NOTE ON LATTICE-POINT-FREE CONVEX BODIES

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Abstract. We prove inequalities relating the inradius of a convex body with interior containing no point of the integral lattice, with the volume or surface area of the body. These inequalities are tight and generalize previous results.

1. Introduction

Let E^d be the d-dimensional Euclidean space equipped with the norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$. The space of all compact convex bodies is denoted by $\mathcal{K}^d$ and $B^d \in \mathcal{K}^d$ denotes the d-dimensional unit ball. For K in $\mathcal{K}^d$ the radius of a largest d-dimensional ball contained in K is called the inradius of K and is denoted by $r(K)$. The interior of $K \in \mathcal{K}^d$ is denoted by $\mathrm{int}(K)$ and $V(S)$ denotes the volume (d-dimensional Lebesgue-measure) of a subset $S \subset E^d$. The surface area of $K \in \mathcal{K}^d$ is denoted by $F(K)$ and x_i denotes the i-th coordinate of a vector $x \in E^d$. For $S \subset E^d$ the number of lattice points of the integral lattice $\mathbb{Z}^d$ contained in S is denoted by $G(S)$, i.e.,

$$G(S) = \#\{z \in \mathbb{Z}^d : z \in S\}.$$

Obviously, the volume (or surface area) of a lattice-point-free convex body K, i.e., $K \in \mathcal{K}^d$ with $G(\mathrm{int}(K)) = 0$, can be arbitrary large. However, the inradius of such a body is bounded above by $\sqrt{d}/2$ and in this paper we study the problem: W hat is the maximal volume (or surface area) of a lattice-point-free convex body with given inradius $r(K)$?

In order to give the answer we need the following notation. For two positive real numbers α, β let $C(\alpha, \beta)$ be the cross polytope with vertices $\{\pm\alpha e^1, \pm\beta e^2, \dots, \pm\beta e^d\}$, where $e^i \in E^d$ denotes the i-th unit vector. For such a cross polytope it is ea sy to verify that

$$\begin{aligned} r(C(\alpha,\beta)) &= \frac{\alpha\beta}{\sqrt{\beta^2 + (d-1)\alpha^2}}, \qquad V(C(\alpha,\beta)) = \frac{2^d}{d!}\,\alpha\beta^{d-1}, \\ F(C(\alpha,\beta)) &= dV(C(\alpha,\beta))\Big/ r(C(\alpha,\beta)). \end{aligned}$$

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Now let $\widehat{p} = (1/2,\dots,1/2)^T \in E^d$ and for $r \in \left(\sqrt{d-1}/2, \sqrt{d}/2\right]$ consider $C(\alpha(r),\beta(r))$ where

$$\alpha(r) = \frac{dr}{2r - \sqrt{d-(2r)^2}\sqrt{d-1}}, \quad \beta(r) = \frac{dr}{2r + \sqrt{d-(2r)^2}/\sqrt{d-1}}.$$

Using our previous expression for $r(C(\alpha,\beta))$ we can check that $r(C(\alpha(r),\beta(r))) = r$, and from the representation

$$\begin{aligned} C(\alpha(r),\beta(r)) \quad &= \quad \Bigg\{ x \in E^d : \left(2r - \sqrt{d-(2r)^2}\sqrt{d-1}\right)|x_1| \\ &\qquad + \left(2r + \frac{\sqrt{d-(2r)^2}}{\sqrt{d-1}}\right)\sum_{i=2}^{d}|x_i| \le dr \Bigg\} \end{aligned}$$

we see that $\widehat{p}$ lies on a defining hyperplane of $C(\alpha(r),\beta(r))$. We deduce that $\mathrm{int}\left(\widehat{p} + C(\alpha(r),\beta(r))\right) \cap \mathbb{Z}^d = \emptyset$ and

$$\{x \in E^d : 0 \le x_i \le 1,\ 1 \le i \le d\} \subset \widehat{p} + C(\alpha(r),\beta(r)).$$

So, $\widehat{p} + C(\alpha(r),\beta(r))$ always contains a fundamental cell of the lattice $\mathbb{Z}^d$, and therefore

$$\mathrm{int}\left(x + C(\alpha(r),\beta(r))\right) \cap \mathbb{Z}^d = \emptyset \Longleftrightarrow x \in \widehat{p} + \mathbb{Z}^d. \tag{1}$$

Now we can state our main result.

Theorem 1. *Let $K \in \mathcal{K}^d$ with $G(\mathrm{int}(K)) = 0$ and $r = r(K) > \sqrt{d-1}/2$. Then*

$$V(K) \quad \le \quad V\left(C(\alpha(r),\beta(r))\right) = \frac{2^d}{d!}\,\alpha(r)\beta(r)^{d-1}, \tag{2}$$

$$F(K) \quad \le \quad F\left(C(\alpha(r),\beta(r))\right) = \frac{2^d}{(d-1)!}\frac{\alpha(r)\beta(r)^{d-1}}{r} \tag{3}$$

and equality holds if and only if K is — up to a permutation of the coordinates — of the from $z + \widehat{p} + C(\alpha(r),\beta(r))$ for some $z \in \mathbb{Z}^d$.

First of all we note that in order to bound the volume (or the surface area) of a lattice-point-free convex body K, it is necessary to assume that $r(K) > \sqrt{d-1}/2$. To see this, let $K(\lambda) = \widehat{p} + \mathrm{conv}\{(\sqrt{d-1}/2)B^d, \pm\lambda e^1\}$, $\lambda > 0$. Obviously, $V(K(\lambda))$ tends to infinity as λ approaches infinity and it is easy to check that $G(K(\lambda)) = 0$.

Based on Theorem 1 we will obtain

Corollary 1. *Let $K \in \mathcal{K}^d$ with $G(\mathrm{int}(K)) = 0$. Then*

$$\begin{aligned} \left(2r(K) - \sqrt{d-1}\right)V(K) \quad &\le \quad \frac{d^d}{d!}\left(\sqrt{d} - \sqrt{d-1}\right), \\ \left(2r(K) - \sqrt{d-1}\right)F(K) \quad &\le \quad \frac{d^d}{(d-1)!}\frac{2}{\sqrt{d}}\left(\sqrt{d} - \sqrt{d-1}\right) \end{aligned}$$

and equality holds if and only if $K = z + \widehat{p} + C(d/2, d/2)$ for some $z \in \mathbb{Z}^d$.

In the 2-dimensional case, Theorem 1 and Corollary 1, as well as other inequalities for lattice-point-free planar convex bodies, have been proved by Awyong & Scott [AS96]. A best possible inequality relating th e volume and the surface area of a convex body K, $G(\mathrm{int}(K)) = 0$, was given by Bokowski, Hadwiger&Wills [BHW72]. They showed $V(K) < (1/2)F(K)$. For more information on lattice-point-free convex bodies we refer t o [GW93].

2. Proofs of Theorem 1 and Corollary 1

For the proof of Theorem 1 we need the following two lemmas from elementary calculus. A proof of the first lemma can be found, e.g., in [Sch86], p. 192, but for completeness we include the short proof.

Lemma 1. *Let $r, R \in \mathbb{R}$ and let $x^* = (x_1^*, \ldots, x_d^*)^T \in E^d$ be an optimal solution of*

$$\min \prod_{i=1}^{d} x_i, \quad \textit{such that} \quad \sum_{i=1}^{d} x_i = r \quad \textit{and} \quad \sum_{i=1}^{d} (x_i)^2 = R.$$

After a suitable permutation of the coordinates one has $x_1^* \le x_2^* = \cdots = x_d^*$.

Proof. Let $x_1^* \le x_2^* \le \cdots \le x_d^*$. For $d = 2$ there is nothing to prove. So let $d \ge 3$ and first we investigate the case $d = 3$.

If we replace x_1, x_2, x_3 by $x_1 - \frac{1}{3}r, x_2 - \frac{1}{3}r, x_3 - \frac{1}{3}r$ the first of the above conditions will be automatically satisfied. However, now the minimum is non-positive and $x_1^* \le 0 \le x_2^* \le x_3^*$. Hence by the geometri c-arithmetic mean inequality we get

$$x_1^* x_2^* x_3^* \ge x_1^* \left(\frac{x_2^* + x_3^*}{2} \right)^2 = \frac{1}{4}(x_1^*)^3. \tag{4}$$

Adding the equalities $(x_1^*)^2 = (x_2^* + x_3^*)^2$ and $2(x_1^*)^2 = 2R - 2(x_2^*)^2 - 2(x_3^*)^2$ gives $3(x_1^*)^2 = 2R - (x_2^* - x_3^*)^2 \le 2R$. So we have $x_1^* \ge -\sqrt{(2/3)R}$ and together with (4) we obtain

$$x_1^* x_2^* x_3^* \ge -\frac{R}{18}\sqrt{6R}. \tag{5}$$

On the other hand $(x_1, x_2, x_3) = (-\frac{1}{3}\sqrt{6R}, \frac{1}{6}\sqrt{6R}, \frac{1}{6}\sqrt{6R})$ is a feasible solution of the problem with $x_1 x_2 x_3 = -\frac{R}{18}\sqrt{6R}$. Thus we have equality in (5) and therefore in (4)

, which shows $x_2^* = x_3^*$.

Now let $d \ge 4$. Then for all $1 \le i < j < k \le d$, the triple (x_i^*, x_j^*, x_k^*) must be an optimal solution of $\min x_1 x_2 x_3$ such that $x_1 + x_2 + x_3 = x_i^* + x_j^* + x_k^*$ and $(x_1)^2 + (x_2)^2 + (x_3)^2 = (x_i^*)^2 + (x_j^*)^2 + (x_k^*)^2$ and by the previous case we know $x_j^* = x_k^*$. Thus we have $x_1^* \le x_2^* = \cdots = x_d^*$. □

Lemma 2. *For* $\rho \in \left(\sqrt{d-1}, \sqrt{d}\right]$ *let* $v(\rho) = V\left(C(\alpha(\rho/2), \beta(\rho/2))\right)$, $f(\rho) = F\left(C(\alpha(\rho/2), \beta(\rho/2))\right)$, *i.e.,*

$$
\begin{aligned}
v(\rho) &= \frac{d^d \sqrt{d-1}^{\,d-1}}{d!} \times \frac{\rho^d}{\left(\rho - \sqrt{d-\rho^2}\sqrt{d-1}\right)\left(\sqrt{d-1}\rho + \sqrt{d-\rho^2}\right)^{d-1}}, \\
f(\rho) &= (2d)\frac{v(\rho)}{\rho}.
\end{aligned}
$$

Then

i) $v(\rho)$ *is strictly monotonously decreasing in* ρ
ii) $(\rho - \sqrt{d-1}) \times f(\rho)$ *is strictly monotonously increasing in* ρ
iii) $(\rho - \sqrt{d-1}) \times v(\rho)$ *is strictly monotonously increasing in* ρ.

Proof. Let $g_1(\rho) = \rho - \sqrt{d-\rho^2}\sqrt{d-1}$ and $g_2(\rho) = \sqrt{d-1}\rho + \sqrt{d-\rho^2}$. First we show that $\widehat{v}(\rho) = \rho^d/(g_1(\rho) \cdot g_2(\rho)^{d-1})$ is strictly monotonously decreasing. To this end we calculate the f
irst derivative of $\widehat{v}$. Since $g_1'(\rho) = g_2(\rho)/\sqrt{d-\rho^2}$ and $g_2'(\rho) = -g_1(\rho)/\sqrt{d-\rho^2}$ we find

$$
\begin{aligned}
\widehat{v}'(\rho) &= \frac{\rho^{d-1}}{g_1(\rho)^2 g_2(\rho)^d}\frac{1}{\sqrt{d-\rho^2}} \\
&\times \left(dg_1(\rho)\left[g_2(\rho)\sqrt{d-\rho^2} + \rho d g_1(\rho)\right] - \rho\left[g_2(\rho)^2 + g_1(\rho)^2\right]\right).
\end{aligned}
$$

Use of the identities $g_1(\rho)^2 + g_2(\rho)^2 = d^2$ and $\sqrt{d-\rho^2}g_2(\rho) + \rho g_1(\rho) = d$ yields

$$
\widehat{v}'(\rho) = \frac{\rho^{d-1}d^2}{g_1(\rho)^2 g_2(\rho)^d}\frac{g_1(\rho) - \rho}{\sqrt{d-\rho^2}} = -\frac{\rho^{d-1}d^2\sqrt{d-1}}{g_1(\rho)^2 g_2(\rho)^d}. \tag{6}
$$

Therefore $\widehat{v}(\rho)$ – and thus $v(\rho)$ — is strictly monotonously decreasing.

Now let $\widehat{f}(\rho) = (\rho - \sqrt{d-1}) \times \widehat{v}(\rho)/\rho$. By (6) we get

$$
\widehat{f}'(\rho) = \frac{\rho^{d-2}}{g_1(\rho)^2 g_2(\rho)^d}\left(\sqrt{d-1}g_1(\rho)g_2(\rho) - \rho d^2\sqrt{d-1} + (d-1)d^2\right).
$$

Let $h(\rho)$ be the function within the large brackets. Substituting for $g_1(\rho)$, $g_2(\rho)$ we obtain

$$
h(\rho) = 2(d-1)\rho^2\sqrt{d-1} - \sqrt{d-1}\rho\sqrt{d-\rho^2}(d-2) - \rho d^2\sqrt{d-1} + (d-1)(d^2-d).
$$

Now $h(\sqrt{d-1}) = 0$ and calculating the first derivative shows that $h(\rho)$ is strictly monotonously increasing for $\rho \in [\sqrt{d-1}, \sqrt{d}]$. Thus $\widehat{f}(\rho)$ — and hence $(\rho - \sqrt{d-1}) \times f(\rho)$ — is strictly monotonously i
ncreasing.

The statement iii) follows directly from ii). □

Proof of Theorem 1. First we prove the theorem for the volume functional. Let $K \in \mathcal{K}^d$ with $G(\mathrm{int}(K)) = 0$ and let $r = r(K) > \sqrt{d-1}/2$. Let $K_0 = K$ and for $1 \le i \le d$ let K_i be the Steiner-symmetral of K_{i-1} w ith respect to the hyperplane $H_i = \{x \in E^d : x_i = 1/2\}$. Then K_d is symmetric with respect to all of these hyperplanes H_i and we claim $G(\mathrm{int}(K_d)) = 0$.

Suppose the contrary and let $z = (z_1, \dots, z_d)^T \in \mathrm{int}(K_d) \cap \mathbb{Z}^d$. Since K_d is symmetric with respect to H_d we also have $(z_1, \dots, z_{d-1}, -z_d + 1)^T \in \mathrm{int}(K) \cap \mathbb{Z}^d$. Hence the length of the intersection of the line $z + \lambda e^d$, $\lambda \in \mathbb{R}$, with K_d is greater than 1 and by the definition of the Steiner-symmetral the same holds for the body K_{d-1}. Hence $G(\mathrm{int}(K_{d-1})) > 0$ and applying the above argumentation recursively to the bodies $K_{d-1}, \dots, K_1$ gives

the contradiction $G(\mathrm{int}(K)) > 0$.

Furthermore, it is well known that $V(K) = V(K_d)$ and $r(K_d) \geq r$ (cf. [BZ88], [Egg58], [Lei79], [SY93]). Hence by Lemma 2 i), it s uffices to prove (2) with respect to K_d. To this end let $K_s = K_d - \widehat{p}$ and $\Lambda = \mathbb{Z}^d - \widehat{p}$.

Obviously, $V(K_s) = V(K_d)$, $r(K_s) = r(K_d)$ and K_s is symmetric with respect to all the coordinate hyperplanes $E_i = \{x \in E^d : x_i = 0\}$, $1 \leq i \leq d$. Thus we have $r(K_s)B^d \subset K_s$. Since $\widehat{p} \notin \mathrm{int}(K_s)$ there exists a $u \in B^d$, $|u| = 1$, such that $K_s \subset H_u^+ = \{x \in E^d : \langle u, x\rangle \leq \frac{1}{2}\sum_{i=1}^d u_i\}$. Since $r(K_s) > \sqrt{d-1}/2$ the points $\widehat{p} - \frac{1}{2}e^i$, $1 \leq i \leq d$, belong to $\mathrm{int}(K_s)$ and thus $u_i > 0$, $1 \leq qi \leq d$. Now let

$$K_u = \left\{ x \in E^d : \sum_{i=1}^d u_i |x_i| \leq \frac{1}{2} \sum_{i=1}^d u_i \right\}.$$

K_u is the cross polytope with vertices $\{\pm\big(\frac{1}{2}(\sum_{i=1}^d u_i)/u_j\big)e^j : 1 \leq j \leq d\}$ and clearly we have $\mathrm{int}(K_u) \cap \Lambda = \emptyset$. Moreover, since K_s is symmetric with respect to the d coordinate hyperplanes

, and $K_s \subset H_u^+$ we have $K_s \subset K_u$. Hence by Lemma 2 i) it suffices to show

$$V(K_u) \leq V\left(C(\alpha(r(K_u)), \beta(r(K_u)))\right). \tag{7}$$

We have $r(K_u) = \frac{1}{2}\sum_{i=1}^d u_i$ and for the volume of K_u we find

$$V(K_u) = \frac{1}{d!}\left(\sum_{i=1}^d u_i\right)^d \Big/ \prod_{i=1}^d u_i.$$

With respect to the constraints $\sum_{i=1}^d u_i = 2r(K_u)$ and $\sum_{i=1}^d (u_i)^2 = 1$ we know from Lemma 1 that the right hand side becomes maximal for a vector $(u_1^*, \dots, u_d^*)^T \in E^d$ satisfying — up to a permutation of the coord inates — $u_1^* \leq u_2^* = \cdots = u_d^*$. Easy calculations yield $u_1^* = r(K_u)/\alpha(r(K_u))$ and $u_2^* = r(K_u)/\beta(r(K_u))$, which establishes (7) and thus we have proved (2).

Suppose we have equality in (2). Then the above argument shows that, up to a permutation of the coordinates, K_d is of the form $\widehat{p} + C(\alpha(r), \beta(r))$. In particular, we have $r(K_d) = r$ and it follows that $V(K_d) = (r(K_d)/d)F(K_d)$ $= (r/d)F(K_d)$. On the other hand we have $V(K) \geq (r/d)F(K)$, hence

$$\frac{r}{d}F(K_d) = V(K_d) = V(K) \geq \frac{r}{d}F(K). \tag{8}$$

It is well known that the surface area of the Steiner-symmetral of a convex body $\widehat{K}$, $\mathrm{int}(\widehat{K}) \neq \emptyset$, with respect to a hyperplane H, say, is not greater than $F(\widehat{K})$ and remains unchanged if and only if $\widehat{K}$ is symmetric in H (cf. [Egg58]). Therefore, (8) implies that K is symmetric in all the planes H_i and thus $K = x + K_d$ for some $x \in E^d$. However, by (1) we get $x \in \mathbb{Z}^d$.

Now since Theorem 1 is true for the volume functional, the statement for the surface area is an immediate consequence of the inequality

$$\frac{r}{d}F(K) \leq V(K) \leq V\left(C(\alpha(r), \beta(r))\right) = \frac{r}{d}F\left(C(\alpha(r), \beta(r))\right).$$

□

Proof of the Corollary 1. For $r(K) \leq \sqrt{d-1}/2$ there is nothing to prove and so let $r = r(K) > \sqrt{d-1}/2$. By Theorem 1 and Lemma 2 iii) we obtain

$$\begin{aligned}(2r - \sqrt{d-1})V(K) &\leq (2r - \sqrt{d-1})V\left(C(\alpha(r), \beta(r)\right) \\ &\leq (\sqrt{d} - \sqrt{d-1})V\left(C(\alpha(\sqrt{d}/2), \beta(\sqrt{d}/2))\right) \\ &= (\sqrt{d} - \sqrt{d-1})\frac{d^d}{d!},\end{aligned}$$

and equality holds if and only if $K = z + \widehat{p} + C(\alpha(\sqrt{d}/2), \beta(\sqrt{d}/2))$ for some $z \in \mathbb{Z}^d$. Using Lemma 2 ii), the result for the surface area can be proved in the same way. □

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