$, the function $g(\nu+i\chi)$ tends to zero as ν tends to infinity for any fixed $\chi\ge 0$. Thus we may replace the integration $\int_{-\infty}^\infty g(t)dt$ along the real line with respect to t by integration along the line $\nu + \chi i$, $\nu \in (-\infty, \infty)$, $\chi = \sqrt{n/2}$, and get

$$\begin{aligned} \Phi(n) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\nu^2+\chi^2-2\chi\nu i} \left(\int_0^{\infty} e^{-\theta s^2 - 2\chi s + 2\nu s i} \, ds \right)^n \\ &= \frac{e^{n/2}}{\sqrt{\pi}(2n)^{n/2}} \int_{-\infty}^{\infty} e^{-\nu^2} \left(2\chi e^{-i\nu/\chi} \int_0^{\infty} e^{-\theta s^2 - 2\chi s + 2\nu s i} \, ds \right)^n d\nu. \end{aligned} \tag{2.5}$$

Let $\Psi(n, \nu)$ be denote the expression taken to the n^{th} power (in the parentheses) on the right hand side. Integration by parts yields that

$$\begin{aligned} |\Psi(n,\nu)| &\leq 2\chi \int_0^{\infty} e^{-\theta s^2 - 2\chi s} \, ds \\ &= 2\chi \left[\frac{1}{2\chi} - \int_0^{\infty} \left(2\theta s \, e^{-\theta s^2} \right) \left(\frac{1}{2\chi} e^{-2\chi s} \right) ds \right] < 1. \end{aligned}$$

Since $e^{-\nu^2} < e^{-\nu}$ for $\nu > 1$, we have

$$\int_{\ln n}^{\infty} e^{-\nu^2} |\Psi(n,\nu)|^n \, d\nu < \frac{1}{n}. \tag{2.6}$$

Therefore let us assume that $|\nu| < \ln n$. By the corresponding formula in [Rog64] we deduce that, if the real part of $w \in \mathbb{C}$ is positive then

$$\int_0^{\infty} e^{-\theta s^2 - ws} \, ds = \frac{1}{w} - \frac{2\theta}{w^3} + O\left(\frac{\theta^2}{|w|^5} \right).$$

Applying this with $w = 2\chi - 2\nu i$ to $\Psi(n, t)$ yields that

$$\begin{aligned} \Psi(n,\nu) &= 2\chi \left[1 - \frac{i\nu}{\chi} - \frac{\nu^2}{n} + \frac{\nu^3 i}{6\chi^3} + O\left(\frac{\nu^4}{n^2} \right) \right] \times \\ &\quad \left[\frac{1}{2\chi - 2\nu i} - \frac{2\theta}{(2\chi - 2\nu i)^3} + O\left(\frac{\theta^2}{n^{5/2}} \right) \right] \\ &= 1 - \frac{\theta}{n} - \frac{\nu^2}{n} - \left(2\theta + \frac{2}{3}\nu^2 \right) \frac{\sqrt{2}\nu i}{n\sqrt{n}} + O\left(\frac{(1+\theta^2)(1+\nu^4)}{n^2} \right). \end{aligned}$$

Observe that $\Phi(n)$ is a real number, so no term of order $1/(n\sqrt{n})$ shows up in its expansion. We conclude by (2.5) and (2.6) that

$$\begin{aligned} \Phi(n) &= \frac{e^{n/2}}{\sqrt{\pi}(2n)^{n/2}} \times \\ &\quad \left(\int_{-\ln n}^{\ln n} e^{-2\nu^2 - \theta} \left(1 + O\left(\frac{(1+\theta^2)(1+\nu^4)}{n} \right) \right) d\nu + O\left(\frac{1}{n} \right) \right) \\ &= \frac{e^{(n/2)-\theta}}{\sqrt{2}(2n)^{n/2}} \left(1 + O\left(\frac{1+\theta^2}{n} \right) \right). \end{aligned}$$

Finally, the assertion follows by (2.3). □

By (2.1) and Lemma 2.1 we conclude

Corollary 2.1.

$$\beta(T^k, T^{d-1}) = \frac{(k+1)^{\frac{d-k-2}{2}} e^{\frac{d-3k-3}{2}}}{\sqrt{2}^{d-k}\sqrt{\pi}^{d-k-1} d^{\frac{d-k-2}{2}}} \cdot \left(1 + O\left(\frac{k^2+1}{d}\right)\right).$$

3. Random projections of regular cross polytopes

In order to proof Theorem 1.1, we need the following statement about the the asymptotic behavior of the external angles $\gamma(F^k, C^n)$ of a k-face of a regular n-crosspolytope C^n.

Lemma 3.1. *If n tends to infinity then*

$$\gamma(F^k, C^n) \sim \frac{1}{2} \frac{(k+1)!}{\sqrt{k+1}} \frac{(\pi \ln(n))^{k/2}}{n^{k+1}}.$$

Proof. The proof follows a proof of Vershik&Sporyshev [VS86] (see also [Ray70]) and is based on the following formula for $\gamma(F^k, C^n)$ (cf. [BH92])

$$\gamma(F^k, C^n) = \sqrt{\frac{k+1}{\pi}} \int_0^\infty e^{-(k+1)x^2} \left(\frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy\right)^{n-k-1} dx. \tag{3.1}$$

Let $\Phi(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2} dy$. By the substitution $x = \Phi^{-1}(1-u)$ we get

$$\gamma(F^k, C^n) = \frac{\sqrt{k+1}}{2} \int_0^1 e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du.$$

Let

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2} \int_0^{1/2} e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du,$$
$$\xi_2(k,n) = \frac{\sqrt{k+1}}{2} \int_{1/2}^1 e^{-k(\Phi^{-1}(1-u))^2} (1-u)^{n-k-1} du.$$

Obviously, we have

$$0 \le \xi_2(k,n) \le \frac{\sqrt{k+1}}{2} \left(\frac{1}{2}\right)^{n-k} \tag{3.2}$$

and in the following we investigate the function $\xi_1(k,n)$. It is well known that $\Phi(x) = 1 - [e^{-x^2}/(x\sqrt{\pi})](1 + O(x^{-2}))$ as $x \to \infty$ (cf. [VS86]) and thus

$$u = \frac{e^{-x^2}}{x\sqrt{\pi}}(1 + O(x^{-2})), \quad u \to 0.$$

Taking the logarithm yields

$$x^2 = -\ln(u) - \ln(x) - \ln(\sqrt{\pi}) + \ln(1 + O(x^{-2})), \tag{3.3}$$

and so $x = \sqrt{-\ln(u)(1+O(x^{-1}))}$. Replacing this expression in (3.3) gives

$$x^2 = -\ln(u) - \frac{1}{2}\ln(-\ln(u)) - \frac{1}{2}\ln(1+O(x^{-1})) - \ln(\sqrt{\pi}) + \ln(1+O(x^{-2})).$$

Hence we may write

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \int_0^{1/2} u^k |\ln(u)|^{k/2}(1-u)^{n-k-1}(1+O(x^{-1}))du.$$

Observe, that $x \geq \Phi^{-1}(1/2)$. Applying the substitution $1-u = e^{-v}$ yields

$$\begin{aligned} \xi_1(k,n) &= \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \times \\ & \int_0^{-\ln(\frac{1}{2})} (1-e^{-v})^k |-\ln(1-e^{-v})|^{k/2} e^{-(n-k)v}(1+O(x^{-1}))dv. \end{aligned}$$

Noting that $(1-e^{-v})^k = v^k(1+O(v))$ and $|\ln(1-e^{-v})|^{k/2} = |\ln(v)|^{k/2}(1+O(v))$ as $v \to 0$ we get

$$\xi_1(k,n) = \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k \int_0^{1/2} v^k |\ln(v)|^{k/2} e^{-(n-k)v}(1+O(x^{-1}))(1+O(v))dv.$$

The asymptotic behavior of such an integral was explicitly determined by Watson (cf. [VS86]) and applying that result gives

$$\begin{aligned} \xi_1(k,n) &\sim \frac{\sqrt{k+1}}{2}\sqrt{\pi}^k (n-k)^{-(k+1)}(\ln(n-k))^{k/2} k!(1+O(\ln(n)^{-1})) \\ &\sim \frac{1}{2}\frac{(k+1)!}{\sqrt{k+1}}\frac{(\pi\ln(n))^{k/2}}{n^{k+1}}. \end{aligned}$$

Together with (3.2) this shows $\gamma(F^k, C^n) \sim \xi_1(k,n)$ as n tends to infinity. □

Now we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $F \in \mathcal{F}_k(C^n)$ and $G \in \mathcal{F}_l(C^n)$ with $F \subset G$. Since every l-face, $l < n$, of a regular n-crosspolytope is a regular simplex we get $\beta(F,G) = \beta(T^k, T^l)$. Furthermore, we have $f_k(C^n) = 2^{k+1}\binom{n}{k+1}$ (cf. [HRZ97]) and the number of l-faces containing a fixed k-face is equal to $2^{l-k}\binom{n-k-1}{l-k}$. By (1.1) we get

$$\begin{aligned} E(f_k(\Pi_d C^n)) = 2\sum_{s\geq 0} 2^{d-2s}\binom{n}{d-2s} \times \\ \binom{d-2s}{k+1}\beta(T^k, T^{d-1-2s})\gamma(T^{d-1-2s}, C^n) \end{aligned} \tag{3.4}$$

and by Lemma 3.1 we obtain for $n \to \infty$

$$2\binom{n}{d-2s}\gamma(T^{d-1-2s}, C^n) \sim \frac{(\pi\ln(n))^{(d-2s-1)/2}}{\sqrt{d-2s}}.$$

The number of nonzero summands in the sum does not depend on n and since the sum is dominated by the term $s = 0$, we obtain Theorem 1.1. □

4. Remarks

As an easy application of (1.2) we determine the probability that the orthogonal projection of C^n onto a randomly chosen $(n-1)$-dimensional plane has $2n$ vertices. Let this probability be denoted P_{2n} and let v be a vertex of C^n. By a result of McMullen [McM75] we have the angle-sum relation

$$\sum_{F \text{ is a face of } C^n} \beta(v,F)\gamma(F,C^n) = 1.$$

Since $\gamma(v,C^n) = 1/(2n)$ this is equivalent to

$$\beta(v,C^n) = \frac{2n-1}{2n} - \sum_{j=1}^{n-1} 2^j \binom{n-1}{j} \beta(v,T^j)\gamma(T^j,C^n) \tag{4.1}$$

and by (1.2) we obtain $E(f_0(\Pi_{n-1}C^n)) = 2n(1-2\beta(v,C^n))$. Thus

$$\begin{aligned} (1-P_{2n})(2n-2) + 2nP_{2n} &= E(f_0(\Pi_{n-1}C^n)) = 2n(1-2\beta(v,C^n)) \\ \iff \qquad P_{2n} &= 1 - 2n\beta(v,C^n). \end{aligned} \tag{4.2}$$

In particular for $n = 3$ we have $\beta(v,T^1) = 1/2$, $\beta(v,T^2) = \arccos(1/2)/(2\pi)$ and $\gamma(T^1,C^3) = \arccos(1/3)/(2\pi)$, $\gamma(T^2,C^3) = 1/2$. Hence by (4.1) we get $\beta(v,C^3) = 1/2 - \arccos(1/3)/\pi$ and therefore (cf. (4.2))

$$P_6 = 1 - 6\beta(v,C^3) \sim 0.35095.$$

The next tables contain some numerical values of $E(f_k(\Pi_d C^n))$ using (3.4), (3.1), (2.4) and (2.3). The calculations were carried out by the program *Maple V Release 4*[1].

n	$\mathbf{d=2}$; $k=0$	$\mathbf{d=3}$; $k=0$	$k=1$	$k=2$
10	6.66	12.15	30.46	20.31
20	7.68	16.21	42.62	28.41
30	8.23	18.68	50.05	33.37
40	8.61	20.47	55.42	39.95
50	8.89	21.88	59.64	39.76
60	9.12	23.04	63.12	42.08
70	9.31	24.02	66.08	44.05
80	9.47	24.88	68.65	45.76
90	9.61	25.64	70.93	47.28
100	9.73	26.326	72.97	48.65

Table 1.

n	3	4	5	6	7	8	9	10
$\mathbf{d = n-1}$ $k = n-2$	4.70	10.67	23.61	51.40	110.54	233.57	498.46	1048.74

Table 2.

We would like to thank R. Vitale for helpful comments.

References

[AS92] F. Affentranger, R. Schneider, *Random projections of regular simplices*, Discrete&Computational Geometry, **7**, 1992, 219-226.

[BV94] Y.M. Baryshnikov and R.A. Vitale, *Regular Simplices and Gaussian Samples*, Discrete Comput. Geom. **11**, 1994, 141–147.

[BH92] U. Betke, M. Henk, *Intrinsic volumes and lattice points of crosspolytopes*, Monatsh. Mathematik, **115**, 1993, 27–33.

[Grü67] B. Grünbaum, *Convex polytopes*, Interscience, London, 1967.

[Had79] H. Hadwiger, *Gitterpunktanzahl im Simplex und Wills'sche Vermutung*, Math. Ann., **239**, 1979, 271-288

[HRZ97] M. Henk, J. Richter-Gebert, G.M. Ziegler, *Basic properties of convex polytopes*, CRC Handbook of Discrete und Computational Geometry, J.E. Goodman und J. O'Rourke eds., CRC Press, Boca Raton, Florida, 1997.

[McM75] P. McMullen, *Non-linear angle-sum relations for polyhedral cones and polytopes*, Math. Proc. Camb. Phil. Soc., **78** , 1975, 247-261.

[Ray70] H. Raynaud, *Sur l'enveloppe convexe des nuages de points aléatoires dans* $\mathbb{R}^n$, J. Appl. Probab. **7**, 1970, 35–48.

[Rog64] C.A. Rogers, *Packing and Covering*, Cambridge University Press, Cambridge, 1964.

[Rub60] H. Ruben, *On the geometrical moments of skew-regular simplices in hyperspherical space; with some applications in geometry and mathematical statistics*, Acta. Math. (Uppsala), **103**, 1960, 1–23.

[VS86] A.M. Vershik, P.V. Sporyshev, *An asymptotic estimate of the average number of steps of the parametric simplex method*, U.S.S.R. Comput. Math. and Math. Phys., **26**, 1986, 104-113. (Russian Orginal: Zh. Vychisl. Mat. i Mat. Fiz., **26**, 1986, 813-826).

Mathematical institute of the Hungarian academy of sciences, Budapest, P.O.B. 127,H-1364, Hungary

E-mail address: carlos@cs.elte.hu

Konrad-Zuse-Zentrum, Berlin, Takutstr. 7, D-14195 Berlin-Dahlem, Germany

E-mail address: henk@zib.de