FINITE AND INFINITE PACKINGS

Ulrich Betke, Martin Henk and Jörg M. Wills

Abstract. Let $K \subset E^d$, $d \geq 2$, be a centrally symmetric convex body with volume $V(K) > 0$ and distance function f. For $n \in \mathbb{N}$ let $\mathcal{P}_n(K) = \{C_n \subset E^d : C_n = \{x^1, \ldots, x^n\}, f(x^i - x^j) \geq 2, i \neq j\}$, i.e. $C_n + K$, $C_n \in \mathcal{P}_n(K)$, is a packing of n translates of K. For $\rho \geq 0$ let

$$\delta(K, n, \rho) = \max\{n \cdot V(K)/V(\mathrm{conv}(C_n + \rho K)) : C_n \in \mathcal{P}_n(K)\}$$

be the density of a densest packing of n translates of K, where ρ controls the influence of the boundary. We show that for $\rho \geq 2$

$$\limsup_{n\to\infty} \delta(K, n, \rho) = \delta(K),$$

where $\delta(K)$ is the classical densest infinite packing density of K. So we get a new approach to classical packings. For $d = 2$ we generalize classical results by Rogers, Oler and Groemer. We further show that there exists a $\rho_d > 0$ only depending on the dimension such that for $\rho \leq \rho_d$ $\delta(K, n, \rho)$ is attained if $\mathrm{conv}(C_n)$ is a segment, i.e. if $\mathrm{conv}(C_n + \rho K)$ is a "sausage". In particular we prove L. Fejes Tth's sausage conjecture for $d \geq 13.387$

0. Introduction

Throughout this paper E^d denotes the d-dimensional Euclidean space and the set of all centrally symmetric convex bodies $K \subset E^d$ — compact convex sets with $K = -K$ — with non-empty interior ($\mathrm{int}(K) \neq \emptyset$) is denoted by $\mathcal{K}_0^d$. B^d denotes the d-dimensional unit ball with boundary S^{d-1} and $\mathrm{conv}(P)$ denotes the convex hull of a set $P \subset E^d$. Further, the volume of P with respect to the affine hull of P is denoted by $V(P)$ and for $K \in \mathcal{K}_0^d$ let $\delta(K)$ be the density of a densest packing of translates of K (cf. [GL], [CS], [FK]).

In this paper we consider finite and infinite packings of translates for $K \in \mathcal{K}_0^d$. To this end we introduce for $n \in \mathbb{N}$ the set $\mathcal{P}_n(K)$ of all possible packing arrangements of n translates of K, which can be defined by

$$\mathcal{P}_n(K) = \{C_n \subset E^d : C_n = \{x^1, \ldots, x^n\}, f_K(x^i - x^j) \geq 2, i \neq j\},$$

where $f_K : E^d \to \mathbb{R}$ denotes the distance function of K, i.e. $f_K(x) = \min\{\lambda \geq 0 : x \in \lambda K\}$. So for $C_n = \{x^1, \ldots, x^n\} \in \mathcal{P}_n(K)$ we have $\mathrm{int}(x^i + K) \cap \mathrm{int}(x^j + K) = \emptyset$,

1991 *Mathematics Subject Classification.* 52C17, 05B40, 11H06, 52A40.

Key words and phrases. finite packing, infinite packing, critical determinant, Voronoi-cell.

We wish to thank K. Böröczky, Jr., L. Danzer, G. Fejes Tóth, J. Rush and the referee for helpful comments and suggestions

$i \neq j$. It turns out that the case $\dim(\mathrm{conv}(C_n)) = 1$ plays an essential role; so we introduce a special notation: For $u \in S^{d-1}$ and $K \in \mathcal{K}_0^d$ we call $S_n(u, K) = \{x^1, \ldots, x^n\} \in \mathcal{P}_n(K)$ a sausage arrangement in direction u iff $x^i = 2i \cdot u / f_K(u)$ for $1 \leq i \leq n$. In the special case $K = B^d$ we have that $V(\mathrm{conv}(S_n(u, B^d) + \rho B^d))$ is apparently independent of u and thus we write $S_n = S_n(u, B^d)$ for some arbitrary $u \in S^{d-1}$. In this paper we consider the following functionals

Definition 0.1. For $K \in \mathcal{K}_0^d$, $n \in \mathbb{N}$ and $\rho \in \mathbb{R}^{\geq 0}$ let

$$\begin{aligned} \delta(K, n, \rho) &= \max\left\{ \frac{n \cdot V(K)}{V(\mathrm{conv}(C_n + \rho K))} : C_n \in \mathcal{P}_n(K) \right\}, \\ \delta(K, \rho) &= \limsup_{n \to \infty} \delta(K, n, \rho). \end{aligned}$$

$\delta(K, n, \rho)$ and $\delta(K, \rho)$ can be interpreted as packing densities with a parameter ρ which controls the influence of the boundary.

In section 1 we show some simple but basic results; in particular the close relation between finite packing densities and classical packing densities. In section 2 we give for $d = 2$ a description of $\delta(K, n, \rho)$ and $\delta(K, \rho)$ by other functionals (Theorem 2.2). The proof is based on a lower bound of $V(\mathrm{conv}(C_n + \rho K))$ (Theorem 2.1), which generalizes classical results by ROGERS, OLER and GROEMER. From Definition 0.1 we get

$$\delta(K) \leq \delta(K, \rho) \leq 1 \text{ for } 1 \leq \rho < \infty. \tag{0.1}$$

For $\rho < 1$ we may have $\delta(K, \rho) > 1$. In section 3 we prove that $\delta(K, \rho) = \delta(K)$ for $\rho \geq 2$ (Corollary 3.1). This statement is an easy consequence of a more general result for arbitrary convex bodies (Theorem 3.1). Thus we obtain a new approach to classical infinite packings by translates.

For $d = 2$ ROGERS [R1] proved $\delta(K, 1) = \delta(K) = \delta_{\mathcal{L}}(K)$, where $\delta_{\mathcal{L}}(K)$ is the density of a densest lattice packing of K. For $d \geq 3$ no such result can be expected for arbitrary K. For this let $Z \in \mathcal{K}_0^d$ be the cartesian product of B^{d-1} and a segment of length 1, say in direction $e^d \in S^{d-1}$, then

$$V(\mathrm{conv}(S_n(e^d, Z) + Z)) \leq V(\mathrm{conv}(C_n + Z)), \quad C_n \in \mathcal{P}_n(Z),$$

and in contrast to ROGERS' result we obtain $\delta(Z, n, 1) = \delta(Z, 1) > \delta(Z)$. Thus the behaviour of $\delta(K, \rho)$ for $\rho = 1$ is completely different from the case $\rho \geq 2$. A nice example for such linear arrangements in usual 3-space is the densest packing of equal coins.

A conjecture in the same spirit is L. FEJES TTH's [F] famous "sausage conjecture": For $d \geq 5$, $\rho = 1$ and $n \in \mathbb{N}$

$$\min\{V(\mathrm{conv}(C_n + \rho B^d)) : C_n \in \mathcal{P}_n(B^d)\} = V(\mathrm{conv}(S_n + \rho B^d)). \tag{0.2}$$

Several partial results support this conjecture (e.g. [BGW], [BG], [FGW], [Bö1], [Bö2], [Bö3], [BöH]), but until now it was not proved for any dimension. In section 4 we show that (0.2) even holds for all $\rho < 2/\sqrt{3}$ and sufficiently large d (Theorem 4.1). In particular we show that the sausage conjecture ($\rho = 1$) is true for $d \geq 13.387$ (Theorem 4.2).

Finally, in section 5 we show that for small ρ a sausage arrangement is not only best possible for B^d with respect to $\delta(K, n, \rho)$, but for every $K \in \mathcal{K}_0^d$ there exists a function $\psi(R(K)/r(K), d) > 0$, only depending on the ratio of circumradius and inradius and the dimension, such that for $\rho \leq \psi(R(K)/r(K), d)$ the maximum of $\delta(K, n, \rho)$ is attained for a certain sausage $S_n(u, K)$ (Theorem 5.1). This result implies that there exists a constant $\rho_d > 0$ only depending on the dimension with the same property (cf. Corollary 5.1).

Remarks.

(1) All results can be generalized to arbitrary (non-symmetric) convex bodies, but only for large ρ this can be done shortly and elegantly (cf. Theorem 3.1 and Corollary 3.1).

(2) Similar results hold for the restriction to finite and infinite lattice packings, which will be considered in a later paper, and similar ideas also work for finite and infinite coverings, but the methods seem to be different.

1. Basic Properties

From Definition 0.1 follows that the calculation of $\delta(K, n, \rho)$ and $\delta(K, \rho)$ requires information on $V(\mathrm{conv}(C_n + \rho K))$, which can be written as polynomial in ρ with the mixed volumes $V_i(\mathrm{conv}(C_n), K)$ (cf. [BF] or [GL]) as coefficients:

$$V(\mathrm{conv}(C_n + \rho K)) = \sum_{i=0}^{d} \binom{d}{i} \rho^i V_i(\mathrm{conv}(C_n), K) \tag{1.1}$$

In particular $V_0(\mathrm{conv}(C_n), K) = V(\mathrm{conv}(C_n))$, $V_d(\mathrm{conv}(C_n), K) = V(K)$ and $V_i(\mathrm{conv}(C_n), K) = 0$ if $\dim(\mathrm{conv}(C_n)) < d - i$. Formula (1.1) is an essential tool in this paper.

For a sausage $S_n(u, K) = \{x^1, \dots, x^n\}$ we have $\dim(\mathrm{conv}(S_n(u, K))) = 1$ and thus $V_i(\mathrm{conv}(S_n(u, K)), K) = 0$, $i = 0, 1, ..., d-2$. Moreover, since $f_K(x^i - x^{i+1}) = 2$, $1 \leq i \leq n-1$, we get $V_{d-1}(\mathrm{conv}(S_n(u, K)), K) = 2(n-1)(f_K(u))^{-1} V(K_u)$, where K_u denotes the orthogonal projection of K onto a hyperplane with normal vector u (cf. [BF, p. 45]). Hence

$$V(\mathrm{conv}(S_n(u, K) + \rho K)) = 2(n-1) \frac{V(K_u)}{f_K(u)} \rho^{d-1} + V(K) \rho^d. \tag{1.2}$$

For $K = B^d$ we get from (1.2) with $V(B^d) = \kappa_d$

$$V(\mathrm{conv}(S_n + \rho B^d)) = 2(n-1)\kappa_{d-1}\rho^{d-1} + \kappa_d \rho^d. \tag{1.3}$$

From (1.2) we obtain that

$$\max \left\{ \frac{n \cdot V(K)}{2(n-1)V(K_u)/f_K(u)\rho^{d-1} + V(K)\rho^d} : u \in S^{d-1} \right\}$$

is the density of a densest sausage arrangement of n translates of K. Since we are interested in infinte packings as limit of finite packings we define

Definition 1.1. *For $K \in \mathcal{K}_0^d$ let*

$$\delta_S(K) = \max \left\{ \frac{f_K(u) \cdot V(K)}{2V(K_u)} : u \in S^{d-1} \right\}.$$

$\delta_S(K)$ is the density of a densest "infinite sausage arrangement" of K with respect to $\rho = 1$. Observe, for arbitrary ρ the appropriate density is given by $\delta_S(K)/\rho^{d-1}$. With (1.1) and (1.2) we obtain some simple but basic results.

Theorem 1.1.

(1) *Let* $n \in \mathbb{N}$ *and* $u \in S^{d-1}$ *such that for* $\rho = \rho_1$

$$\delta(K, n, \rho) = n \cdot V(K)/V(\mathrm{conv}(S_n(u, K) + \rho K)). \tag{1.4}$$

Then (1.4) *holds for each* $\rho \in [0, \rho_1]$.

(2) *Let* $u \in S^{d-1}$ *such that for* $\rho = \rho_1$

$$\delta(K, \rho) = \delta_S(K)/\rho^{d-1}. \tag{1.5}$$

Then (1.5) *holds for each* $\rho \in [0, \rho_1]$.

(3) *Let* $\rho_2 \in \mathbb{R}^{>0}$ *such that for* $\rho = \rho_2$

$$\delta(K, \rho) = \delta(K). \tag{1.6}$$

Then (1.6) *holds for each* $\rho \in [\rho_2, \infty)$.

(4) *Let* $\rho \in \mathbb{R}^{\geq 0}$, $n \in \mathbb{N}$ *and* $C_n \in \mathcal{P}_n(K)$ *such that* $\delta(K, n, \rho) = n \cdot V(K)/V(\mathrm{conv}(C_n + \rho K))$. *For every nonsingular affine transformation* $T : E^d \to E^d$ *we have*

$$\delta(T(K), n, \rho) = n \cdot V(T(K))/V(\mathrm{conv}(T(C_n) + \rho T(K))).$$

Proof. From (1.1), (1.2) and (1.4) follows

$$2(n-1)\frac{V(K_u)}{f_K(u)}\rho_1^{d-1} \leq \sum_{i=0}^{d-1} \binom{d}{i} \rho_1^i V_i(\mathrm{conv}(C_n), K).$$

Since the mixed volumes are nonnegative the inequality holds for each $\rho \in [0, \rho_1]$ and thus we obtain (1). (2) follows from (1) with $n \to \infty$ and (3) is a consequence of the observation (see (0.1)), that for each K and C_n $V(\mathrm{conv}(C_n + \rho K))$ increases with ρ. (4) is an immediate consequence of $T(C_n) \in \mathcal{P}_n(T(K))$. □

Theorem 1.1 motivates the following

Definition 1.2. For $K \in \mathcal{K}_0^d$ let

$$\rho_s(K) = \sup\left\{\rho : \delta(K, \rho) = \delta_S(K)/\rho^{d-1}\right\}$$

be the *sausage radius* of K and

$$\rho_c(K) = \inf\{\rho : \delta(K, \rho) = \delta(K)\}$$

be the *critical radius* of K.

Remark. Clearly $\rho_s(K) \leq \rho_c(K)$, if $\rho_s(K)$ and $\rho_c(K)$ exist at all.

In Theorem 3.1 we show $\rho_c(K) \leq 2$ for $K \in \mathcal{K}_0^d$ and in Theorem 5.1: $\rho_s(K) > 0$; so for $K \in \mathcal{K}_0^d$

$$0 < \rho_s(K) \leq \rho_c(K) \leq 2.$$

From Definition 1.1, 1.2 and (1.2) one gets the following simple result, which shows the close relations between ρ_s, ρ_c, δ_S and δ

Theorem 1.2.

$$\delta_S(K)\,(\rho_c(K))^{1-d} \le \delta(K) \le \delta_S(K)\,(\rho_s(K))^{1-d}.$$

For $K = B^d$ follows with $\delta_S(B^d) = \kappa_d/(2\kappa_{d-1})$

Corollary 1.1.

$$\frac{\kappa_d}{2\kappa_{d-1}}\left(\rho_c(B^d)\right)^{1-d} \le \delta(B^d) \le \frac{\kappa_d}{2\kappa_{d-1}}\left(\rho_s(B^d)\right)^{1-d}.$$

Remark. Corollary 1.1 implies that any upper bound of $\rho_c(B^d)$ and any lower bound of $\rho_s(B^d)$ gives a lower and an upper bound for $\delta(B^d)$. In particular Corollary 3.2, Corollary 4.2 and the inequality $\sqrt{2\pi/(d+1)} < (\kappa_d/\kappa_{d-1}) < \sqrt{2\pi/d}$ (cf. [BGW]) imply that for every $\epsilon > 0$ exists a $d(\epsilon)$ such that for $d \ge d(\epsilon)$

$$\sqrt{\frac{2\pi}{d+1}}2^{-d} < \delta(B^d) < \sqrt{\frac{2\pi}{3d}}\left(\frac{2}{\sqrt{3}} - \epsilon\right)^{-d}.$$

Though this is much weaker than the best known bounds for $\delta(B^d)$ it shows that finite packings are not only of interest in their own but also give a new approach to the study of infinite packings.

Theorem 2.2 iii) implies that $\rho_s(K) = \rho_c(K)$ holds for each $K \in \mathcal{K}_0^2$. For the unit cube C^d we obviously get: $\rho_s(C^d) = \rho_c(C^d) = 1$. But in general we can not expect $\rho_s(K) = \rho_c(K)$ as the next simple result shows (cf. [Gr, pp. 43]):

Theorem 1.3. *For each $d \ge 4$ there is a $K \in \mathcal{K}_0^d$ with*

$$\rho_s(K) < 1 < \rho_c(K).$$

Proof. Let $K \in \mathcal{K}_0^d$ be the cartesian product of the regular hexagon H and B^{d-2} embedded in the orthogonal complement of H. Clearly H generates a tiling of the plane. So for sufficiently large $n \in \mathbb{N}$ $V(\mathrm{conv}(C_n + K))$ is minimal if $\dim(\mathrm{conv}(C_n))$ $= 2$. As this minimal C_n is neither 1-dimensional nor d-dimensional, we have $\rho_s(K) < 1 < \rho_c(K)$. □

But for the unit ball B^d we conjecture

Strong Sausage Conjecture. *For $d \ge 1$*

$$\rho_s(B^d) = \rho_c(B^d).$$

This conjecture would imply the equivalence of the two problems of the determination of $\delta(B^d)$ and of $\rho_c(B^d)$.

2. The 2-dimensional case

For $K \in \mathcal{K}_0^2$ let $\overline{K}$ be a minimal circumscribed parallelogram of K. Obviously we have $V(\overline{K}) = V(K)/\delta_S(K)$. Further for two convex bodies $C, D \subset E^2$ let $A(C, D)$ be the mixed area. Then (1.1) becomes

$$V(\mathrm{conv}(C_n + \rho K)) = V(\mathrm{conv}(C_n)) + 2\rho A(\mathrm{conv}(C_n), K) + \rho^2 V(K). \tag{2.1}$$

For abbreviation we set $\Delta(K) = V(K)/\delta(K)$. With this notation we have

Theorem 2.1. *Let* $K \in \mathcal{K}_0^2$, $n \in \mathbb{N}$ *and* $C_n \in \mathcal{P}_n(K)$. *For each* $\rho \geq 0$

$$V(\mathrm{conv}(C_n + \rho K)) \geq (n-1)\Delta(K) + 2(\rho - \gamma(K))A(\mathrm{conv}(C_n), K) + \rho^2 V(K), \quad (2.2)$$

with $\gamma(K) = \delta_S(K)/\delta(K)$. *Further* $3/4 \leq \gamma(K) \leq 1$ *with* $\gamma(K) = 1$, *if* K *is a parallelogram and* $\gamma(K) = 3/4$, *if* K *is an affinely regular hexagon.*

Remark. With slightly different notation special cases of (2.2) were already proved before. The first one was ROGERS [R1]. He proved (2.2) for $\rho = 1$ and without the summand with $A(C_n, K)$. Rogers later [R2] gave a weaker inequality for arbitrary convex bodies $K \subset E^2$. GROEMER [G] proved (2.2) for the special case $K = B^2$ and $\rho = \gamma(B^2) = \sqrt{3}/2$. OLER [O] proved (2.2) for $\rho = 0$; from his result we deduce the general case. FOLKMAN & GRAHAM [FG] and GRAHAM, WITSENHAUSEN and ZASSENHAUS [GWZ] gave easier proofs and generalizations to Minkowski planes of OLER's theorem. None of these authors made a remark on general ρ.

Before we start with the proof we deduce from Theorem 2.1 the following results on $\delta(K, n, \rho)$ and $\delta(K, \rho)$

Theorem 2.2. *For* $K \in \mathcal{K}_0^2$ *and* $n \in \mathbb{N}$

$$\begin{aligned} &i)\ (\delta(K,n,\rho))^{-1} = (1-1/n)\rho(\delta_S(K))^{-1} + (1/n)\rho^2, \quad 0 \leq \rho \leq \gamma(K), \\ &ii)\ (\delta(K,n,\rho))^{-1} \geq (1-1/n)(\delta(K))^{-1} + (1/n)\rho^2, \quad \gamma(K) \leq \rho < \infty, \\ &iii)\ \delta(K,\rho) = \begin{cases} \delta_S(K)/\rho, & 0 \leq \rho \leq \gamma(K), \\ \delta(K), & \gamma(K) \leq \rho < \infty. \end{cases} \end{aligned}$$

Corollary 2.1. *For* $K \in \mathcal{K}_0^2$ *holds*

$$\rho_c(K) = \rho_s(K) = \gamma(K).$$

Remark. For $\rho = \gamma(K)$ various minimal configurations are possible; in particular also linear arrangements ('sausages'). For $K = B^2$ and $\gamma(B^2) = \sqrt{3}/2$ this was already observed by GROEMER (cf. also WEGNER [W]).

Proof. On account of (0.1) iii) is an immediate consequence of i) and ii). Further, ii) is an immediate consequence of (2.2) since $\rho \geq \gamma(K)$. To prove i) we first consider $\rho = \gamma(K)$. Let $u \in S^{d-1}$ such that $2V(K_u)/f_K(u) = V(K)/\delta_S(K)$. Then by (1.2)

$$V(S_n(u,K) + \gamma(K)K) = (n-1)\gamma(K)V(K)/\delta_S(K) + (\gamma(K))^2 V(K).$$

This shows that for $S_n(u,K)$ we have equality in (2.2). As for $\rho = \gamma(K)$ the right hand side in (2.2) is independent of C_n i) follows. Now for $\rho \leq \gamma(K)$ i) follows by Theorem 1.1 (1). □

The proof of Theorem 2.1 is prepared by the following simple result.

Lemma 2.1. $\overline{K}$ *meets* K *in the four midpoints of its edges.*

Proof. Let $\overline{K}$ be minimal and let a, b, a', b' be the four edges of $\overline{K}$ with a parallel to a' and b parallel to b'. Assume that the midpoint of b and hence of b' does not meet K. Then let a'' and a''' be segments parallel to a and a' through the centre of K, with a'' having its endpoints on b and b' and a''' having its endpoints on the boundary of K. If l denotes the length, then $l(a'') > l(a''')$.

Now let c and c' be the two parallel segments tangent to K at the endpoints of a''' with their endpoints lying on aff$\{a\}$ and aff$\{a'\}$. They generate with aff$\{a\}$ and aff$\{a'\}$ a parallelogram set $\overline{K}'$. It follows

$$V(\overline{K}') = l(a''') \cdot \mathrm{dist}(a, a') < l(a'') \cdot \mathrm{dist}(a, a') = V(\overline{K}).$$

which contradicts the minimality of $\overline{K}$. □

Proof of Theorem 2.1. First we consider $\gamma(K)$. Obviously $\delta(K) \geq \delta_S(K)$ and thus $\gamma(K) \leq 1$. Now, let H be a convex hexagon of minimal area circumscribed about K and let $\overline{H}$ be a smallest circumscribed parallelogram of H, i.e. $V(\overline{H}) = V(H)/\delta_S(H)$. Then $\Delta(H) = \Delta(K)$ (cf. [EGH, p. 44] and $\gamma(H) \leq \gamma(K)$.

By Lemma 2.1 it follows that the edges of $\overline{H}$ meet H at their midpoints. If one (and hence two) of these midpoints is a vertex of H, we can choose the corresponding edges of $\overline{H}$ such that they contain a corresponding pair of edges of H. So without restriction we can assume that each of the four edges of $\overline{H}$ has a common affine hull with one of the edges of H. So H and $\overline{H}$ have two common vertices, whereas the four remaining vertices of H lie on the four edges of $\overline{H}$.

Obviously $\delta_S(H)/\delta(H)$ is minimal if these four vertices are the midpoints of these edges. In this case H is an affine image of the regular hexagon and an elementary calculation shows $\delta_S(H)/\delta(H) = 3/4$. Thus we obtain the required properties of $\gamma(K)$.

From Oler's Theorem 1 (cf. [O], p. 20) follows with a suitable change of notation $V(\mathrm{conv}(C_n))/\Delta(K) + (1/2)M_K(\mathrm{conv}(C_n)) + 1 \geq n$ or

$$V(\mathrm{conv}(C_n)) \geq (n-1)\Delta(K) - \frac{1}{2} M_K(\mathrm{conv}(C_n)) V(K)/\delta_S(K)\gamma(K) \qquad (2.3)$$

where M_K denotes the perimeter in the Minkowski-space with gauge body K. In fact we do not need M_K explicitly, because Oler showed in formula 6 on p. 48 of [O]:

$$V(\mathrm{conv}(C_n + K)) \geq V(\mathrm{conv}(C_n)) + \frac{1}{2} M_K(C_n) V(K)/\delta_S(K) + V(K).$$

Hence $(1/2)M_K(\mathrm{conv}(C_n))V(K)/\delta_S(K) \leq 2A(\mathrm{conv}(C_n), K)$ (cf. (2.1)) and together with (2.3) it follows

$$V(\mathrm{conv}(C_n)) \geq (n-1)\Delta(K) - 2A(\mathrm{conv}(C_n), K)\gamma(K)$$

which implies (2.2) by (2.1). □

3. Relations between $\delta(K, \rho)$ and $\delta(K)$

In this section we prove a result for arbitrary (non-symmetric) convex bodies K. For this we give a definition of $\mathcal{P}_n(K)$ without the distance function. Let $\mathcal{K}^d \subset E^d$ be the set of all convex bodies and for $K \in \mathcal{K}^d$ let

$$\mathcal{P}_n(K) = \{C_n \subset E^d : C_n = \{x^1, \ldots, x^n\},\ \mathrm{int}(x^i + K) \cap \mathrm{int}(x^j + K) = \emptyset,\ i \neq j\}.$$

Now, let $\delta(K, n, \rho)$ and $\delta(K, \rho)$ for $K \in \mathcal{K}^d$ be defined in the same way as in Definition 0.1. Further let $\delta(K)$ be the density of a densest packing of translates of K.

Theorem 3.1. *Let $K \in \mathcal{K}^d$ and $\rho \in \mathbb{R}^{>0}$ such that* $\mathrm{int}(K) \cap \mathrm{int}(y+K) = \emptyset$ *holds for all $y \notin \rho K$. Then for $n \in \mathbb{N}$*

$$\delta(K, n, \rho) \leq \delta(K).$$

On account of (0.1), which is also valid for $K \in \mathcal{K}^d$, we have for such a ρ as in Theorem 3.1 $\delta(K, \rho) = \delta(K)$. Hence we get

Corollary 3.1. *For $K \in \mathcal{K}_0^d$ and $\rho \geq 2$ or for $K \in \mathcal{K}^d$ and $\rho \geq d+1$ holds*

$$\delta(K, \rho) = \delta(K).$$

Proof. Let $K \in \mathcal{K}_0^d$, $\rho \geq 2$ and $y \notin \rho K$. Assume $\mathrm{int}(K) \cap \mathrm{int}(y+K) \neq \emptyset$. Then there exist $x, z \in \mathrm{int}(K)$ with $x = y + z$ and thus $y \in K + (-K) = 2K$ which contradicts the choice of ρ.

Let $K \in \mathcal{K}^d$. Since $\delta(K, n, \rho)$ is invariant with respect to translations of K we may assume $K + (-K) \subset (d+1)K$ (cf. [R3, p. 43]). As above we obtain $\mathrm{int}(K) \cap \mathrm{int}(y+K) = \emptyset$ for all $y \notin (d+1)K$. □

Corollary 3.2. *For $K \in \mathcal{K}_0^d$ holds*

$$\rho_c(K) \leq 2.$$

The proof of Theorem 3.1 is based on the following idea: Assume that $C_n + \rho K$ is a finite packing with $\delta(K, n, \rho) > \delta(K)$. Then a packing lattice Λ of $\mathrm{conv}(C_n + \rho K)$ with elementary cell Z is chosen. For every $x \in Z$ the lattice packing $L(\mathrm{conv}(C_n + \rho K) + x) = \{(\mathrm{conv}(C_n + \rho K) + x) + g : g \in \Lambda\}$ is superposed on a densest infinite packing $\{K + a : a \in P(K)\}$ with density $\delta(K)$. Further all $K + a$, $a \in P(K)$, which meet $L(\mathrm{conv}(C_n + \rho K) + x)$ are deleted.

A standard averaging argument with respect to x shows the existence of an infinite packing of translates of K with density $> \delta(K)$ which contradicts the definition of $\delta(K)$. Hence $\delta(K, n, \rho) \leq \delta(K)$. The proof gives a careful analysis of this idea.

Proof of Theorem 3.1. Let $\Delta(K) = V(K)/\delta(K)$. Assume there exist $K \in \mathcal{K}^d$, $\rho \in \mathbb{R}^{>0}$ satisfying the assumption and an integer n with $\delta(K, n, \rho) > \delta(K)$. Then there is a $C_n \in \mathcal{P}_n(K)$ and an $\epsilon > 0$ with

$$V(\mathrm{conv}(C_n + \rho K)) = n \cdot \Delta(K) - \epsilon. \tag{3.1}$$

Let Λ be a packing lattice of $\mathrm{conv}(C_n + \rho K)$. We may assume that $\mathrm{conv}(C_n + \rho K)$ is contained in a fixed elementary cell Z of Λ. From (3.1) follows

$$\left(1 - \frac{V(\mathrm{conv}(C_n + \rho K))}{\det(\Lambda)}\right) \frac{\det(\Lambda)}{\det(\Lambda) + \epsilon} + \frac{n\Delta(K)}{\det(\Lambda) + \epsilon} = 1.$$

From this we get with $\Delta(K) < \det(\Lambda)$ and multiplication with $\delta(K)$:

$$\left(1 - \frac{V(\mathrm{conv}(C_n + \rho K))}{\det(\Lambda)}\right) \frac{V(K)}{\Delta(K) + \epsilon} + \frac{nV(K)}{\det(\Lambda) + \epsilon} > \delta(K). \tag{3.2}$$

Now, for $\lambda > 0$ let $W_\lambda \in \mathcal{K}_0^d$ be the cube of edge length 2λ. Apparently there is a constant μ only depending on Z such that for every $\lambda > 0$ there is a subset $L_\lambda \subset \Lambda$ such that $W_\lambda + Z \subset L_\lambda + Z$ and $L_\lambda + 2Z \subset W_{\lambda+\mu}$.

By the definition of $\delta(K)$ (cf. e.g. [GL]) for every $\lambda > 0$ there exists a set $C_{m(\lambda)} \in \mathcal{P}_{m(\lambda)}(K)$ such that $C_{m(\lambda)} + K \subset W_\lambda$ and

$$\lim_{\lambda\to\infty} \frac{m(\lambda)V(K)}{V(W_\lambda)} = \delta(K).$$

Obviously $\lim_{\lambda\to\infty} V(W_{\lambda+\mu})/V(W_\lambda) = 1$, so there exists a $\zeta > 0$ and a set $C_{m(\zeta)} \in \mathcal{P}_{m(\zeta)}(K)$ with $C_{m(\zeta)} + K \subset W_\zeta$ such that

$$\frac{V(K)}{\Delta(K)+\epsilon} \le \frac{m(\zeta)V(K)}{V(W_{\zeta+\mu})} \quad \text{and} \quad \frac{nV(K)}{\det(\Lambda)+\epsilon} \le \frac{nV(K)\,\mathrm{card}(L_\zeta)}{V(W_{\zeta+\mu})}. \tag{3.3}$$

For every $x \in Z$ we construct a finite packing $C_{n(x)} \in \mathcal{P}_{n(x)}(K)$ – for a suitable $n(x) \in \mathbb{N}$ – with $C_{n(x)} + K \subset W_{\zeta+\mu}$ in the following way:

$$C_{n(x)} = \{x + L_\zeta + C_n\} \cup \{y \in C_{m(\zeta)} : y \notin x + L_\zeta + \mathrm{conv}(C_n + \rho K)\}.$$

The choice of ρ guarantees that $C_{n(x)}$ is a packing. While it is difficult to determine the cardinality $n(x)$ of $C_{n(x)}$ for fixed x it is easy to calculate $\int_{x\in Z} n(x)dx$. To this end for every $y \in C_{m(\zeta)}$ let $\chi_y(x) = 1$ for $y \notin x + L_\zeta + \mathrm{conv}(C_n + \rho K)$ and $\chi_y(x) = 0$ else. Then

$$\begin{aligned}\int_{x\in Z} n(x)dx &= \int_{x\in Z}\left(n\,\mathrm{card}(L_\zeta) + \sum\nolimits_{y\in C_{m(\zeta)}} \chi_y(x)\right)dx \\ &= n\det(\Lambda)\,\mathrm{card}(L_\zeta) + m(\zeta)\left(\det(\Lambda) - V(\mathrm{conv}(C_n + \rho K))\right).\end{aligned}$$

So there is a $z \in Z$ with

$$n(z) \ge m(\zeta)\left(1 - \frac{V(\mathrm{conv}(C_n + \rho K))}{\det(\Lambda)}\right) + n\,\mathrm{card}(L_\zeta)$$

or

$$\frac{n(z)V(K)}{V(W_{\zeta+\mu})} \ge \frac{m(\zeta)V(K)}{V(W_{\zeta+\mu})}\left(1 - \frac{V(\mathrm{conv}(C_n + \rho K))}{\det(\Lambda)}\right) + \frac{nV(K)\,\mathrm{card}(L_\zeta)}{V(W_{\zeta+\mu})}.$$

From (3.2) and (3.3) follows

$$\frac{n(z)V(K)}{V(W_{\zeta+\mu})} > \delta(K).$$

But this contradicts the definition of $\delta(K)$ since $C_{n(z)} + K \subset W_{\zeta+\mu}$. □

4. The sausage conjecture

For the sausage arrangement of n-balls with radius ρ we have (cf. (1.3))

$$V(\mathrm{conv}(S_n + \rho B^d)) = 2(n-1)\kappa_{d-1}\cdot\rho^{d-1} + \kappa_d\cdot\rho^d. \tag{4.1}$$

The purpose of this section is to prove

Theorem 4.1. *For every $\rho < \rho_m := 2/\sqrt{3}$ exists a 'sausage'-dimension $d(\rho)$ such that for $d \geq d(\rho)$*

$$\min\{V(\mathrm{conv}(C_n + \rho B^d)) : C_n \in \mathcal{P}_n(B^d)\} = V(\mathrm{conv}(S_n + \rho B^d)).$$

Corollary 4.1.

$$\liminf_{d\to\infty} \rho_s(B^d) \geq 2/\sqrt{3}.$$

Theorem 4.2. $d(1) \leq 13.387$.

Hence the 'sausage'-conjecture of L. Fejes Tóth is verified for $d \geq 13.387$.
The rather lengthy proofs of the results in this and the following section are based on the following observation: Assume that $C_n = \{x^1, \dots, x^n\}$ is not a sausage arrangement. We consider the Dirichlet-Voronoi-cell (DV-cell) H of x for some $x \in \{x^1, \dots, x^n\}$. The deviation from the sausage arrangement is measured by a parameter ϕ. Then $C_n + \rho B^d$ has a sausage part and a part which is the cartesian product of a 2-dimensional set and essentially a $(d-2)$-ball of radius ρ.
The sausage part is of size $\kappa_{d-1}\rho^{d-1}(2 - \mathrm{const}_1\phi)$ and the other part is of size $\mathrm{const}_2\kappa_{d-2}\rho^{d-2}\phi$, where const_1 and const_2 are constants independent of d. The results follow from $\kappa_{d-2}/\kappa_{d-1} \to \infty$ as $d \to \infty$.

Proof of Theorem 4.1

In the sequel let $C_n \in \mathcal{P}_n(B^d)$ with $C_n = \{x^1, \dots, x^n\}$ be an arbitrary but fixed arrangement and let $\rho \in [1, \rho_m)$ (cf. Theorem 1.1). The proof is based on a careful analysis of the volume part which belongs to a DV-cell

$$H_i = \{x \in E^d : 2\langle x, x^j - x^i\rangle \leq |x^j|^2 - |x^i|^2,\ 1 \leq j \leq n\}, \quad 1 \leq i \leq n,$$

of the considered arrangement. H_i is called the DV-cell with respect to x^i.

Obviously, $V(\mathrm{conv}(C_n + \rho B^d)) = \sum_{i=1}^n V(H_i \cap (\mathrm{conv}(C_n + \rho B^d)))$ and thus by (4.1) it is sufficient to show $V(H_i \cap (\mathrm{conv}(C_n + \rho B^d))) \geq 2\kappa_{d-1}\rho^{d-1}$ for $(n-2)$ DV-cells and $V(H_i \cap (\mathrm{conv}(C_n + \rho B^d))) \geq \kappa_{d-1}\rho^{d-1} + \kappa_d\rho^d/2$ for 2 DV-cells. To this end we consider a fixed DV-cell, say $H = H_n$ with respect to $x^n = 0$, and $D_\rho = H \cap (\mathrm{conv}(C_n + \rho B^d))$. In order to obtain good lower bounds for $V(D_\rho)$ we need a large 2-dimensional section. To measure this we introduce an angle ϕ:

Definition 4.1. *Let $y^j = x^j/|x^j|$, $1 \leq j \leq n-1$ and let*

$$\phi = \max_{1\leq k,l\leq n-1}\{\arccos(|\langle y^k, y^l\rangle|)\},$$

where $\arccos(\cdot)$ *is choosen in* $[0, \pi/2]$. *Further let y^{j_1}, y^{j_2} be defined by*

$$\arccos(|\langle y^{j_1}, y^{j_2}\rangle|) = \begin{cases} \phi, & \text{if } \phi \geq \pi/3 \text{ or } \langle y^k, y^l\rangle \geq 0 \text{ for } 1 \leq k,l \leq n-1, \\ \max\limits_{1\leq k,l\leq n-1}\{\arccos(|\langle y^k, y^l\rangle|) : \langle y^k, y^l\rangle \leq 0\}, & \text{else.} \end{cases}$$

We may assume $y^{j_1} = y^1, y^{j_2} = y^2$ and observe that in the second case we have $-\cos(\phi/2) \leq \langle y^1, y^2\rangle \leq -\cos(\phi)$.

Thus a small ϕ indicates that in a neighbourhood of 0 the arrangement is like the middle of a sausage (for $\langle y^1, y^2\rangle < 0$) or like the end of a sausage (for $\langle y^1, y^2\rangle > 0$). Clearly, for $\phi < \pi/3$ the last case can occur at most twice.

Now, let L be the plane spanned by y^1, y^2 and $C(\phi) = \mathrm{conv}\{0, 2y^1, 2y^2\} \cap B^d$. We observe $C(\phi) \subset H \cap \mathrm{conv}(C_n)$. We distinguish several parts of D_ρ according to their position relative to $C(\phi)$. To this end we use the nearest point map (cf. [McS]): For a convex body $K \subset E^d$ the nearest point map $p : E^d \to E^d$ with respect to K is given by

$$p(x) = y \in K \text{ with } |x - y| = \min\{|x - z| : z \in K\}.$$

Using the nearest point map with respect to $C(\phi)$ we define

Definition 4.2.

$$\begin{aligned} D^1_\rho &= \mathrm{cl}\{x \in D_\rho : p(x) \in \mathrm{relint}\, C(\phi)\},\\ D^2_\rho &= \mathrm{cl}\{x \in D_\rho : p(x) \in \mathrm{relint}\,\mathrm{conv}\{0, y^1\} \cup \mathrm{relint}\,\mathrm{conv}\{0, y^2\}\},\\ D^3_\rho &= \mathrm{cl}\{x \in D_\rho : p(x) = 0\},\\ D^4_\rho &= \mathrm{cl}\{x \in D_\rho : p(x) \in \mathrm{relint}\,\mathrm{conv}\{2y^1, 2y^2\}\}. \end{aligned}$$

Clearly, $V(D_\rho) \geq \sum_{i=1}^4 V(D^i_\rho)$. The proof of Theorem 4.1 depends on various estimates of the $V(D^i_\rho)$. These estimates are prepared by the following two Lemmas.

Lemma 4.1. *Let $w \in H \cap S^{d-1}$, $v \in w^- \cap S^{d-1}$, $\mu, \epsilon > 0$ with $\mu \geq 1/(\mu + \epsilon)$ and $(\mu + \epsilon)v \in H$. Then*

$$c_1(\mu, \epsilon) \cdot \mathrm{conv}\{0, w\} + \mu v \in H,$$

with $c_1(\mu, \epsilon) = \epsilon / \sqrt{(\mu + \epsilon)^2 - 1}$.

Proof. The assertion follows with some elementary calculation from $B^d \subset H$ and the convexity of H. □

Lemma 4.2. $V(C(\phi)) \geq \phi/2$.

Proof. Let $\gamma = \langle y^1, y^2\rangle$, $\delta = \arccos(|\gamma|)$ and $\mathrm{cone}\{y^1, y^2\}$ be the positive hull of y^1, y^2. First, suppose $\gamma \geq -1/2$. Then $\mathrm{cone}\{y^1, y^2\} \cap B^d \subset C(\phi)$ and thus

$$V(C(\phi)) \geq \delta/2. \tag{4.2}$$

Next, assume $\gamma < -1/2$ and let M be the set of points x with $x \in \mathrm{cone}\{y^1, y^2\} \cap B^d$ and $x \notin C(\phi)$. Obviously, we have $V(C(\phi)) = V(\mathrm{cone}\{y^1, y^2\} \cap B^d) - V(M)$ and by elementary calculation we get

$$V(C(\phi)) = \frac{\pi - \delta}{2} - \Big(\arccos(2\sin(\delta/2)) - 2\sin(\delta/2)\sqrt{1 - (2\sin(\delta/2))^2}\,\Big).$$

On account of $\arcsin(x) = \pi/2 - \arccos(x)$ substituting $x = 2\sin(\delta/2)$ in the right hand side yields $V(C(\phi)) - \delta \geq \min\{f(x) : x \in [0,1]\}$ with $f(x) = \arcsin(x) - 3\arcsin(x/2) + x\sqrt{1 - x^2}$. Now, $f(0) = f(1) = 0$, and for the second derivate $f''(x)$ we have $f''(x) \leq 0$ for $x \in [0,1]$. Hence $f(x) \geq 0$ for $x \in [0,1]$ and thus we get

$$V(C(\phi)) \geq \delta. \tag{4.3}$$

If $\langle y^1, y^2\rangle \geq -1/2$ we have $\delta = \phi$ and in the case $\langle y^1, y^2\rangle < -1/2$ we have $\delta \geq \phi/2$. Thus the assertion follows by (4.2) and (4.3). □

Now, we start with the estimates

Lemma 4.3. *Let* $\sin(\phi) \le 1/\rho$. *Then*

$$V(D^1_\rho) \ge \frac{\phi}{2} \cdot c_1(\rho, 1/\sin(\phi) - \rho)^2 \rho^{d-2} \kappa_{d-2}.$$

Proof. By the definition of ϕ we have $|\langle y^j, y^i \rangle| \ge \cos(\phi)$ for $1 \le j \le n-1$, $i = 1, 2$. This implies $\langle y^j, v_{y^i} \rangle \le \sin(\phi)$ for all $v_{y^i} \in ((y^i)^- \cap B^d)$, $i = 1, 2$. Hence by the definition of H

$$(1/\sin(\phi)) \cdot ((y^i)^- \cap B^d) \subset H, \quad i = 1, 2. \tag{4.4}$$

Thus $(1/\sin(\phi))(L^- \cap B^d) \subset H$ and by Lemma 4.1 we get $c_1(\rho, 1/\sin(\phi) - \rho) \cdot C(\phi) + \rho(B^d \cap L^-) \subset D^1_\rho$. Now the assertion follows from Lemma 4.2. □

Lemma 4.4. *Let* $\sin(\phi) \le 1/\rho$. *Then*

$$V(D^2_\rho) \ge c_1(\rho, 1/\sin(\phi) - \rho)\rho^{d-1}\kappa_{d-1}.$$

Proof. From (4.4) and Lemma 4.1 follows

$$c_1(\rho, 1/\sin(\phi) - \rho) \cdot \operatorname{conv}\{0, y^i\} + \rho((y^i)^- \cap B^d) \subset D_\rho. \tag{4.5}$$

Now, let $a^i \in L$ be the outward normal vector of $\operatorname{conv}\{0, y^i\}$ with respect to $\operatorname{conv}\{0, y^1, y^2\}$, $i = 1, 2$. Then $\{x \in E^d : \langle a^i, x \rangle \ge 0\} \cap D_\rho \subset D^2_\rho$ and by (4.5) we get the assertion. □

Lemma 4.5. *Let* $\sin(\phi) \le 1/\rho$, $\phi < \pi/3$ *and* $\langle y^1, y^2 \rangle > 0$. *Then*

$$V(D^3_\rho) \ge \frac{1 - \phi/\pi}{2} \rho^d \kappa_d.$$

Proof. Let $F \subset L$ be the set of all outward unit normal vectors of supporting lines at 0 with respect to $\operatorname{conv}\{0, y^1, y^2\}$. By the definition of y^1, y^2 we have $\langle y^i, y^k \rangle \ge \cos(\phi)$ for $1 \le k \le n-1$, $i = 1, 2$, and thus $\langle y^i, a \rangle \le 0$, $1 \le i \le n-1$, for all $a \in F$. Now $\rho v \in D_\rho$ for $v \in L^- \cap B^d$ (cf. (4.4)) and hence we get $(F + L^-) \cap \rho B^d \subset D^3_\rho$. Since $V(F) = (1 - \phi/\pi)/2$ we obtain the required estimate. □

Lemma 4.6. *Let* $\tan(\phi) \le 1/\rho$, $\phi < \pi/3$ *and* $\langle y^1, y^2 \rangle < 0$. *Then*

$$V(D^4_\rho) \ge \frac{\cos(\phi) - \rho\sin(\phi)}{\cos(\phi/2)} \cdot \rho^{d-1}\kappa_{d-1}.$$

Proof. Let $w = (y^1 - y^2)/|y^1 - y^2|$. In particular we have $\phi < \pi/3$ and thus $\langle y^j, y^1 \rangle \ge \cos(\phi) \Leftrightarrow \langle y^j, y^2 \rangle \le -\cos(\phi)$, $1 \le j \le n-1$. It follows $|\langle y^j, w \rangle| \ge \cos(\phi)$ which implies $\langle y^j, v \rangle \le \sin(\phi)$ for all $v \in w^- \cap B^d$. Thus for $v \in w^- \cap B^d$ and for $\lambda \in [0, 1]$ we obtain

$$\langle \lambda 2y^1 + (1-\lambda)2y^2 + \rho v, y^j \rangle \le \begin{cases} \lambda(2\cos(\phi) + 2) - 2\cos(\phi) + \rho\sin(\phi), & \langle y^j, y^1 \rangle \ge \cos(\phi), \\ -\lambda(2\cos(\phi) + 2) + 2 + \rho\sin(\phi), & \langle y^j, y^1 \rangle \le -\cos(\phi). \end{cases}$$

Hence by the definition of H

$$\lambda 2y^1 + (1-\lambda)2y^2 + \rho(w^- \cap B^d) \subset H, \qquad \text{for } \lambda \in [c_2(\phi,\rho), 1 - c_2(\phi,\rho)],$$

with $c_2(\phi,\rho) = (1+\rho\sin(\phi))/(2+2\cos(\phi))$. Observe, by assumption the given interval is nonempty and $\lambda 2y^1 + (1-\lambda)2y^2 \in C(\phi)$. Thus

$$\lambda 2y^1 + (1-\lambda)2y^2 + \rho(w^- \cap B^d) \subset D_\rho,\ \ \lambda \in [c_2(\phi,\rho), 1 - c_2(\phi,\rho)].$$

Let $u \in L$ be the outward unit normal vector of $\operatorname{conv}\{2y^1, 2y^2\}$ with respect to $C(\phi)$ and let $v' \in \{x \in E^d : x \in (w^- \cap B^d),\ \langle u, x\rangle \geq 0\}$. We have $(\operatorname{conv}\{2y^1, 2y^2\} + \rho v') \cap D_\rho \subset D_\rho^4$ and thus we obtain

$$V(D_\rho^4) \geq (1 - 2c_2(\phi,\rho)) \cdot |2y^1 - 2y^2| \rho^{d-1} \frac{\kappa_{d-1}}{2}.$$

□

Lemma 4.7. *Let $\phi > 0$. Then for every $\epsilon > 0$ such that $\rho + \epsilon < \rho_m$*

$$V(D_\rho^1) \geq \frac{\phi}{2} \cdot c_1(\rho,\epsilon)^2 \rho^{d-2} \cdot \frac{\kappa_{d-2}}{1 + c_3(\rho+\epsilon, d)}$$

with

$$c_3(\mu, d) = \int_{1/\mu}^{1} (1-x^2)^{(d-5)/2} dx \Big/ \int_{1/\rho_m}^{1/\mu} (1-x^2)^{(d-5)/2} dx, \qquad \mu \in [1, \rho_m).$$

Proof. We have

$$V(D_\rho^1) \geq \frac{\rho^{d-2}}{d-2} \int_{C(\phi)} \int_{\{z \in L^{\perp} \cap S^{d\perp 1} : w + \rho z \in H\}} dv dw. \tag{4.6}$$

In the sequel we show that for a certain set $G \subset C(\phi)$ with $V(G) > 0$ the above inner integral is of order κ_{d-2}. For this purpose we first consider the inner integral at $w = 0$ and set $M_\rho = \{z \in L^- \cap S^{d-1} : \rho z \notin H\}$, $K_\rho = \{z \in L^- \cap S^{d-1} : \rho z \in H\}$. Assume $M_\rho \neq \emptyset$. Then $L^- \cap \rho S^{d-1}$ intersects the affine hull of certain facets F_{i_j} of the DV-cell H, $j = 1, \ldots, k$. Let $v^{i_j} \in L^-$ be the outer unit normal vector of $\operatorname{aff}\{F_{i_j}\} \cap L^-$. Since the distance of a $(d-2)$-dimensional face of H from 0 is at least ρ_m ([R3]) we get $\operatorname{aff}\{F_{i_j}\} \cap (L^- \cap \rho S^{d-1}) \subset \operatorname{relint}\{F_{i_j} \cap L^-\}$ and there exists an $\alpha_{i_j} \in [1, \rho]$ such that $\alpha_{i_j} v^{i_j} \in \operatorname{relint}\{F_{i_j} \cap L^-\}$. With

$$M_{i_j} = \{z \in (L^- \cap S^{d-1}),\ \langle z, v^{i_j}\rangle > \alpha_{i_j}/\rho\}$$

we have $M_\rho = \cup_{j=1}^k M_{i_j}$. Now for $1 \leq j \leq k$ define

$$K_{i_j} = \{z \in (L^- \cap S^{d-1}),\ \alpha_{i_j}/\rho_m \leq \langle z, v^{i_j}\rangle \leq \alpha_{i_j}/\rho\},$$

and let $z \in K_{i_j}$. With $\gamma_z = \alpha_{i_j}/\langle z, v^{i_j}\rangle \geq \rho$ we get $\gamma_z z \in \operatorname{aff}\{F_{i_j}\} \cap L^-$ and $|\gamma_z z - \alpha_{i_j} v^{i_j}|^2 \leq \rho_m^2 - \alpha_{i_j}^2$. With the same reasoning as above we obtain $\gamma_z z \in$

$F_{i_j} \cap L^-$. This shows $\operatorname{relint}\{K_{j_l}\} \cap \operatorname{relint}\{K_{j_k}\} = \emptyset$, $k \neq l$, and $\cup_{j=1}^{k} K_{i_j} \subset K_\rho$. Thus we may write

$$\int_{K_\rho} dv = \frac{\int_{K_\rho} dv + \int_{M_\rho} dv}{1 + \int_{M_\rho} dv / \int_{K_\rho} dv} \geq \frac{(d-2)\kappa_{d-2}}{1 + \int_{M_l} dv / \int_{K_l} dv},$$

for a suitable index $l \in \{j_1, \dots, j_k\}$. Let $g(x) = (1-x^2)^{(d-5)/2}$ and for $\gamma \in [1, \rho)$ set $f_1(\gamma) = \int_{\gamma/\rho}^{1} g(x)dx$, $f_2(x) = \int_{\gamma/\rho_m}^{\gamma/\rho} g(x)dx$. Thus $\int_{M_l} dv / \int_{K_l} dv = f_1(\alpha_l)/f_2(\alpha_l)$. The first derivative of $f_1(\gamma)/f_2(\gamma)$ shows that this function is monotonely decreasing and thus

$$\int_{K_\rho} \geq \frac{(d-2)\kappa_{d-2}}{1 + c_3(\rho, d)}. \tag{4.7}$$

Now, by Lemma 4.1 we know: $z \in L^- \cap S^{d-1}$ with $(\rho+\epsilon)z \in H \Rightarrow c_1(\rho,\epsilon)C(\phi) + \rho z \in H$. Hence we obtain for every $\epsilon > 0$ with $\rho + \epsilon < \rho_m$ from (4.6) and (4.7)

$$V(D_\rho^1) \geq \frac{\phi}{2} \cdot c_1(\rho,\epsilon)^2 \rho^{d-2} \cdot \frac{\kappa_{d-2}}{1 + c_3(\rho+\epsilon, d)}$$

□

Proof of Theorem 4.1. Having Lemmas 4.3 – 4.7 the proof is an easy consequence of $\lim_{d\to\infty} \kappa_{d-1}/\kappa_d = \infty$. Let $\rho < \rho_m = 2/\sqrt{3}$ and $\phi_0 = \min\{\arctan(1/\rho), \pi/3\}$. We distinguish three cases depending on ϕ and the sign of $\langle y^1, y^2 \rangle$. For simplification we use $V(D_\rho^2) \geq (1 - \rho\sin(\phi))\rho^{d-1}\kappa_{d-1}$ (cf. Lemma 4.4).

I). $\phi < \phi_0$ and the assumptions of Lemma 4.5 hold. Then we have by Lemma 4.3, 4.4 and 4.5

$$\begin{aligned} V(D_\rho) &\geq V(D_\rho^1) + V(D_\rho^2) + V(D_\rho^3) \\ &\geq \rho^{d-1}\kappa_{d-1} + \frac{\rho^d \kappa_d}{2} + \phi\rho^{d-2}\left(\frac{1}{2}\frac{(1-\rho\sin(\phi_0))^2}{1-\sin^2(\phi_0)}\kappa_{d-2} - \rho^2\kappa_{d-1} - \frac{\rho^2\kappa_d}{2\pi}\right) \\ &\geq \rho^{d-1}\kappa_{d-1} + \frac{\rho^d\kappa_d}{2}, \end{aligned} \tag{4.8}$$

for all sufficiently large d.

II). $\phi < \phi_0$ and the assumptions of Lemma 4.6 hold. Then we have by Lemma 4.3, 4.4 and 4.6 and $\cos(\phi) \geq 1 - \phi^2/2$

$$\begin{aligned} V(D_\rho) &\geq V(D_\rho^1) + V(D_\rho^2) + V(D_\rho^4) \\ &\geq 2\rho^{d-1}\kappa_{d-1} + \phi\rho^{d-2}\left(\frac{1}{2}\frac{(1-\rho\sin(\phi_0))^2}{1-\sin^2(\phi_0)}\kappa_{d-2} - 2\rho^2\kappa_{d-1} - \frac{\phi_0}{2}\rho\kappa_{d-1}\right) \\ &\geq 2\rho^{d-1}\kappa_{d-1}, \end{aligned} \tag{4.9}$$

for all sufficiently large d.

III). $\phi \geq \phi_0$. Choose an ϵ such that the assumption of Lemma 4.7 holds. Then by Lemma 4.7.

$$\begin{aligned} V(D_\rho) &\geq V(D_\rho^1) \\ &\geq \frac{\phi_0}{2} \cdot c_1(\rho,\epsilon)^2\rho^{d-2} \cdot \frac{\kappa_{d-2}}{1 + c_3(\rho+\epsilon, d)} \geq 2\rho^{d-1}\kappa_{d-1}, \end{aligned}$$

for all sufficiently large d. As the first case occurs at most twice everything is proved. □

Remark. The constant $2/\sqrt{3}$ in Theorem 4.1 is almost certainly not best possible. In fact a quick review of the proof shows that it can be replaced by any constant such that (4.7) holds. It should be possible to prove this inequality for any $\rho < \sqrt{2}$ by methods used by Rogers [R3].

Proof of Theorem 4.2

For the proof of Theorem 4.2 we use the well know relation

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}. \tag{4.10}$$

Proof of Theorem 4.2. We use the same argumentation as in the proof of Theorem 4.1. We only evaluate $V(D_1)$ ($\rho = 1$) more carefully. Let $\phi_0 = \pi/4$.
I). $\phi \le \phi_0$ and the assumptions of Lemma 4.5 hold. Then we have by (4.8)

$$\begin{aligned} V(D_1) &\ge V(D_1^1) + V(D_1^2) + V(D_1^3) \\ &\ge \kappa_{d-1} + \frac{\kappa_d}{2} + \phi\kappa_d \left(\frac{1}{2}\frac{1-\sin(\phi_0)}{1+\sin(\phi_0)}\frac{\kappa_{d-2}}{\kappa_d} - \frac{\kappa_{d-1}}{\kappa_d} - \frac{1}{2\pi} \right) \\ &= \kappa_{d-1} + \frac{\kappa_d}{2} + \phi\kappa_d \cdot f_1(d) \end{aligned}$$

By (4.10) we get $f_1(d) \ge 0$ for $d \ge d_1 = 879$ and thus

$$V(D_1) \ge \kappa_{d-1} + \frac{\kappa_d}{2}, \quad d \ge d_1. \tag{4.11}$$

II). $\phi \le \phi_0$ and the assumptions of Lemma 4.6 hold. Then we have by (4.9)

$$\begin{aligned} V(D_1) &\ge V(D_1^1) + V(D_1^2) + V(D_1^4) \\ &\ge 2\kappa_{d-1} + \phi\kappa_{d-1} \left(\frac{1}{2}\frac{1-\sin(\phi_0)}{1+\sin(\phi_0)}\frac{\kappa_{d-2}}{\kappa_{d-1}} - (2 + \frac{\phi_0}{2}) \right) \\ &= 2\rho^{d-1}\kappa_{d-1} + \phi\kappa_{d-1} \cdot f_2(d) \end{aligned}$$

By (4.10) we get $f_2(d) \ge 0$ for $d \ge d_2 = 4889$ and thus

$$V(D_1) \ge 2\kappa_{d-1}, \quad d \ge d_2. \tag{4.12}$$

Let $\phi^* = \pi/3 + 0.00053$.
III). $\phi_0 \le \phi \le \phi^*$. By Lemma 4.3 and 4.4 we obtain

$$\begin{aligned} V(D_1) &\ge V(D_1^1) + V(D_1^2) \\ &\ge 2\kappa_{d-1} + \kappa_{d-1} \left(\frac{\phi}{2}\frac{1-\sin(\phi)}{1+\sin(\phi)}\frac{\kappa_{d-2}}{\kappa_{d-1}} - \frac{2\cos(\phi) - 1 + \sin(\phi)}{\cos(\phi)} \right) \\ &= 2\kappa_{d-1} + \kappa_{d-1} \cdot f_3(\phi, d). \end{aligned}$$

The first partial derivative of f_3 with respect to ϕ shows that f_3 is monotonely decreasing in ϕ and thus $f_3(\phi, d) \geq f_3(\phi^*, d)$. By (4.10) we get $f_3(\phi^*, d) \geq 0$ for $d \geq d_3 = 13.387$ and thus

$$V(D_1) \geq 2\kappa_{d-1}, \quad d \geq d_3. \tag{4.13}$$

IV). $\phi^* \leq \phi \leq \pi/2$. By Lemma 4.4 and Lemma 4.7 we may write with $\epsilon^* = 0.1545 < \rho_m - 1$

$$\begin{aligned} V(D_1) &\geq V(D_1^1) + V(D_1^2) \\ &\geq 2\kappa_{d-1} + \kappa_{d-1}\left(\frac{\phi}{2} \cdot c_1(1,\epsilon^*)^2 \cdot \frac{\kappa_{d-2}/\kappa_{d-1}}{1 + c_3(1+\epsilon^*, d)} - \frac{2\cos(\phi) - 1 + \sin(\phi)}{\cos(\phi)}\right) \\ &= 2\kappa_{d-1} + \kappa_{d-1} \cdot f_4(\phi, d). \end{aligned}$$

The mean value theorem of integral calculus shows $c_3(\mu, d) \geq c_3(\mu, d+1)$ and thus $f_4(\phi, d) < f_4(\phi, d+1)$. By numerical calculations which can be carried out in any desired precision we obtain $f_4(\phi^*, d) \geq 0$ for $d \geq d_4 = 13.387$. In particular we obtain $c_1(1,\epsilon^*)^2 \cdot (\kappa_{d-2}/\kappa_{d-1})/(1+c_3(1+\epsilon^*, d)) \geq 1$ for all $d \geq d_4$. But this implies that f_4 is monotonely increasing in ϕ for $\phi \in [\phi^*, \pi/2]$ and $d \geq d_4$. Hence it follows

$$V(D_1) \geq 2\kappa_{d-1}, \quad d \geq d_4. \tag{4.14}$$

From (4.11)—(4.14) we obtain the assertion (cf. the Proof of Theorem 4.1.) □

5. Sausages for centrally symmetric convex bodies

In this section we prove an analogue of Theorem 4.1 for general centrally symmetric convex bodies.

Theorem 5.1. *There is a positive function $\psi(x, y)$ on $[1, \infty) \times \mathbb{N}$ with the properties*

$$\lim_{y \to \infty} \psi(1, y) = \frac{2}{\sqrt{3}} \quad \text{and} \quad \psi(x_1, y) > \psi(x_2, y) \text{ if } x_1 < x_2,$$

such that for $K \in \mathcal{K}_0^d$ with inradius r and circumradius R and for $\rho \leq \psi(R/r, d)$

$$\min_{C_n \in \mathcal{P}_n(K)} \{V(\mathrm{conv}(C_n + \rho K))\} = \min_{u \in S^{d \perp 1}} \{V(\mathrm{conv}(S_n(u, K) + \rho K))\}.$$

By a theorem of John ([J], [GL, pp. 13]) we can always achieve $R(AK)/r(AK) \leq \sqrt{d}$ by a suitable linear transformation A. Thus we can deduce from Theorem 5.1 (cf. Theorem 1.1 (4))

Corollary 5.1. *Let $K \in \mathcal{K}_0^d$. There exists a constant $\rho_d > 0$ only depending on d such that for $\rho \leq \rho_d$*

$$\min_{C_n \in \mathcal{P}_n(K)} \{V(\mathrm{conv}(C_n + \rho K))\} = \min_{u \in S^{d \perp 1}} \{V(\mathrm{conv}(S_n(u, K) + \rho K))\}.$$

For the sausage radius we obtain

Corollary 5.2. *For each d there is a constant $\rho_d > 0$ such that*

$$\rho_s(K) \geq \rho_d \qquad \text{for all } K \in \mathcal{K}_0^d.$$

Proof of Theorem 5.1

In the sequel let $K \in \mathcal{K}_0^d$ be a centrally symmetric convex body with distance function $f : E^d \to \mathbb{R}$, inradius r and circumradius R. As in the proof of Theorem 4.1 let H be a fixed DV-cell of the arrangement $C_n = \{x^1, \ldots, x^n\} \in \mathcal{P}_n(K)$ with respect to $x^n = 0$ and let ϕ, y^j, L be defined as in section 4. Now, let $D_\rho = H \cap (\mathrm{conv}(C_n + \rho K))$ and $C(\phi) = \mathrm{conv}\{0, 2y^1/f(y^1), 2y^2/f(y^2)\} \cap H$. With respect to $C(\phi)$ we define D_ρ^i, $1 \le i \le 4$, as in Definiton 4.2. Since the arrangement is admissible, we have $|x^i| \ge 2/f(y^i)$ and thus $2y^i/f(y^i) \in \mathrm{conv}(C_n)$. In particular

$$\{x \in E^d : \langle x, y^j \rangle \le 1/f(y^j),\ 1 \le j \le n-1\} \subset H. \tag{5.1}$$

Observe that $1/R \le f(v) \le 1/r$ for $v \in S^{d-1}$. Further we define

Definition 5.1. *Let $u \in S^{d-1}$. For $y \in K_u$ let*

$$\lambda(y,u) = \min\{|\lambda| : \lambda \in \mathbb{R} \text{ and } y + \lambda u \in K\}, \quad s(u) = \max\{\lambda(y,u) : y \in K_u\},$$
$$\rho_K = \max\{1/(s(u)f(u)) : u \in S^{d-1}\},$$
$$s(y,u) = \begin{cases} -\lambda(y,u), & \text{if } y - \lambda(y,u)u \in K \\ \lambda(y,u), & \text{if } y + \lambda(y,u)u \in K. \end{cases}$$

Obviously $\lambda(y,u) \le \sqrt{R^2 - |y|^2}$ and for $|y| \le r$ the point y belongs to K and thus $\lambda(y,u) = 0$. Hence

$$|s(u)| \le \sqrt{R^2 - r^2} \text{ and } \rho_K \ge r/\sqrt{R^2 - r^2}. \tag{5.2}$$

From the definition we have $s(y,u) = -s(-y,u)$. In the case $K = B^d$ we obtain $s(u) = 0$ and we may set $\rho_K = \infty$. For the proof we need the following two Lemmas

Lemma 5.1. *Let $K \in \mathcal{K}_0^d$ with distance function f, inradius r and circumradius R. Then for $v, w \in S^{d-1}$*

$$|\langle v, w \rangle| \ge \cos(\phi) \Rightarrow f(v) \le (1 + \phi \cdot g(R/r)) \cdot f(w),$$

where $g : [1, \infty) \to \mathbb{R}$ is a monotonely increasing function with $g(1) = 0$.

Proof. Assume $\langle v, w \rangle \ge \cos(\phi)$ and let $f(w) \le f(v)$. Further let $a \in E^d$ be a unit outward normal vector of a supporting hyperplane S of K with $v/f(v) \in K \cap S$. Assume $\langle a, v/f(v) \rangle = \gamma = \cos(\eta)/f(v)$. On account of $\langle v, w \rangle \ge \cos(\phi)$ and $\langle a, w/f(w) \rangle \le \langle a, v/f(v) \rangle$ we obtain $\gamma f(w) \ge \langle a, w \rangle \ge \cos(\phi)\cos(\eta) - \sin(\phi)(1 - \cos^2(\eta))^{1/2}$ or

$$1 \ge (f(v)/f(w)) \left(\cos(\phi) - \sin(\phi)\sqrt{(1/\cos(\eta))^2 - 1} \right).$$

Now, $\cos(\eta) \ge r/R$, $f(v)/f(w) \le R/r$ and thus

$$\frac{f(v)}{f(w)} \le \begin{cases} (\cos(\phi) - \sin(\phi)\sqrt{(R/r)^2 - 1})^{-1}, & \phi \le \arccos((r/R)^2) - \arccos(r/R) \\ R/r, & \text{else.} \end{cases}$$

From this it is not hard to deduce an appropriate function g. □

For abbreviation we set $\overline{R} = R/r$ and $\nu(\phi, \overline{R}) = 1 + \phi g(\overline{R})$. Moreover we define the functions

$$\underline{c}_2(\phi,\overline{R}) = \max\left\{\frac{2\nu(\phi,\overline{R}) - 1 + (\rho_m\overline{R})\sin(\phi)\nu(\phi,\overline{R})}{2\nu(\phi,\overline{R}) + 2\cos(\phi)}, \frac{\nu(\phi,\overline{R})}{2\sqrt{2+2\cos(\phi)}}\right\},$$

$$\overline{c}_2(\phi,\overline{R}) = \min\left\{\frac{2\cos(\phi) + 1/\nu(\phi,\overline{R}) - (\rho_m\overline{R})\sin(\phi)}{2\nu(\phi,\overline{R}) + 2\cos(\phi)}, 1 - \frac{\nu(\phi,\overline{R})}{2\sqrt{2+2\cos(\phi)}}\right\},$$

$$\gamma(\rho,\phi,\overline{R}) = (1 - (\rho\overline{R})\sin(\phi))/\nu(\phi,\overline{R}).$$

Lemma 5.2. *Let* $\sin(\phi) \le 1/(\rho_m\overline{R})$*. Then for* $\rho \le \rho_m$ *and* $i = 1, 2$

$$\gamma(\rho,\phi,\overline{R}) \cdot \mathrm{conv}\{0, y^i/f(y^i)\} + \rho R\left((y^i)^- \cap B^d\right) \subset H. \tag{5.3}$$

Proof. Apparently, this statement is closely related to Lemma 4.4, but in contrast to Lemma 4.4 we can not make use of Lemma 4.1 because in general $y^j/f(y^j) \notin rS^{d-1}$. Now, let $v_{y^i} \in ((y^i)^- \cap B^d)$, $i = 1, 2$. By the definition of ϕ we have $\langle y^j, v_{y^i}\rangle \le \sin(\phi)$ (cf. Lemma 4.3) and on account of the definition of $\nu(\phi,\overline{R})$ we obtain $\langle \lambda y^i/f(y^i) + \rho R v_{y^i}, y^j\rangle \le 1/f(y^j)$, $i = 1, 2$, if $\lambda \le \gamma(\rho,\phi,\overline{R})$ and hence by (5.1) we get (5.3). □

Now, we transfer the results of the Lemmas 4.3 – 4.7 to the centrally symmetric convex body K.

Lemma 5.3. *Let* $\sin(\phi) \le 1/(\rho_m\overline{R})$*. Then for* $\rho \le \rho_m$

$$V(D^1_\rho) \ge r^d \cdot \frac{\phi}{2} \cdot c_1(\rho, 1/\sin(\phi) - \rho)^2 \rho^{d-2} \cdot \kappa_{d-2}.$$

Proof. Immediate consequence of $rB^d \subset D_\rho$ and Lemma 4.3. □

The next two Lemmas correspond to Lemma 4.4 and Lemma 4.5. Since in general the set $\{x \in D^2_\rho : \langle y^i, x\rangle = \gamma\}$ for $\gamma \in [0, \gamma(\rho,\phi,\overline{R})/f(y^i)]$ does not contain a set of volume $\rho^{d-1}V(K_{y^i})/2$ as in the case $K = B^d$ we have to evaluate these sections more carefully. To this end we distinguish two cases depending on the sign of $\langle y^1, y^2\rangle$. In the following let $z^i \in L$, $i = 1, 2$, be the outward unit normal vectors of $\mathrm{conv}\{0, 2y^i/f(y^i)\}$ with respect to $C(\phi)$. Further let $\overline{u} \in S^{d-1}$ with

$$2(n-1)\frac{V(K_{\overline{u}})}{f(\overline{u})}\rho^{d-1} + \rho^d V(K) = \min\{V(\mathrm{conv}(S_n(u,K) + \rho K)) : u \in S^{d-1}\},$$

and let $\beta(K) = V(K_{\overline{u}})/f(\overline{u})$.

Lemma 5.4. *Let* $\langle y^1, y^2\rangle \le -\cos(\phi)$ *and let* ϕ *satisfy:* $\sin(\phi) \le 1/(\rho_m\overline{R})$*, and* $\nu(\phi,\overline{R}) \le 2\cos(\phi)$*. Then for* $\rho \le \min\{\rho_m, \rho_K\}$

$$V(D^2_\rho) \ge \gamma(\rho,\phi,\overline{R})\left(\rho^{d-1}\beta(K) - 2\frac{\sin(\phi)}{\cos(\phi)}\sqrt{R^2 - r^2}\rho^{d-1}R^{d-1}\kappa_{d-2}\right).$$

Proof. For $i=1,2$ let $M^i_\gamma(\rho)=\{x\in \mathrm{conv}(C_n+\rho K) : \langle y^i,x\rangle=\gamma,\, \langle z^i,x\rangle \geq 0,\, |x-\gamma y^i|\leq \rho R\}$ with $\gamma\in[0,\gamma(\rho,\phi,\overline{R})/f(y^i)]$. In the following we show

$$V(M^i_\gamma(\rho))\geq \rho^{d-1}\frac{V(K_{y^i})}{2}-\frac{\rho s(y^i)\sin(\phi)}{\cos(\phi)}(\rho R)^{d-2}\kappa_{d-2}. \tag{5.4}$$

By (5.3) we have $M^i_\gamma(\rho)\subset D^2_\rho$ and on account of the definition of $s(y^i)$ (5.4) implies the assertion. For the proof of (5.4) we will only consider the case $i=1$; the other case can be treated similarly. Before we start we introduce some notation:

$$\begin{aligned}
T&=\{x\in\rho K_{y^1} : \langle z^1,x\rangle\geq 0\},\\
T^{\leq 0}&=\{x\in T : s(x/\rho,y^1)\leq 0\}, & T^{>0}&=\{x\in T : s(x/\rho,y^1)>0\},\\
M_\gamma&=\{x\in \mathrm{conv}(C_n+\rho K) : \langle y^1,x\rangle=\gamma\}, & M^0_\gamma&=M^1_\gamma(\rho)-\gamma y^1,\\
\alpha(x,\phi,\gamma)&=\rho\cdot s(x/\rho,y^1)\frac{2\sin(\phi)}{\gamma f(y^2)+2\cos(\phi)}, & \alpha(\phi)&=\rho\cdot s(y^1)\frac{\sin(\phi)}{\cos(\phi)}.
\end{aligned}$$

Obviously we have $V(T)=\rho^{d-1}V(K_{y^1})/2$ and $\alpha(\phi)\geq\alpha(x,\phi,\gamma)$. First we claim

$$T^{\leq 0}+\gamma y^1\subset M^1_\gamma(\rho). \tag{5.5}$$

To prove this, it suffices to show $T^{\leq 0}+\gamma y^1\subset \mathrm{conv}(C_n+\rho K)$. Let $x\in T^{\leq 0}$ and let $\mu=(\rho\cdot s(x/\rho,y^1)+\langle x^1,y^1\rangle-\gamma)/\langle x^1,y^1\rangle$. Since $\rho\cdot s(x/\rho,y^1)-\gamma\leq 0$ we have $\mu\leq 1$. Further $\rho\leq\rho_K$ yields $\rho\cdot s(x/\rho,y^1)\geq -1/f(y^1)$ and on account of $\gamma\leq 1/f(y^1)$, $\langle x^1,y^1\rangle=|x^1|\geq 2/f(y^1)$ it follows $\mu\geq 0$. Now, $x+\gamma y^1=\mu(x+\rho s(x/\rho,y^1)y^1)+(1-\mu)(x+\rho s(x/\rho,y^1)y^1+x^1)\in\mathrm{conv}(C_n+\rho K)$. This shows (5.5). Next we claim

$$x\in T^{>0}\Rightarrow x-\alpha(x,\phi,\gamma)z^1+\gamma y^1\in M_\gamma. \tag{5.6}$$

Let $x\in T^{>0}$ and let $\mu=(\rho\cdot s(x/\rho,y^1)f(y^2))/(\gamma f(y^2)+2\cos(\phi))$. It is clear that $\mu\geq 0$. By the definition of ρ_K and by the choice of ϕ we obtain $\rho\cdot s(x/\rho,y^1)f(y^2)\leq f(y^2)/f(y^1)\leq\nu(\phi,\overline{R})\leq 2\cos(\phi)$, where the second inequality follows from Lemma 5.1. Hence $\mu\leq 1$ and on account of $y^2=-\cos(\phi)y^1-\sin(\phi)z^1$ we get $x-\alpha(x,\phi,\gamma)z^1+\gamma y^1=\mu(2y^2/f(y^2)+x+\rho s(x/\rho,y^1)y^1)+(1-\mu)(\gamma y^1+x+\rho s(x/\rho,y^1)y^1)$ $\in\mathrm{conv}(C_n+\rho K)$, which implies (5.6).

Now, let U be the orthogonal projection of T onto the hyperplane $\{x\in E^d : \langle z^1,x\rangle=0\}$. For $x\in U$ let $\overline{v}_x=\max\{v\in\mathbb{R} : x+vz^1\in M^0_\gamma\}$, $\underline{v}_x=\min\{v\in\mathbb{R} : x+vz^1\in M^0_\gamma\}$ and let $\overline{\sigma}_x$, $\underline{\sigma}_x$ be defined in the same way with respect to T instead of M^0_γ. Since $V(M^1_\gamma(\rho))\geq\int_U\overline{v}_x-\underline{v}_x dx$ and $\rho^{d-1}V(K_{y^1})/2=V(T)=\int_U\overline{\sigma}_x-\underline{\sigma}_x dx$ it suffices to show for (5.4) that for $x\in U$

$$\overline{v}_x-\underline{v}_x\geq\overline{\sigma}_x-\underline{\sigma}_x-\alpha(\phi). \tag{5.7}$$

Observe, if $\langle z^1,x\rangle-\alpha(x,\phi,\gamma)\geq 0$ holds for $x\in T^{>0}$ then we also have $x-\alpha(x,\phi,\gamma)z^1+\gamma y^1\in M^1_\gamma(\rho)$. Thus (5.7) follows immediately from (5.5), (5.6) and the convexity of $M^1_\gamma(\rho)$. □

Lemma 5.5. *Let* $\sin(\phi) \leq 1/(\rho_m \overline{R})$*,* $\phi < \pi/3$ *and* $\langle y^1, y^2\rangle > 0$*. Then for* $\rho \leq \min\{\rho_m, \rho_K\}$

$$V(D_\rho^2) + V(D_\rho^3) \geq \rho^d \frac{V(K)}{2} + \gamma(\rho, \phi, \overline{R})\rho^{d-1}\beta(K) - \frac{\phi}{\pi}(\rho R)^d \kappa_d.$$

Proof. Again we introduce some sets

$$\begin{aligned} U^r &= \{x \in \rho K : \langle y^2, x\rangle > 0 \wedge \langle y^1, x\rangle \leq 0\}, \\ U^l &= \{x \in \rho K : \langle y^2, x\rangle \leq 0 \wedge \langle y^1, x\rangle \leq 0\}, \\ T_i^r &= \{x \in \mathrm{conv}\{\rho K, x^i + \rho K\} : 0 \leq \langle y^i, x\rangle \leq \gamma(\rho, \phi, \overline{R}) \wedge \langle z^i, x\rangle \geq 0\},\ i = 1, 2, \\ T_i^l &= \{x \in \mathrm{conv}\{\rho K, x^i + \rho K\} : \langle y^i, x\rangle \leq 0 \wedge \langle z^i, x\rangle \leq 0 \wedge x \notin \rho K\},\ i = 1, 2. \end{aligned}$$

Obviously, $U^l \subset D_\rho^3$ and on account of (5.3) we get $T_i^r \subset D_\rho^2$. First we show that T_i^l belongs to $D_\rho^2 \cup D_\rho^3$. Let $x \in T_i^l$. Then $x \in \mathrm{conv}\{\rho R B^d, x^i + \rho R B^d\}$ and $\langle y^i, x\rangle \leq 0$ yields $|x| \leq \rho R$. Since $\phi < \pi/3$ we have $\langle y^i, y^j\rangle \geq \cos(\phi)$ for $1 \leq j \leq n-1$ and so $\langle y^j, x\rangle \leq \sin(\phi)|x| \leq \sin(\phi)\rho R \leq r$. Here the last inequality follows by the choice of ϕ. Now $r \leq 1/f(y^j)$ and from (5.1) we obtain $x \in H$ which implies $x \in D_\rho^2 \cup D_\rho^3$. On account of $V(U^l \cup U^r) = \rho^d V(K)/2$ and $\dim(U^l \cap (T_i^r \cup T_i^l)) \leq d-1$, $\dim(T_1^l \cap T_2^l) \leq d-1$, $\dim(T_1^r \cap T_2^r) \leq d-1$ we may write

$$\begin{aligned} V(D_\rho^2 \cup D_\rho^3) &\geq V(U^l \cup (T_1^r \cup T_1^l) \cup (T_2^r \cup T_2^l)) \\ &\geq V(U^l) + V(T_1^r \cup T_1^l) + V(T_2^r \cup T_2^l) - V(T_1^l \cap T_2^r) - V(T_1^r \cap T_2^l) \\ &\geq \rho^d \frac{V(K)}{2} + \sum_{i=1}^{2} V(T_i^r + T_i^l) - V(U^r) - V(T_1^l \cap T_2^r) - V(T_1^r \cap T_2^l). \end{aligned} \tag{5.8}$$

For $i = 1, 2$ we claim

$$V(T_i^r \cup T_i^l) \geq \gamma(\rho, \phi, \overline{R}) \cdot \rho^{d-1} \frac{V(K_{y^i})}{2 f(y^i)}. \tag{5.9}$$

To prove this, it suffices to show that we have for $M_\gamma^i = \{x \in T_i^r : \langle y^i, x\rangle = \gamma\} \cup \{x \in T_i^l : \langle y^i, x\rangle = -\gamma\}$ and $0 \leq \gamma \leq \gamma(\rho, \phi, \overline{R})/f(y^i)$

$$V(M_\gamma^i) \geq \rho^{d-1} \frac{V(K_{y^i})}{2}. \tag{5.10}$$

To this end let $x \in \rho K_{y^i}$ with $\langle z^i, x\rangle \geq 0$.
a) $\rho \cdot s(x/\rho, y^i) \leq \gamma$.
Let $\mu = (\langle x^i, y^i\rangle + \rho \cdot s(x/\rho, y^i) - \gamma)/\langle x^i, y^i\rangle$. As in the proof of (5.5) we may deduce $\mu \in [0, 1]$ and obtain $x + \gamma y^i = \mu(x + \rho \cdot s(x/\rho, y^i)y^i) + (1-\mu)(x + \rho \cdot s(x/\rho, y^i)y^i + x^i) \in \mathrm{conv}\{\rho K, x^i + \rho K\}$. Hence $x + \gamma y^i \in T_i^r$ and $x + \gamma y^i \in M_\gamma^i$.
b) $\rho \cdot s(x/\rho, y^i) > \gamma$.
Assume $x + \gamma y^i \in \rho K$. Then we have $s(x/\rho, y^i) \leq \gamma/\rho$ which contradicts the assumption. Hence $-x - \gamma y^i \notin \rho K$ and further $\langle z^i, -x - \gamma y^i\rangle \leq 0$. Now let $\mu = (\langle x^i, y^i\rangle + \gamma - \rho \cdot s(x/\rho, y^i))/\langle x^i, y^i\rangle$. Again $\mu \in [0, 1]$. Since $-x - \rho \cdot s(x/\rho, y^i)y^i =$

$-x+\rho\cdot s(-x/\rho,y^i)y^i\in\rho K$ we obtain $-x-\gamma y^i=\mu(-x-\rho\cdot s(x/\rho,y^i)y^i)+(1-\mu)(-x-\rho\cdot s(x/\rho,y^i)y^i+x^i)\in\operatorname{conv}\{\rho K,x^i+\rho K\}$. Thus we have $-x-\gamma y^1\in T_i^l$ and $-x-\gamma y^i\in M_\gamma^i$.

Altogether we get (5.10) and thus (5.9). Now by definition we have $U^r\cap(T_1^l\cap T_2^r)=\emptyset$ and $U^r\cup(T_1^l\cap T_2^r)\subset\{x\in E^d:|x|\le\rho R\wedge\langle y^2,x\rangle\ge 0\wedge\langle y^1,x\rangle\le 0\}$. Further $(T_1^r\cap T_2^l)\subset\{x\in E^d:|x|\le\rho R\wedge\langle y^2,x\rangle\le 0\wedge\langle y^1,x\rangle\ge 0\}$. By (5.9) and (5.8) we get the assertion. □

Lemma 5.6. *Let $\langle y^1,y^2\rangle<0$ and let ϕ satisfy: $\phi<\pi/3$ and $\underline{c}_2(\phi,\overline{R})\le\overline{c}_2(\phi,\overline{R})$. Then for $\rho\le\min\{\rho_m,\rho_K\}$*

$$V(D_\rho^4)\ge(\overline{c}_2(\phi,\overline{R})-\underline{c}_2(\phi,\overline{R}))\cdot\frac{\sqrt{2+2\cos(\phi)}}{\nu(\phi,\overline{R})}\cdot\rho^{d-1}\cdot\beta(K).$$

Proof. Let $\underline{u}(\phi,\overline{R})$ ($\overline{u}(\phi,\overline{R})$) be the first expression in the definition of $\underline{c}_2(\phi,\overline{R})$ ($\overline{c}_2(\phi,\overline{R})$) and let $w=(y^1/f(y^1)-y^2/f(y^2))/|y^1/f(y^1)-y^2/f(y^2)|$. On account of the definition of $\nu(\phi,\overline{R})$ and (5.1) we obtain with the method used in the proof of Lemma 4.6

$$\lambda\frac{2y^1}{f(y^1)}+(1-\lambda)\frac{2y^2}{f(y^2)}+R\rho(w^-\cap B^d)\subset H,\ \lambda\in[\underline{u}(\phi,\overline{R}),\overline{u}(\phi,\overline{R})]. \tag{5.11}$$

By assumption the above interval is well defined. Without loss of generality we assume $f(y^2)\le f(y^1)$ and thus $|y^1/f(y^1)-y^2/f(y^2)|\ge|y^1-y^2|/f(y^1)$. On account of $|\langle y^1,w\rangle|\ge\cos(\phi)$ we obtain with respect to Lemma 5.1

$$f(w)\cdot\left|\frac{y^1}{f(y^1)}-\frac{y^2}{f(y^2}\right|\ge\frac{|y^1-y^2|}{f(y^1)}\cdot\frac{f(y^1)}{\nu(\phi,\overline{R})}\ge\frac{\sqrt{2+2\cos(\phi)}}{\nu(\phi,\overline{R})}. \tag{5.12}$$

Let z^3 be the outward normal vector of $\operatorname{conv}\{2y^1/f(y^1),2y^2/f(y^2)\}$ with respect to $C(\phi)$ and for $\gamma\in[\underline{c}_2(\phi,\overline{R}),\overline{c}_2(\phi,\overline{R})]$ let $M_\gamma=\{x\in\operatorname{conv}\{2y^2/f(y^2)+\rho K,2y^1/f(y^1)+\rho K\}:x=\gamma 2y^1/f(y^1)+(1-\gamma)2y^2/f(y^2)+y$ with $\langle w,y\rangle=0\wedge\langle z^3,y\rangle\ge 0\}$. From (5.11) we get $M_\gamma\subset D_\rho^4$ and we claim

$$V(M_\gamma)\ge\rho^{d-1}\frac{V(K_w)}{2}. \tag{5.13}$$

Let $T=\{y\in\rho K_w:\langle z^3,y\rangle\ge 0\}$. For $y\in T$ let $\mu=\gamma-(\rho\cdot s(y/\rho,w))(2|y^1/f(y^1)-y^2/f(y^2)|)$. On account of (5.12), the choice of γ and the definition of ρ_K we have $\mu\in[0,1]$. Hence $\gamma 2y^1/f(y^1)+(1-\gamma)2y^2/f(y^2)+y=\mu(2y^1/f(y^1)+y+\rho\cdot s(y/\rho,w)w)+(1-\mu)(2y^2/f(y^2)+y+\rho\cdot s(y/\rho,w)w)\in M_\gamma$. Thus $\gamma 2y^1/f(y^1)+(1-\gamma)2y^2/f(y^2)+T\subset M_\gamma$ and we obtain (5.13). Hence

$$\begin{aligned}V(D_\rho^4)&\ge(\overline{c}_2(\phi,\overline{R})-\underline{c}_2(\phi,\overline{R}))\cdot\left|\frac{2y^1}{f(y^1)}-\frac{2y^2}{f(y^2)}\right|\rho^{d-1}\frac{V(K_w)}{2}\\&\ge(\overline{c}_2(\phi,\overline{R})-\underline{c}_2(\phi,\overline{R}))\cdot\frac{|y^1-y^2|}{\nu(\phi,\overline{R})}\cdot\rho^{d-1}\frac{V(K_w)}{f(w)}\end{aligned}$$

□

Lemma 5.7. *Let $\phi > 0$ and $c_3(\mu, d)$ be defined as in Lemma 4.7. Then for every $\rho < \rho_m$ and $\epsilon > 0$ such that $\rho + \epsilon < \rho_m$*

$$V(D_\rho^1) \geq r^d \frac{\phi}{2} \cdot c_1(\rho, \epsilon)^2 \rho^{d-2} \cdot \kappa_{d-2}/(1 + c_3(\rho + \epsilon, d)).$$

Proof. Immediate consequence of $rB^d \subset D_\rho$ and Lemma 4.7. □

Proof of Theorem 5.1. Let K be a centrally symmetric convex body with circumradius R and inradius r. Further let $\overline{\phi} \in (0, \pi/2]$ satisfy $\overline{c}_2(\overline{\phi}, \overline{R}) \geq \underline{c}_2(\overline{\phi}, \overline{R})$, $\sin(\overline{\phi}) \leq 1/(\rho_m \overline{R})$ and $\nu(\overline{\phi}, \overline{R}) \leq 2\cos(\overline{\phi})$. Set $\phi_0 = \min\{\overline{\phi}, \pi/3\}$. Observe that ϕ_0 depends only on the ratio R/r.

As in the proof of Theorem 4.1 we distinguish three cases depending on ϕ and the sign of $\langle y^1, y^2 \rangle$. Since the proof is completely analogous to the proof of Theorem 4.1 we only give the essential steps.

I). $\phi < \phi_0$ and the assumptions of Lemma 5.5 hold. By Lemma 5.3 and Lemma 5.5 we get for $\rho \leq \min\{\rho_m, \rho_K\}$

$$V(D_\rho) \geq V(D_\rho^1) + V(D_\rho^2) + V(D_\rho^3) \geq \frac{V(K)}{2}\rho^d + \beta(K)\rho^{d-1} + \phi\rho^{d-2} r^d \cdot f_1(\rho, \overline{R}, d),$$

where f_1 is a function with the following properties: f_1 is continuous in ρ and $\overline{R}$, monotonely decreasing in $\overline{R}$ and $f_1(0, \overline{R}, d) > 0$. Thus there exists a $\psi_1(\overline{R}, d) > 0$ such that

$$V(D_\rho) \geq \frac{V(K)}{2}\rho^d + \beta(K)\rho^{d-1}, \quad \rho \in [0, \psi_1(\overline{R}, d)].$$

II). $\phi < \phi_0$ and the assumptions of Lemma 5.6 hold. By Lemma 5.3, 5.4 and 5.6 we get for $\rho \leq \{\rho_m, \rho_K\}$

$$V(D_\rho) \geq V(D_\rho^1) + V(D_\rho^2) + V(D_\rho^4) \geq 2\beta(K)\rho^{d-1} + \phi\rho^{d-2} r^d \cdot f_2(\rho, \overline{R}, d),$$

for a certain function f_2 with the same properties as f_1. Thus there exists a $\psi_2(\overline{R}, d) > 0$ such that

$$V(D_\rho) \geq 2\beta(K)\rho^{d-1}, \quad \rho \in [0, \psi_2(\overline{R}, d)].$$

III). $\phi \geq \phi_0$. For $0 \leq \rho < \rho_m$ let $\gamma(\rho) = \max\{c_1(\rho, \epsilon)^2/(1 + c_3(\rho + \epsilon, d)) : \epsilon > 0, \rho + \epsilon < \rho_m\}$. Then by Lemma 5.7

$$V(D_\rho) \geq V(D_\rho^1) \geq r^d \frac{\phi}{2}\rho^{d-2}\kappa_{d-2}\gamma(\rho) \geq 2\beta(K)\rho^{d-1} + r^d\rho^{d-2} \cdot f_3(\rho, \overline{R}, d),$$

for a certain function f_3 with the same properties as f_1. Thus there exists a $\psi_3(\overline{R}, d) > 0$ such that

$$V(D_\rho) \geq 2\beta(K)\rho^{d-1}, \quad \rho \in [0, \psi_3(\overline{R}, d)].$$

Now, let $\psi(\overline{R}, d) = \min\{\psi_1(\overline{R}, d), \psi_2(\overline{R}, d), \psi_3(\overline{R}, d)\}$. As the first case occurs at most twice the assertion of Theorem 5.1 is proved for $\rho \leq \psi(\overline{R}, d)$. Since the functions $\psi_i(\overline{R}, d)$ are monotonely decreasing in $\overline{R}$ we also have this property for $\psi(\overline{R}, d)$.

Assume $\overline{R} = 1$. Thus K is a ball with radius 1, say. We obtain $\rho_K = \infty$, $g(\overline{R}) = 0$, $\nu(\phi, \overline{R}) = 1$, $(\rho\overline{R}) = \rho$ and $\underline{c}_2(\phi, \overline{R}) = c_2(\phi, \rho_m)$, $\overline{c}_2(\phi, \overline{R}) = 1 - c_2(\phi, \rho_m)$ (cf. Lemma 4.6). Thus Lemmas 5.3 – 5.7 become the appropriate Lemmas of section 4, where Lemma 5.5 is a combination of Lemma 4.4 and 4.5. Since fixing $\rho = \rho_m$ in Lemma 4.6 has no influence on the proof of Theorem 4.1 we can choose the functions $f_i(\rho, \overline{R}, d)$, $1 \leq i \leq 3$, in such a way that $\lim_{d\to\infty} \psi(1, d) = \rho_m$. □

References

[BG] U. Betke and P. Gritzmann, *ber L. Fejes Tth's Wurstvermutung in kleinen Dimensionen*, Acta Math. Hungar. **43** (1984), 299–307.

[BGW] U. Betke, P. Gritzmann and J.M. Wills, *Slices of L. Fejes Tth's Sausage Conjecture*, Mathematika **29** (1982), 194–201.

[BHW] U. Betke, M. Henk and J.M. Wills, *Finite and Infinite Lattice Packings*, in preparation.

[BF] T. Bonnesen and W. Fenchel, *Theorie der konvexen Krper*, Springer, Berlin, 1934.

[Bö1] K. Böröczky, Jr., *Intrinsic Volumes of finite ball-packings*, PhD thesis, University of Calgary, 1992.

[Bö2] ———, *About four-ball packings*, Mathematika (to appear).

[Bö3] ———, *Proceedings of the conference on intuitive Geometry*, Szeged, 1991.

[BöH] K. Böröczky, Jr. and M. Henk, *Radii and the Sausage Conjecture* (to appear).

[CS] J.H. Conway and N.J.A. Sloane, *Sphere Packings, Lattices and Groups*, Springer, New York, 1988.

[EGH] P. Erds, P.M. Gruber and J. Hammer, *Lattice Points*, Longman, Essex, 1989.

[FGW] G. Fejes Tth, P. Gritzmann and J.M. Wills, *Finite Sphere Packing and Sphere Covering*, Discrete Comp. Geom. **4** (1989), 19–40.

[FK] G. Fejes Tth and W. Kuperberg, *Packing and Covering (Chapter 3.3)*, Handbook of Convex Geometry (P.M. Gruber and J.M. Wills, eds.), North Holland, Amsterdam, 1993.

[F] L. Fejes Tth, *Research Problem 13*, Period. Math. Hungar. **6** (1975), 197–199.

[FG] J.H. Folkman and R.L. Graham, *A Packing Inequality for Compact Convex Subsets of the Plane*, Canad. Math. Bull. **12** (1969), 745–752.

[GWZ] R.L. Graham, H.S. Witsenhausen and H.J. Zassenhaus, *On Tightest Packings in the Minkowski Plane*, Pacific J. Math. **41** (1972), 699–715.

[Gr] P. Gritzmann, *Finite Packungen und Überdeckungen*, Habilitationsschrift, Universität-GH Siegen, 1984.

[GW] P. Gritzmann and J.M. Wills, *Finite Packing and Covering (Chapter 3.4)*, Handbook of Convex Geometry (P.M. Gruber and J.M. Wills, eds.), North Holland, Amsterdam, 1993.

[G] H. Groemer, *ber die Einlagerungen von Kreisen in einem konvexen Bereich*, Math. Z. **73** (1960), 285–294.

[GL] P.M. Gruber and C.G. Lekkerkerker, *Geometry of Numbers*, North Holland, Amsterdam, 1987.

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and essays presented to R. Courant (1948), 187–204.

[McS] P. McMullen and G.C. Shephard, *Convex polytopes and the upper bound conjecture*, Cambridge University Press, 1971.

[O] N. Oler, *An Inequality in the Geometry of Numbers*, Acta Math. **105** (1961a), 19–48.

[R1] C.A. Rogers, *The Closest Packing of Convex Two-Dimensional Domains*, Acta Math. **86** (1951), 309–321.

[R2] ———, *The Closest Packing of Convex Two-Dimensional Domains, Corrig.*, Acta Math. **104** (1960), 305–306.

[R3] ———, *Packing and Covering*, Cambridge University Press, 1964.

[W] G. Wegner, *Über endliche Kreispackungen in der Ebene*, Studia Sci. Math. Hung. **21** (1986), 1–28.

Mathematisches Institut, Universitt Siegen, Hlderlinstrasse 3, D-57068 Siegen, Federal Republic of Germany.

E-mail address: `henk@hrz.uni-siegen.dbp.de` or `wills@hrz.uni-siegen.dbp.de`