# INRADII OF SIMPLICES

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Abstract. We study the following generalization of the inradius: For a convex body $K$ in the $d$-dimensional Euclidean space and a linear $k$-plane $L$ we define the inradius of $K$ with respect to $L$ by $r_L(K) = \max\{r(K; x + L) : x \in E^d\}$, where $r(K; x + L)$ denotes the ordinary inradius of $K \cap (x + L)$ with respect to the affine plane $x + L$. We show how to determine $r_L(P)$ for polytopes and use the result to estimate $\min\{r_L(T_r^d) : L \text{ is a } k\text{-plane}\}$ for the regular $d$-simplex $T_r^d$. These estimates are optimal for all $k$ in infinitely many dimensions and for certain $k$ in the remaining dimensions.

## 1. Introduction

We denote by $E^d$ the $d$-dimensional Euclidean space equipped with the norm $||\cdot||$ and inner product $\langle\cdot,\cdot\rangle$. The space of all compact convex bodies is denoted by $\mathcal{K}^d$ and $B^d \in \mathcal{K}^d$ denotes the $d$-dimensional unit ball. By $\mathcal{L}_k^d$ we denote the space of all $k$–dimensional linear subspaces of $E^d$ and for $L \in \mathcal{L}_k^d$, $M \subset E^d$ we write $M|L$ for the orthogonal projection of $M$ onto $L$. The orthogonal complement of $L \in \mathcal{L}_k^d$ is denoted by $L^\perp$.

The inradius $r(K)$, the circumradius $R(K)$, the diameter $D(K)$ and the minimal width $\Delta(K)$ are classical fundamental functionals of a convex body $K \in \mathcal{K}^d$. For a detailed description we refer to [BF34]. More information on the body can be obtained if we link each pair of the functionals by a series of $d-2$ intermediate functionals. It turns out that these functionals allow generalizations and analogues of many classical results. A first systematic study of these series can be found in [Hen91] and for recent work see e.g. [Bal92], [Hen92], [BH92], [BH93].

Some of these intermediate functionals are well known functionals in approximation theory, called Bernstein and Kolmogorov diameters, (cf. e.g. [Pin85], [Puk79]) and they are also of interest for computational aspects of convex bodies (cf. e.g. [GK93], [GHK90], [BH93]).

Here we study a series linking the inradius and minimal width. There are two natural series of this kind. For the definition we need some more notation: For a $k$-dimensional plane $L$ and a set $M \subset E^d$ we write $r(M; L)$ for the inradius of $M$ with respect to the space $L$, i.e., $r(M; L)$ is the radius of the largest $k$-dimensional ball contained in $M \cap L$. A ball with radius $r(M; L)$ contained in $M \cap L$ is called an inball. For $x \in E^d$ the translate $\{l + x : l \in L\}$ is denoted by $L(x)$.

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**Definition 1.** *For $K \in \mathcal{K}^d$ and $1 \leq k \leq d$ let*

$$r_k^\pi(K) = \min_{L \in \mathcal{L}_k^d} r(K|L; L) \quad \textit{and} \quad r_k^\sigma(K) = \min_{L \in \mathcal{L}_k^d} \max_{x \in E^d} r(K; L(x)).$$

Obviously, we have

$$r(K) = r_d^\pi(K) \leq r_{d-1}^\pi(K) \leq \cdots \leq r_1^\pi(K) = \Delta(K)/2,$$
$$r(K) = r_d^\sigma(K) \leq r_{d-1}^\sigma(K) \leq \cdots \leq r_1^\sigma(K) = \Delta(K)/2.$$

Furthermore, we have $r_k^\pi(K) \geq r_k^\sigma(K)$ and there exist convex bodies $K \in \mathcal{K}^d$, $d \geq 3$, with $r_k^\pi(K) > r_k^\sigma(K)$ [Tsi96].

For a $d$-simplex $T^d \in \mathcal{K}^d$ an explicit formula for $r_k^\pi(T^d)$ was given in [BH93]. Apparently the situation for the $r_k^\sigma(K)$ is much more complicated, as we have a twofold optimization. Here we give formulae for the inner optimization and then we use these formulae to compute the $r_k^\sigma(T_r^d)$ for the regular $d$-simplex $T_r^d$.

For the analogous series $r_\sigma^k(K) = \max_{L \in \mathcal{L}_k^d} \max_{x \in E^d} r(K; L(x))$ this was done by K. Ball (cf. [Bal92]). There it turned out that the $r_\sigma^k(T_r^d)$ are the ordinary inradii of $k$–dimensional faces. Unfortunately, the problem to characterize the optimal planes for $r_k^\sigma(T_r^d)$ is much more intractable. In particular, we will see that the value of $r_k^\sigma(T_r^d)$ depends on the parity of the dimension.

Based on a formula for $r_k^\sigma(T^d)$ for arbitrary simplices $T^d$ we will give estimates for $r_k^\sigma(T_r^d)$ which are optimal for infinitely many dimensions and for certain $k$ in the remaining dimensions but do not hold in general. The first case that the estimate is not tight occurs for $d$ even and $k = 1$ or $k = d - 1$. However, in this instance we have again an exact formula.

In more detail our results are as follows: For $L \in \mathcal{L}_k^d$ and $K \in \mathcal{K}^d$ let

$$r_{L_k}(K) = \max_{x \in E^d} r(K; L_k(x)). \tag{1}$$

For polytopes we have

**Theorem 1.** *Let $P \subset E^d$ be a $d$-polytope and $L_k \in \mathcal{L}_k^d$. Then $r_{L_k}(P)$ is attained for a plane $L_k(x)$ such that an inball of $P \cap L_k(x)$ touches at least $d + 1$ facets of $P$.*

By Theorem 1 the computation of $r_k^\sigma(P)$ is reduced to the computation of $r_k^\sigma(T^d)$ for finitely many simplices $T^d \subset P$. For a $d$-simplex $T^d$ Theorem 1 yields an effective formula: Let $u^0, \ldots, u^{d+1}$ be the outer normal vectors of the facets of $T^d$ such that $||u_i||$ is equal to the area of the corresponding facet. $u^0, \ldots, u^d$ are called the facets vectors of $T^d$. With this notation we have

**Theorem 2.** *Let $T^d \subset E^d$ be a $d$-simplex with facets vectors $u^0, \ldots, u^d$ and $L_k \in \mathcal{L}_k^d$. Then*

$$r_{L_k}(T^d) = r(T^d) \cdot \frac{\sum_{i=0}^d ||u^i||}{\sum_{i=0}^d ||u^i|L_k||}.$$

To state our last Theorem about the inradii of a regular simplex we need Hadamard numbers:

**Definition 2.** *A $p \times p$–matrix $H = (h_{ij})$ is called a Hadamard matrix if $h_{ij} = \pm 1$ for all $i, j$ and $HH^T = pI_p$ where $H^T$ denotes the transpose of $H$ and $I_p$ denotes the unit matrix. A number $p$, for which a Hadamard matrix exists, is called a Hadamard number.*

It is well known that $p = 2^m$ is a Hadamard number for all $m \in \mathbb{N}$ and it is conjectured that $p = 4m$ is a Hadamard number for all $m \in \mathbb{N}$. The conjecture has been verified for $m \leq 106$ and in many other cases as well (cf. [Miy91]).

**Theorem 3.** *For a regular $d$-simplex $T_r^d$ with $r(T_r^d) = 1$ one has*

$$r_k^\sigma(T_r^d) \geq \sqrt{\frac{d}{k}} \text{ for } k = 1, \dots, d.$$

*Equality holds (at least) in the following cases:*

1. *for $k = m$ if $m + 1$ divides $d + 1$,*
2. *for all $k$ if $d + 1$ is a Hadamard number,*
3. *for $k = 2$ and for $d$ odd, $k = 3$.*

*If equality holds for the pair $(k, d)$ then we also have equality in the cases $(d - k, d)$, $(k, n(d + 1) - 1)$, $(nk, n(d + 1) - 1)$, $n \in \mathbb{N}$.*
*For $d$ even it holds*

$$r_1^\sigma(T_r^d) = \frac{d+1}{\sqrt{d+2}}, \quad r_{d-1}^\sigma(T_r^d) = \frac{2\sqrt{d+1}}{\sqrt{d+2} + \sqrt{d-2}}.$$

It is certainly of interest to compare $r_k^\pi(T_r^d)$ and $r_k^\sigma(T_r^d)$. This shows a strange behaviour of these numbers. If we take the dimension $d$ such that $d+1$ is a Hadamard number we see (cf. (10)) that $r_k^\pi(T_r^d)$ and $r_k^\sigma(T_r^d)$ coincide if and only if $k + 1$ divides $d + 1$. We observe that our result implies many examples for $r_k^\pi(K) \neq r_k^\sigma(K)$ and the relation of $r_k^\pi(T_r^d)$ and $r_k^\sigma(T_r^d)$ depends rather on the number-theoretical properties of $k$ and $d$ than on their sizes.

In section 2 we prove Theorem 1 and Theorem 2 and in section 3 we study $r_k^\sigma(T_r^d)$. In order to simplify the proof of Theorem 3 we split it in several lemmas. In the last section we point out a relation between the inradii of arbitrary convex bodies and simplices.

Finally, we remark that the problem to determine $r_k^\sigma(K)$ may be considered as a special case of the more general problem to obtain information about a convex body via inscribed "largest" convex bodies. For questions of this type we refer to [GKL95] and [HKL95].

## 2. Optimal planes and Inradii of simplices

We start with the proof of the characterization of the planes for which $r_{L_k}(P)$ is attained (cf. (1)).

*Proof of Theorem 1.* Let $w^1, \dots, w^m$ be the outward unit normal vectors of the facets $F^1, \dots, F^m$ of the polytope $P$ and let $b_1, \dots, b_m \in \mathbb{R}$ such that

$$P = \{x \in E^d : \langle w^i, x \rangle \leq b_i,\ 1 \leq i \leq m\}.$$

Moreover, let $v^1, \dots, v^d$ be an orthonormal basis of $E^d$ such that $L_k$ is generated by $v^1, \dots, v^k$. For abbreviation we write $\widehat{w}^i$ instead of $w^i|L_k$. Now, for $x \in E^d$ and a positive number $r$ we consider the functions

$$f_i(x,r) = \langle w^i, x\rangle + r||\widehat{w}^i||, \quad 1 \le i \le m.$$

First, we prove the following relation

$$f_i(x,r) \le b_i,\ 1 \le i \le m,\ \Leftrightarrow\ x + r(B^d \cap L_k) \subset P \cap L_k(x). \tag{2}$$

In order to show "$\Leftarrow$" we may assume $\widehat{w}^i \neq 0$. In this case we have $x + r\widehat{w}^i/||\widehat{w}^i|| \in P$ and hence

$$f_i(x,r) = \langle w^i, x + r\widehat{w}^i/||\widehat{w}^i||\rangle \le b_i. \tag{3}$$

For the reverse direction we note that $x \in P \cap L_k(x)$ and $P \cap L_k(x)$ is a nonempty polytope. W.l.o.g. let $F^1, \dots, F^n$ be the facets of $P$ such that $F^i \cap L_k(x)$, $1 \le i \le n$, are the facets of the polytope $P \cap L_k(x)$. Then we have $\widehat{w}^i \neq 0$ and the outward unit normal vector of the facet $F^i \cap L_k(x)$ with respect to $P \cap L_k(x)$ is given by $\widehat{w}^i/||\widehat{w}^i||$. Hence it suffices to show $x + r\widehat{w}^i/||\widehat{w}^i|| \in P$, $1 \le i \le n$, what is equivalent to $f_i(x,r) \le b_i$, $1 \le i \le n$, and (2) is proved.

Now, suppose $f_i(x,r) \le b_i$, $1 \le i \le m$. By our last argument we have further

$$f_i(x,r) = b_i \Leftrightarrow \Big(x + r(B^d \cap L_k)\Big) \cap F^i \neq \emptyset. \tag{4}$$

In order to prove the theorem we choose an $x \in E^d$ such that the inball $x + r_{L_k}(P)(B^d \cap L_k) \subset P$ touches a maximal number of facets of $P$. W.l.o.g. let $F^1, \dots, F^l$ be these facets. By (2), (4) we have

$$f_i(x, r_{L_k}(P)) = b_i,\ 1 \le i \le l, \text{ and } f_i(x, r_{L_k}(P)) < b_i,\ l+1 \le i \le m.$$

Now, we assume $l < d+1$ and distinguish two cases.
i) The vectors $w^1, \dots, w^l$ are linearly independent. Then we can find a $z \in E^d$ with $\langle w^i, z\rangle < 0$, $1 \le i \le l$. Hence for sufficiently small $\epsilon > 0$ we get $f_i(x + \epsilon z, r_{L_k}(P)) < b_i$, $1 \le i \le m$, which contradicts the definition of $r_{L_k(P)}$ (cf. (2)).
ii) The vectors $w^1, \dots, w^l$ are linearly dependent. Then there exists a $z \in E^d \backslash \{0\}$ with $\langle w^i, z\rangle = 0$, $1 \le i \le l$. So we have $f_i(x + \epsilon z, r_{L_k}(P)) = b_i$, $1 \le i \le l$, for $\epsilon \in \mathbb{R}$. However, since $P$ is bounded there exists a $\epsilon > 0$ such that $f_i(x + \epsilon z, r_{L_k}(P)) \le b_i$, $1 \le i \le m$, but at least $l+1$ inequalities are satisfied. This contradicts the choice of the point $x$ and the proof is completed. □

Applied to a $d$-dimensional simplex $T^d$ the last theorem says that all facets are touched by an optimal $k$-dimensional inball and, in particular, this inball is uniquely determined. This property is the key tool for the proof of Theorem 2.

*Proof of Theorem 2.* W.l.o.g. let $r(T^d) = 1$ and let $T^d$ be given by

$$T^d = \{x \in E^d : \langle u^i/||u^i||, x\rangle \le 1\}.$$

Let $v^1, \dots, v^d$ be an orthonormal basis of $E^d$ such that $L_k$ is generated by $v^1, \dots, v^k$. Furthermore, let $x \in E^d$ such that $r_{L_k}(T^d) = r(T^d; L_k(x))$ and $x$

is the center of the inball of $T^d \cap L_k(x)$. By Theorem 1 $x + r_{L_k}(T^d)(B^d \cap L_k)$ touches all facets of $T^d$ and we have (cf. (4))

$$\langle u^i, x\rangle + r_{L_k}(T^d)||u^i|L_k|| = ||u^i||.$$

Summing up gives

$$\langle\sum_{i=0}^d u^i, x\rangle + r_{L_k}(T^d)\sum_{i=0}^d ||u^i|L_k|| = \sum_{i=0}^d ||u^i||.$$

Now, it is well known that $\sum_{i=0}^d u^i = 0$ (cf. [BF34]) and thus we obtain the formula. □

So, by Theorem 2 the problem to determine $r_k^\sigma(T^d)$ for a $d$-simplex $T^d$ is reduced to the determination of

$$\min\left\{\frac{\sum_{i=0}^d ||u^i||}{\sum_{i=0}^d ||u^i|L_k||} : L_k \in \mathcal{L}_k^d\right\}. \tag{5}$$

In other words, we have to find a $k$-dimensional plane $L_k$ such that the sum $\sum_{i=0}^d ||u^i|L_k||$ becomes maximal. For regular simplices and certain cases of $k$ and $d$ we can explicitly compute the exact maximum.

## 3. Regular simplices

Let $T_r^d$ be a regular $d$-simplex with $r(T_r^d) = 1$ and let 0 be the center of the $d$-dimensional inball. Hence,

$$T_r^d = \{x \in E^d : \langle w^i, x\rangle \le 1,\ 1 \le i \le d\},$$

where $w^i$ are the outward unit normal vectors of the facets of $T_r^d$. In this case Theorem 2 gives

$$r_k^\sigma(T_r^d) = \frac{d+1}{\max\{\sum_{i=0}^d ||w^i|L_k|| : L_k \in \mathcal{L}_k^d\}}. \tag{6}$$

**Lemma 1.**

$$r_k^\sigma(T_r^d) \ge \sqrt{d/k} \tag{7}$$

*and equality holds if and only if there exists a $L_k \in \mathcal{L}_k^d$ such that*

$$||w^0|L_k|| = ||w^1|L_k|| = \cdots = ||w^d|L_k|| = \sqrt{k/d}.$$

*Proof.* For the proof we use the fact that for $v \in E^d$, $||v|| = 1$, it holds (cf. [Bal92], [Tsi96])

$$\sum_{i=0}^d \langle w^i, v\rangle^2 = \frac{d+1}{d}. \tag{8}$$

Let $L_k \in \mathcal{L}_k^d$ be an arbitrary $k$-plane and let $v^1, \dots, v^d$ be an orthonormal basis of $E^d$ such that $L_k$ is spanned by $v^1, \dots, v^k$. Then we have $||w^i|L_k||^2 = \sum_{j=1}^k \langle w^i, v^j\rangle^2$ and by (8)

$$\sum_{i=0}^d ||w^i|L_k||^2 = \sum_{j=1}^k\sum_{i=0}^d \langle w^i, v^j\rangle^2 = k\cdot\frac{d+1}{d}.$$

Application of the Cauchy-Schwarz inequality yields

$$(\sum_{i=0}^{d} ||w^i|L_k||)^2 \leq (d+1)\sum_{i=0}^{d} ||w^i|L_k||^2 = k \cdot \frac{(d+1)^2}{d}. \tag{9}$$

Together with (6) this shows (7). Furthermore, (9) is satisfied with equality if and only if all $w^i|L_k$ have the same length. □

In order to bound $r_k^\sigma(T_r^d)$ from above we use the explicit formula for $r_k^\pi(T_r^d)$ given in [BH93] (Lemma 3.2)

$$\begin{aligned} & r_k^\sigma(T_r^d) \leq r_k^\pi(T_r^d) \\ & = \left( m \cdot \sqrt{\frac{\lceil l \rceil (d+1-\lceil l \rceil)}{(d+1)^2 d}} + (k+1-m) \cdot \sqrt{\frac{\lfloor l \rfloor (d+1-\lfloor l \rfloor)}{(d+1)^2 d}} \right)^{-1}, \end{aligned} \tag{10}$$

where $l = (d+1)/(k+1)$, $d+1 \equiv m \pmod{k+1}$, $m \in \{0, \dots, k\}$, and $\lceil x \rceil$ ($\lfloor x \rfloor$) denotes the smallest (largest) integer $\geq (\leq)\, x$. Since $l \leq (d+1)/2$ we obtain

$$r_k^\sigma(T_r^d) \leq \left( (k+1) \cdot \sqrt{\frac{\lfloor l \rfloor (d+1-\lfloor l \rfloor)}{(d+1)^2 d}} \right)^{-1}$$

and writing $\lfloor l \rfloor = (d+1)/(k+1) - \mu(d+1)/(k+1)$ gives

$$r_k^\sigma(T_r^d) \leq \sqrt{\frac{d}{k}} \cdot \sqrt{\frac{k}{(1-\mu)(k+\mu)}}.$$

Thus, by Lemma 1 we have

**Lemma 2.**

$$r_k^\sigma(T_r^d) = \sqrt{\frac{d}{k}} \qquad \textit{if } k+1 \textit{ divides } d+1.$$

Moreover, since the parameter $\mu$ is bounded from above by $\min\{k, d-k\}/(d+1)$ it is not hard to see that

$$r_k^\sigma(T_r^d) \leq \sqrt{\frac{d}{k}} \cdot \sqrt{2}\frac{d+1}{d+2}.$$

In general we can not expect to find a $k$-plane such that all the projections of $w^i$ onto that plane have equal lengths as required by Lemma 1. However, for certain constellations of $k$ and $d$ we can construct such planes. To this end we use the following approach: Let $L_k$ be a $k$-plane given by an orthogonal basis $v^1, \dots, v^k$, $||v^i|| = 1$. Each $v^i$ can be uniquely represented as

$$v^i = \sum_{j=0}^{d} x_j^i w^j \text{ with } \sum_{j=0}^{d} x_j^i = 0, \quad 1 \leq i \leq k.$$

Using the identity $\langle w^m, w^n\rangle = -1/d$, $m \neq n$, we obtain for the coordinates $x^i_j$ the relations

$$||v^i|| = 1 \quad \Leftrightarrow \quad \sum_{j=0}^{d} (x^i_j)^2 = d/(d+1), \quad 1 \leq i \leq k,$$

$$\langle v^m, v^n\rangle = 0 \quad \Leftrightarrow \quad \sum_{j=0}^{d} x^m_j x^n_j = 0, \quad m \neq n.$$

Furthermore, we find for the length of the projection $w^j|L_k$

$$||w^j|L_k|| = ||\sum_{i=1}^{k} \langle w^j, v^i\rangle v^i|| = \frac{d+1}{d}\sqrt{\sum_{i=1}^{k}(x^i_j)^2}.$$

In view of Lemma 1 we get with a suitable normalization

**Lemma 3.** $r^\sigma_k(T^d_r) = \sqrt{d/k}$ *iff there exist* $x^1, \ldots, x^k \in \mathbb{R}^{d+1}$ *with*

$$\text{(11)} \qquad \begin{array}{ll} \textit{i) } ||x^i|| = 1,\ 1 \leq i \leq k, & \textit{ii) } \langle x^m, x^n\rangle = 0,\ 1 \leq m < n \leq k, \\ \textit{iii) } \displaystyle\sum_{j=1}^{d+1} x^i_j = 0,\ 1 \leq i \leq k, & \textit{iv) } \displaystyle\sum_{i=1}^{k} (x^i_j)^2 = \frac{k}{d+1},\ 1 \leq j \leq d+1. \end{array}$$

An immediate consequence is

**Lemma 4.** *If* $d+1$ *is a Hadamard number then one has equality in* (7) *for all* $k = 1, \ldots, d$.

*Proof.* Since $d+1$ is a Hadamard number there exist pairwise orthogonal vectors $y^0, y^1, \ldots, y^d \in \mathbb{R}^{d+1}$ with coordinates $y^i_j \in \{-1, 1\}$. W.l.o.g. let $y^0 = (1, 1, \ldots, 1)^T$. Then we have $\sum_{j=1}^{d+1} y^i_j = 0$, $1 \leq i \leq d$, and each subset of the vectors $x^i = (d+1)^{-1/2} y^i$, $1 \leq i \leq d$, satisfies (11). □

The vectors $x^i$ constructed in the previous proof are of a special type, because all coordinates have the same absolute value. Obviously, for odd numbers there do not exist even two vectors of this type satisfying (11). However, as the next lemma shows, in all dimensions we can find two vectors satisfying the conditions of (11).

**Lemma 5.** *Equality holds in* (7) *for* $k = 2$ *and for* $d$ *odd,* $k = 3$.

*Proof.* First, we study the case $k = 2$. For $j = 1, \ldots, d+1$ let

$$\alpha_j = \frac{2\pi j}{d+1}, \quad x^1_j = \sqrt{\frac{2}{d+1}}\cos\alpha_j, \quad x^2_j = \sqrt{\frac{2}{d+1}}\sin\alpha_j.$$

In the following we verify the properties i)–iv) of (11) for the vectors $x^1, x^2$. Obviously, $(x^1_j)^2 + (x^2_j)^2 = 2/(d+1)$, which shows iv). Since $\cos\alpha_j + i\sin\alpha_j$ are the complex roots of the equation $x^{d+1} - 1 = 0$ we also have iii), i.e.,

$\sum_{j=1}^{d+1} x_j^1 = \sum_{j=1}^{d+1} x_j^2 = 0$. Moreover,

$$\begin{aligned}
\sum_{j=1}^{d+1} x_j^1 x_j^2 &= \frac{1}{d+1}\sum_{j=1}^{d+1} 2\cos\alpha_j \sin\alpha_j = \frac{1}{d+1}\sum_{j=1}^{d+1}\sin 2\alpha_j,\\
\sum_{j=1}^{d+1}(x_j^1)^2 &= \frac{2}{d+1}\sum_{j=1}^{d+1}\cos^2\alpha_j = \frac{1}{d+1}\sum_{j=1}^{d+1}(1+\cos 2\alpha_j)\\
&= 1 + \frac{1}{d+1}\sum_{j=1}^{d+1}\cos 2\alpha_j,\\
\sum_{j=1}^{d+1}(x_j^2)^2 &= \ldots = 1 + \frac{1}{d+1}\sum_{j=1}^{d+1}\sin 2\alpha_j.
\end{aligned}$$

It remains to show $\sum_{j=1}^{d+1}\sin 2\alpha_j = \sum_{j=1}^{d+1}\cos 2\alpha_j = 0$. Now, if $d+1$ is odd then the numbers $\cos 2\alpha_j + i\sin 2\alpha_j$, $1 \le j \le d+1$, are the roots of $x^{d+1} - 1 = 0$, otherwise $\cos 2\alpha_j + i\sin 2\alpha_j$, $1 \le j \le (d+1)/2$, and $\cos 2\alpha_j + i\sin 2\alpha_j$, $(d+1)/2 < j \le d+1$ are the roots of $x^{(d+1)/2} - 1 = 0$.

For $d$ odd and $k = 3$ we choose the coordinates

$$\begin{aligned}
x_j^1 &= \sqrt{\frac{2}{d+1}}\cos 2\alpha_j, \quad x_j^2 = \sqrt{\frac{2}{d+1}}\sin 2\alpha_j, \quad 1 \le j \le d+1,\\
x_j^3 &= \begin{cases} 1/\sqrt{d+1}, & j = 1,\ldots,\frac{d+1}{2},\\ -1/\sqrt{d+1}, & j = \frac{d+3}{2},\ldots,d+1. \end{cases}
\end{aligned}$$

This case can be treated completely similar to the case $k = 2$. □

The next lemma shows that equality in (7) for some $(k, d)$ implies equality for certain other values of $k$ and $d$.

**Lemma 6.** *If equality holds in* (7) *for the pair* $(k, d)$ *then we also have equality for in the cases* $(d-k, d)$, $(k, n(d+1)-1)$, $(nk, n(d+1)-1)$, $n \in \mathbb{N}$.

*Proof.* Suppose, we have equality for $(k, d)$. By Lemma 1 there exists a $k$-plane $L_k$ such that $||w^i|L_k|| = \sqrt{k/d}$, $0 \le i \le d$. Obviously, for the orthogonal complement $L_k^\perp \in \mathcal{L}_{d-k}^d$ of $L_k$ we have $||w^i|L_k^\perp||^2 = 1 - ||w^i|L_k||^2$ and hence by Lemma 1 we also have equality for $(d-k, d)$.

Now, let $x^1, \ldots, x^k \in \mathbb{R}^{d+1}$ be vectors satisfying the conditions (11). For an integer $n$ let $y^i = (x^i, x^i, \ldots, x^i) \in \mathbb{R}^{n(d+1)}$ be the vector consisting of $n$ copies of $x^i$. With $\widehat{x}^i = n^{-1/2}y^i$ we obtain $k$ vectors satisfying (11) with respect to the dimension $n(d+1)-1$. In order to show equality for the pairs $(nk, n(d+1)-1)$, $n \in \mathbb{N}$, let $y^{i,m} \in \mathbb{R}^{n(d+1)}$, $1 \le i \le k$, $1 \le m \le n$, be the vectors with coordinates

$$y_j^{i,m} = \begin{cases} x^i_{j-(m-1)(d+1)}, & j \in \{(m-1)(d+1)+1, \ldots, m(d+1)\}\\ 0, & \text{otherwise.} \end{cases}$$

□

Now, we are ready for the proof of Theorem 3.

*Proof of Theorem 3.* By the previous lemmas it remains to study the case $d$ even and $k = 1, d-1$. Since $r_1^\sigma(T_r^d)$ is one half of the minimal width of the regular simplex we may deduce the value of $r_1^\sigma(T_r^d)$ from the Theorem of Steinhagen (cf. [Ste22], [BH93])

$$r(K) \geq \frac{\Delta(K)}{2} \begin{cases} \sqrt{d+2}/(d+1), & \text{d even,} \\ 1/\sqrt{d}, & \text{d odd,} \end{cases}, \quad K \in \mathcal{K}^d, \tag{12}$$

where, for example, equality holds for a regular simplex. Unfortunately, for $k = d-1$ the proof is rather lengthy and tedious, and thus we omit it here and we refer to [Tsi96]. □

## 4. Concluding Remarks

By Theorem 3 we know the exact values of $r_k^\sigma(T_r^d)$ for $d \leq 5$ and the first unknown value is $r_3^\sigma(T_r^6)$. Moreover, since we have equality for $k = 2$ in all dimensions Theorem 3 shows that we also have equality for $k = 4$ in all odd dimensions.

Next we want to show a relation between the inradii of an arbitrary convex body and the inradii of simplices. To this end we define

$$\sigma(k,d) = \inf\{r(T)/r_k^\sigma(T) : T \text{ is a } d\text{-simplex}\}.$$

Theorem 3 implies the upper bound $\sigma(k,d) \leq \sqrt{k/d}$ and we conjecture that $\sigma(k,d)$ is attained for the regular simplex, i.e.,

**Conjecture 1.** *For $1 \leq k \leq d$ it holds*

$$\sigma(k,d) = r(T_r^d)/r_k^\sigma(T_r^d).$$

This conjecture is not only of interest in its own, but it would also lead to a generalization of the Theorem of Steinhagen (12) as the next lemma shows.

**Lemma 7.** *Let $K \in \mathcal{K}^d$ be a convex body with nonempty interior. It holds*

$$r(K)/r_k^\sigma(K) \geq \sigma(k,d).$$

*Proof.* It is well known that there exists a $m$-dimensional plane $L_m$ and a $m$-simplex $T^m \subset L_m$ with $K \cap L_m \subset T^m$, $r(K) = r(T^m; L_m)$ and the cylinder $P = T^m + L_m^\perp$ contains $K$ (cf. [BF34]).

Now, let $T_j^d \subset P$, $j \in \mathbb{N}$, be a sequence of $d$-simplices such that $K \subset T_j^d + \epsilon_j B^d$ for some $\epsilon_j \geq 0$ with $\lim_{j\to\infty} \epsilon_j = 0$. By construction we have $r(T_j^d) \leq r(T^m; L_m) = r(K)$ and $r_k^\sigma(K) \leq \mu_j r_k^\sigma(T_j^d)$ for some $\mu_j \geq 1$ with $\lim_{j\to\infty} \mu_j = 1$. Altogether we get

$$\frac{r(K)}{r_k^\sigma(K)} \geq \liminf_{j\to\infty} \frac{1}{\mu_j} \frac{r(T_j^d)}{r_k^\sigma(T_j^d)} \geq \sigma(k,d).$$

□

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