On the Classification of Willmore Spheres

diplomarbeit

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Matrikel–Nummer: 199903

von der Fakultät II
—Mathematik und Naturwissenschaften—
der Technischen Universität Berlin

Berlin, im Oktober 2004

Contents

Zusammenfassung

In dieser Arbeit werden diejenigen Werte bestimmt, welche das Willmore Funktional für immersierte Willmore Sphären im S^3 annehmen kann. Dabei wird detailliert auf den Zusammenhang zwischen immersierten kompakten Willmore Flächen und Minimalflächen mit flachen Enden eingegangen.

Das klassiche Willmore Funktional einer Immersion $f\colon M \to \mathbb{R}^3$ einer kompakten Riemannschen Fläche M ist definiert als $\int_M (H^2 - K) dA$, wobei H und K die Hauptkrümmung bzw. Gausskrümmung und dA die induzierte Volumenform von f sind. Obwohl das Willmore Funktional durch Euklidische Größen gegeben ist, sind der Integrand und damit auch das Funktional Möbius invariant. Die in den ersten beiden Kapiteln vorgestellte Theorie der quaternionischen holomorphen Linienbündel und deren Anwendung auf die Untersuchung konform immersierter Riemannscher Flächen in der konformen 4-Sphäre S^4 werden dann im Kapitel 3 dazu benutzt eine Definition des Willmore Funktionals in Größen der konformen Geometrie zu geben. Wir definieren Willmore Flächen als kritische Punkte dieses Funktionals.

Im zweiten Teil des 3. Kapitels gehen wir auf Minimalflächen $f\colon M \to \mathbb{R}^3$ mit flachen Enden ein, wobei M eine punktierte kompakte Riemannsche Fläche ist. Daraufhin wird unter Zuhilfenahme der quaternionischen Theorie die Spinordarstellung dieser Minimalflächen hergeleitet. Es wird ein besonderes Augenmerk auf die analytischen Eigenschaften der Spinoren gelegt und die Ergebnisse werden unter anderem dazu benutzt, das Willmore Funktional für diese Klasse von Flächen zu berechnen. Anschließend wird gezeigt, wie Minimalflächen mit flachen Enden kompakte Willmore Flächen in S^3 ergeben. Im Fall von Sphären konnte Bryant in [**Br84**] zeigen, dass auch die Umkehrung davon gilt. Als direkte Folgerung davon erhält man, dass das Willmore Funktional für kompakte Willmore Flächen vom Geschlecht 0 nur natürliche Vielfache von 4π annehmen kann. Wir machen den ersten Schritt in Richtung einer algebraischen Charakterisierung von Willmore Sphären.

Im vierten Kapitel wird zunächst die notwendige Theorie von holomorphen Kurven in komplex projektiven Räumen vorgestellt, insbesondere werden Kontaktkurven in $\mathbb{CP}^3$ und Nullkurven in der komplexen 3–Quadrik Q^3 betrachtet. Wir folgen dem Weg von Bryant in [**Br88**] wenn wir herleiten wie Kontaktkurven vom Geschlecht 0 mit Willmore Sphären zusammenhängen. Dadurch ist es möglich zu zeigen, dass keine Willmore Sphären mit Willmore Funktional $4\pi(n-1)$ existieren, wobei $n \in \{2, 3, 5, 7\}$. Wir geben unter anderem einen detaillierten Beweis für den Fall $n = 7$, der bei Bryant nicht aufgeschrieben ist, und klassifizieren den Raum der Willmore Sphären mit Funktional 12π. Abschließend benutzen wir eine Idee von Peng in [**Pe86**], um Beispiele für Willmore Sphären mit Willmore Funktional $4\pi(2k + 1)$, $k \geq 4$, zu konstruieren.

Introduction

In his celebrated paper "A Duality Theorem for Willmore Surfaces" [**Br84**] Bryant showed that every immersed Willmore sphere in S^3 is a minimal surface of genus 0 with flat ends under a suitable stereographic projection. As a first consequence he obtained that the critical values of the Willmore functional on the space of spherical immersions are nonnegative multiples of 4π. In [**Br88**] he proved that the values $4\pi(n-1)$ for $n \in \{2, 3, 5, 7\}$ cannot occur.

The subject of the present text is the study of the spaces $\mathcal{M}_n$ of immersed Willmore spheres with Willmore functional $4\pi(n-1)$ and the correspondence between immersed compact Willmore surfaces and immersed minimal surfaces with flat ends.

The classical Willmore functional of an immersion $f\colon M \to \mathbb{R}^3$ of a Riemann surface M into Euclidean 3–space is given by $\int_M (H^2 - K) dA$, where H and K are the mean respectively the Gaussian curvature of f and dA is the induced volume form on M. Since this functional is Moebius invariant, the right setting for the study of Willmore surfaces is that of Moebius geometry rather then Euclidean geometry.

The quaternionic approach to the theory of conformally immersed surfaces developed in [**PP98**], [**FLPP01**] and [**BFLPP02**] proved to be very useful for studying surfaces in the conformal 4–sphere. In Chapter I and Chapter II we briefly discuss parts of quaternionic theory used throughout this work. For example, we give a definition of the Willmore functional of immersed compact Riemann surfaces in terms of quantities of the conformal geometry of S^4. Willmore surfaces are defined as critical points of the Willmore functional.

We study a class of minimal surfaces in $\mathbb{R}^3$ which extend to immersed compact Willmore surfaces in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ in the second part of Chapter III. They are called minimal surfaces with flat ends because of their behavior at the ends. We use quaternionic theory to develop the spinor representation of minimal surfaces with flat ends and to study the analytical properties of the spinors. This representation is a coordinate–free version of the Weierstrass representation of minimal surfaces, see [**KuSch95**]. In the case of genus 0, every compact Willmore surface arises from a minimal surface with flat ends.

Since minimal immersions $f\colon M \to \mathbb{R}^3$ of a Riemann surface M into 3–space are given by holomorphic data, it is only a small step from minimal surfaces with flat ends of genus zero to special holomorphic curves in projective space. We use the approach of Bryant in [**Br88**] when we discuss the relationship between Willmore spheres and contact curves of genus 0 in $\mathbb{CP}^3$. With this correspondence the problem of determining the spaces

$\mathcal{M}_n$ is reduced to an algebraic problem and one can show that $\mathcal{M}_n = \emptyset$ for $n \in \{2, 3, 5, 7\}$. We give the proofs, especially in the case of $n = 7$, which is not written down by Bryant. Moreover, we use the developed techniques to determine the space $\mathcal{M}_4$. Our way differs from the way of Bryant [**Br84**] and from the way of Kusner and Schmitt [**KuSch95**]. Finally we give examples of immersed Willmore spheres with Willmore functional $4\pi(n-1)$ for $n \notin \{2, 3, 5, 7\}$. In the case $n = 2k + 1$ we use an idea of Peng in [**Pe86**] to construct the corresponding minimal surfaces with flat ends.

I am thankful to my supervisor Ulrich Pinkall for suggesting the interesting subject, for sharing his ideas and his encouragement.

Special thanks to Christoph Bohle and Paul Peters for their great help in understanding the subject and cooperation.

I thank Markus Schmies for explaining the usage of the programme Oorange, which I used to produce the pictures at the end of this text, and Ulrich Krähmer, Paul Peters and Martin Seehafer for proofreading.

CHAPTER I

Quaternionic Theory

In the first chapter we briefly discuss the basics of the theory of quaternionic holomorphic vector bundles over Riemann surfaces that are essential for the approach to Willmore surface presented here. This theory was developed in [**PP98**] and [**FLPP01**].

1. Quaternionic Vector Spaces

1.1. Quaternions. Let $\mathbb{H}$ be the 4–dimensional real vector space generated by 1, $\mathbb{i}$, $\mathbb{j}$ and $\mathbb{k}$. There is an unique real bilinear multiplication such that 1 is the neutral element, $\mathbb{i}^2 = \mathbb{j}^2 = \mathbb{k}^2 = -1$ and $\mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}$. This defines a skew symmetric field which is called the *quaternions*.

There are some basic constructions: For any $q = a + b\mathbb{i} + c\mathbb{j} + d\mathbb{k} \in \mathbb{H}$ define $\bar{q} = a - b\mathbb{i} - c\mathbb{j} - d\mathbb{k}$ to be the *conjugate*. For two quaternions $\lambda,\ \mu \in \mathbb{H}$ we observe $\overline{\lambda\mu} = \bar{\mu}\bar{\lambda}$. The *real* and *imaginary part* of $q \in \mathbb{H}$ are given by

$$\mathrm{Re}(q) = \frac{1}{2}(q + \bar{q}) \quad \text{and} \quad \mathrm{Im}(q) = \frac{1}{2}(q - \bar{q}).$$

We identify the quaternions as a real vector space canonically with $\mathbb{R}^4 \cong \mathbb{H}$ and the subspace of *imaginary quaternions* $\mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. The natural inner product on $\mathbb{R}^4$ is according to this identification given by

$$< a, b >_{\mathbb{R}} = \mathrm{Re}(\bar{a}b) = \mathrm{Re}(a\bar{b}) = \frac{1}{2}(\bar{a}b + \bar{b}a).$$

We define the *absolute value* of $q \in \mathbb{H}$ by $|q| = \sqrt{< q, q >_{\mathbb{R}}} = \sqrt{q\bar{q}}$.

1.2. Linear Algebra over the Quaternions. We collect the basic notions of linear algebra over the skew symmetric field of quaternions.

Definition. A *quaternionic vector space* is a right module over the quaternions $\mathbb{H}$. *Dimension, basis* and *linear map* between quaternionic vector spaces are defined as in commutative linear algebra.

Normally the space of $\mathbb{H}$–linear maps between quaternionic vector spaces is not a quaternionic vector space itself (in a natural way). But there is an important exception, the dual space:

Definition. Let V be a quaternionic (right)vector space. The space $V^* = \mathrm{Hom}_{\mathbb{H}}(V, \mathbb{H})$ of V is in a natural way a quaternionic left vector space, but we always consider the *dual space* V^* as a right vector space:

$$\omega\lambda := \bar{\lambda}\omega$$

for any $\omega \in V^*$ and $\lambda \in \mathbb{H}$.

For lack of commutativity of the quaternions it is not possible to define a tensor product as in the commutative linear algebra. Although given a real vector space V and a quaternionic vector space W, we can define the real tensor product $V \otimes_{\mathbb{R}} W$ to obtain a quaternionic vector space.

1.3. Complex Quaternionic Vector Spaces. Any quaternionic vector space V is a complex vector space, but not in a canonical way. So it is useful to define an additional structure, acting from the left, i.e. commuting with quaternionic multiplication: If we fix an endomorphism $J \in \mathrm{End}(V)$ with $J^2 = -\mathrm{Id}$, we can define

$$(x + yi)v = vx + (Jv)y.$$

Then we call (V, J) a *complex quaternionic (bi–)vector space.*

Given a complex vector space W, the space $V = W \oplus W$ inherits the structure of a complex quaternionic vector space: For any $v,\ w \in W$ we define

$$\begin{aligned} J(v \oplus w) &= (iv) \oplus (iw) \\ (v \oplus w)\mathbb{i} &= (iv) \oplus (-iw) \\ (v \oplus w)\mathbb{j} &= (-w) \oplus v \\ \mathbb{k} &= \mathbb{i}\mathbb{j}. \end{aligned}$$

Note that W is the $\mathbb{i}$–eigenspace of J.

In general, we can decompose a complex quaternionic vector space into $\pm\mathbb{i}$–eigenspaces:

$$V = V_+ \oplus V_- = \{v \in V | Jv = v\mathbb{i}\} \oplus \{v \in V | Jv = -v\mathbb{i}\}.$$

The decomposition of $v \in V$ is given by $v = \frac{1}{2}(v - Jv\mathbb{i}) + \frac{1}{2}(v + Jv\mathbb{i})$. The eigenspaces $V_\pm$ are isomorphic as $\mathbb{C}$– vector spaces, where the complex multiplication is given by J. An isomorphism is given by (right) multiplication with $\mathbb{j}$.

To summarize, every complex quaternionic vector space is the double of a complex vector space and the above two operations are inverse to each other up to isomorphism.

1.4. Example (The Dual Space)**.** Let (V, J) be a complex quaternionic vector space. The dual space V^* becomes a complex quaternionic vector space in the usual way, i.e. the complex structure on V^* is the dual of the complex structure on V :

$$(J\alpha, v) = (\alpha, Jv),$$

for any $v \in V,\ \alpha \in V^*$, where $(,)$ denotes the evaluation pairing. For our further studies it turns out to be useful to determine the decomposition of V^* into $\pm\mathbb{i}$ eigenspaces of J: For any $\alpha \in (V^*)_+,\ \beta \in (V^*)_-$ and $v \in V_+,\ w \in V_-$ we find out

$$\begin{aligned} -\mathbb{i}\alpha(v) &= (J\alpha)(v) = \alpha(Jv) = \alpha(v)\mathbb{i} \\ -\mathbb{i}\alpha(w) &= (J\alpha)(w) = \alpha(Jw) = -\alpha(w)\mathbb{i} \\ \mathbb{i}\beta(v) &= (J\beta)(v) = \beta(Jv) = \beta(v)\mathbb{i} \\ \mathbb{i}\beta(w) &= (J\beta)(w) = \beta(Jw) = -\beta(w)\mathbb{i}. \end{aligned}$$

Therefore $\alpha(w), \beta(v) \in \operatorname{span}(1, \mathbb{i})$, and if we identify $\operatorname{span}(1, \mathbb{i})$ with $\mathbb{C}$ by $(a + b\mathbb{i}) \mapsto (a - ib) \in \mathbb{C}$ respectively $(a + b\mathbb{i}) \mapsto (a + ib) \in \mathbb{C}$, then $(V_+)^* = (V^*)_-$ and $(V_-)^* = (V^*)_+$ as $\mathbb{C}$–vector spaces in a natural way.

Conversely, given a $\mathbb{C}$–vector space W and its dual W^*, we can regard the complex quaternionic vector space $W^* \oplus W^*$ as the dual space of $V = W \oplus W$ in the following manner:

$$(\alpha \oplus \beta, v \oplus w) = \mathbb{j}(\alpha, v) + (\beta, v) + (\beta, w)\mathbb{j} + \mathbb{j}(\alpha, w)\mathbb{j}, \tag{1.4.1}$$

where $v,\ w \in W,\ \alpha,\ \beta \in W^*$ and the values of the evaluation between W and W^* are treated as elements of $\mathbb{C} \cong \operatorname{span}(1, \mathbb{i}) \subset \mathbb{H}$.

1.5. Example (Tensor Products)**.** Consider a complex vector space U, a complex quaternionic vector space V and the tensor product $U \otimes_{\mathbb{C}} V$. Of course, this tensor product is also a quaternionic vector space and both structures, i.e. the quaternionic and the complex one, are compatible, so this is a new complex quaternionic vector space.

Let $V = W \oplus W$ be the decomposition of V with respect to J. Then we obtain $U \otimes_{\mathbb{C}} V = (U \otimes W) \oplus (U \otimes W)$.

Given two complex quaternionic vector spaces (V, J) and (W, J), we can decompose

$$\operatorname{Hom}(V, W) = \operatorname{Hom}_+(V, W) \oplus \operatorname{Hom}_-(V, W),$$

where $\operatorname{Hom}_\pm(V, W) = \{\Phi \in \operatorname{Hom}(V, W) \mid J\Phi = \pm\Phi J\}$ are the complex and anti–complex linear homomorphism. These three spaces get complex structures by post–composition with $J \in \operatorname{End}(W)$.

1.6. Definition. For a quaternionic vector space of dimension n and $A \in \operatorname{End}(V)$ we define

$$< A >:= \frac{1}{4n} \operatorname{tr}_{\mathbb{R}} A,$$

where the trace is taken over the real endomorphism A.

1.7. Example. Consider an anticomplex endomorphism $A \in \operatorname{End}_-(\mathbb{H})$, where J is given by $J\lambda = \mathbb{i}\lambda$. We compute $< AA >$. Because $A \in \operatorname{End}_-(V)$ it follows that $A(1) = c\mathbb{j} + d\mathbb{k}$ for some $d, c \in \mathbb{R}$. So $A^2(1) = -c^2 - d^2$ and therefore $< AA >= -c^2 - d^2$. We conclude that for any complex quaternionic vector space V of rank 1 the mapping

$$A \in \operatorname{End}_-(V) \mapsto < AA > \in \mathbb{R}$$

is negative definite.

2. Complex Quaternionic Vector Bundles

2.1. Quaternionic Vector Bundles.

Definition. A *quaternionic vector bundle* (V, M, π, F) over a manifold M of rank n is a fiberbundle $\pi\colon V \to M$ with typical fiber F, where F is a n–dimensional quaternionic vector space, such that the transition functions may be chosen to have values in $\operatorname{GL}_{\mathbb{H}}(F) \subset \operatorname{DIFF}(F)$, where the action of $\operatorname{GL}_{\mathbb{H}}(F)$ on F is the canonical one.

A *section* φ in V is a mapping $\varphi\colon M \to V$ such that $\pi \circ \varphi = \operatorname{Id}_M$. The space of sections in V is denoted by $\Gamma(V)$.

By the definition any fiber of a quaternionic vector bundle is a quaternionic vector space. Of course any quaternionic vector bundle is also a real vector bundle. A *homomorphism* $\Phi\colon V \to W$ between quaternionic vector bundles over the same manifold M is a homomorphism between the real vector bundles V and W which is pointwise quaternionic linear. *Endomorphism*, *Isomorphism* and so on are defined as usual.

Example. Let (L, M, π, F) be a *line bundle*, i.e. a vector bundle of rank 1, over a Riemann surface M. Then the total space has real dimension $2 + 4 = 6$ and every section has codimension 4. By transversality theory, every section $\varphi\colon M \to L$ can be slightly deformed in such a way that it does not intersect the 0–section. Thus there exists a global nowhere vanishing section φ. Then the quaternionic linear mapping

$$\Phi\colon M \times \mathbb{H} \to V, \quad (x, \lambda) \mapsto \varphi(x)\lambda$$

is an isomorphism. Vector bundles V which are isomorphic to a product bundle $M \times \mathbb{H}^n$ are called *trivial bundles.*

2.2. Complex Quaternionic Vector Bundles.

Definition. A *complex quaternionic vector bundle* (V, J) over a Riemann surface M is a quaternionic vector bundle V over M together with a *complex structure* $J \in \operatorname{End}(V)$ satisfying $J^2 = -\operatorname{Id}$.

We can do the same things with quaternionic vector bundles as with quaternionic vector spaces. For example, every complex quaternionic vector bundle (V, J) is the double of a complex vector bundle W : Since J is smooth, the $\pm\mathbb{i}$ eigenbundles are smooth subbundles of V. In this situation we call W the *underlying complex vector bundle* of (V, J) and (V, J) the *associated complex quaternionic vector bundle* to W.

Example. On each Riemann surface M we have two important complex vector bundles: The *canonical bundle*

$$K = (T'M)^* = \{\omega \in \Omega^1(M, \mathbb{C}) | {*\omega} = i\omega\},$$

where $T'M$ is the holomorphic tangent bundle and $*$ denotes the precomposition of the complex structure on TM, and the *anticanonical bundle*

$$\bar{K} = \{\omega \in \Omega^1(M, \mathbb{C}) | -{*\omega} = i\omega\},$$

for more details see [**GriHa**].

Given a complex quaternionic vector bundle (V, J) we can tensorize it with the complex bundles $(K, *)$ or $(\bar{K}, -*)$ to obtain two complex quaternionic vector bundles:

$$KV = K \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(V) | {*\omega} = J\omega\}$$
$$\bar{K}V = \bar{K} \otimes_{\mathbb{C}} V = \{\omega \in \Omega^1(V) | -{*\omega} = J\omega\}.$$

Similar to complex geometry they will be of some importance in the quaternionic holomorphic geometry. We get the *decomposition*

$$T^*M \otimes_{\mathbb{R}} V = KV \oplus \bar{K}V.$$

The decomposition of a V–valued 1–form $\omega \in \Omega^1(V)$ is denoted by $\omega = \omega' + \omega''$, where $w' = \frac{1}{2}(\omega - J{*\omega}) \in \Gamma(KV)$ and $w'' = \frac{1}{2}(\omega + J{*\omega}) \in \Gamma(\bar{K}V)$.

A *homomorphism* $\Phi\colon (V,J) \to (W,J)$ between complex quaternionic vector bundles over M is a homomorphism between the quaternionic vector bundles satisfying $J \circ \Phi = \Phi \circ J$.

Example (The Dual Bundle)**.** If (V, M, π, F) is a quaternionic vector bundle over a manifold M, we can define the *dual bundle* (V^*, M, π, F^*) in the usual way. Every fiber V_p^* is the (quaternionic) dual vector space of the fiber V_p. If (V,J) is a complex quaternionic bundle, the dual bundle inherits the dual complex structure $J = J^*$. Hence it is also a complex quaternionic bundle. Furthermore, we obtain the same decomposition as in 1.4. We denote the dual bundle of a line bundle L by L^{-1}.

3. Holomorphic Quaternionic Bundles

We consider quaternionic bundles only over Riemann surfaces in this section. As in complex geometry there are important differential operators on these bundles, which are also called holomorphic structures.

3.1. Holomorphic Structures.

Definition. A *holomorphic structure* on a complex quaternionic vector bundle (V,J) is a quaternionic linear operator

$$D\colon \Gamma(V) \to \Gamma(\bar{K}V)$$

satisfying the Leibniz rule

$$D(\psi\lambda) = (\psi)\lambda + (\psi d\lambda)''$$

for any $\psi \in \Gamma(V)$ and $\lambda\colon M \to \mathbb{H}$, where $\psi d\lambda \in \Omega^1(V)$ is given by $(\psi d\lambda)(X) := \psi(d\lambda(X))$.

A complex quaternionic vector bundle (V,J) together with a holomorphic structure D is called a *holomorphic quaternionic vector bundle.*

A *homomorphism* $\Phi\colon V \to W$ between holomorphic quaternionic vector bundles (V,J,D) and (W,J,D) is a homomorphism between the complex quaternionic bundles (V,J) and (W,J) satisfying $D \circ \Phi = \Phi \circ D$.

Example. Consider a complex holomorphic vector bundle W, i.e. a complex vector bundle W together with a $\mathbb{C}$–linear operator

$$\bar\partial\colon \Gamma(W) \to \Gamma(\bar{K}W)$$

satisfying the Leibniz rule $\bar\partial(f\varphi) = f\bar\partial\varphi + (\bar\partial f)\varphi$, where $\bar\partial f = \frac{1}{2}(df + i{*}df)$. Let (V,J) be the associated complex quaternionic vector bundle to W. We define a holomorphic structure $D = \bar\partial$ on (V,J) as the double of $\bar\partial$:

$$\bar\partial(\varphi_+ \oplus \varphi_-) = (\bar\partial\varphi_+) \oplus (\bar\partial\varphi_-)$$

for $\varphi_\pm \in \Gamma(V_\pm) \cong \Gamma(W)$. Since $\bar{K}V \cong \bar{K}W \oplus \bar{K}W$ this defines a holomorphic structure on (V,J). Note that in this case the holomorphic structure and the complex structure on V commute: $\bar\partial \circ J = J \circ \bar\partial$.

In general, the holomorphic structure does not commute with the complex structure. This gives rise to the following definition:

3.2. Definition. The *Hopf field* Q of a holomorphic structure D on a complex quaternionic line bundle (V, J) is the J–anti–commuting part of D:

$$Q := \frac{1}{2}(D + JDJ) \in \Gamma(\bar{K}\,\mathrm{End}_-(V)).$$

We shall verify that the Hopf field Q is indeed a section in $\bar{K}\,\mathrm{End}_-(V)$: It is enough to check that Q is tensorial and that

$$-{*}Q = JQ = -QJ.$$

Let $\lambda\colon M \to \mathbb{H}$ and $\psi \in \Gamma(V)$ be given, then we compute

$$2Q(\psi\lambda) = (D + JDJ)(\psi\lambda) = (D + JDJ)(\psi)\lambda = 2Q(\psi)\lambda$$

and

$$2(-{*}Q)(\psi) = -{*}(D\psi + JDJ\psi) = JD\psi - DJ\psi = 2JQ\psi = -2QJ\psi.$$

Almost the same computations show that

$$\bar{\partial} := \frac{1}{2}(D - JDJ)$$

is again a holomorphic structure on (V, J) and that $\bar{\partial}$ commutes with J. Therefore $\bar{\partial}$ maps the $\pm\mathbb{i}$–eigenspaces of $J \in \mathrm{End}(V)$ onto the $\pm\mathbb{i}$–eigenspaces of $J \in \mathrm{End}(\bar{K}V)$. Thus, restricted to the underlying complex bundle W, we can regard $\bar{\partial}$ as an $\mathbb{C}$–linear operator $\bar{\partial}\colon \Gamma(W) \to \Gamma(\bar{K}W)$. Of course $\bar{\partial}$ satisfies the Leibniz rule, so $(W, \bar{\partial})$ is a complex holomorphic vector bundle.

We want to measure how far a holomorphic quaternionic bundle is away from being the double of a complex holomorphic bundle.

3.3. Definition. Let V be a holomorphic quaternionic bundle over a compact Riemann surface M with complex structure $D = \bar{\partial} + Q$. The *Willmore energy* of (V, J, D) is defined as

$$W(V, J, D) := 2\int_M < Q \wedge {*}Q >,$$

where $<>$ is the trace form introduced in 1.6.

Remark. We also denote the Willmore energy of (V, J, D) by $W(V)$ if it is clear which complex and holomorphic structures on V we mean.

Note that $Q \wedge {*}Q$ is an $\mathrm{End}_+(V)$–valued 2–form. So it does make sense to integrate $< Q \wedge {*}Q >$ on M.

3.4. Lemma. *Let L be a holomorphic quaternionic line bundle with holomorphic structure $D = \bar{\partial} + Q$. The integrand $< Q \wedge {*}Q >$ of the Willmore energy vanishes if and only if Q vanishes. Moreover $W(V) \geq 0$ with equality if and only if $Q \equiv 0$.*

Proof. Let $p \in M$ and $z = x + iy$ be a holomorphic coordinate around p. We have to show $< Q \wedge {*}Q > (\frac{\partial}{\partial x}_{|p}, \frac{\partial}{\partial y}_{|p}) \geq 0$ with equality if and only if $Q(p) = 0$. Since $Q \in \Gamma(\bar{K}\,\mathrm{End}_-(V))$ we compute

$$Q \wedge {*}Q(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -Q(\frac{\partial}{\partial x})Q(\frac{\partial}{\partial x}) - Q(\frac{\partial}{\partial y})Q(\frac{\partial}{\partial y}) = -2Q(\frac{\partial}{\partial x})Q(\frac{\partial}{\partial x}).$$

Because $Q(\frac{\partial}{\partial x}_{|p}) \in \mathrm{End}_-(V_p)$ the lemma follows by 1.7. $\square$

3.5. Example. Let (L, J) be a complex quaternionic line bundle with holomorphic structure D. Assume $\psi \in \Gamma(L)$ is a nowhere vanishing section, then there exists a function $R\colon M \to \mathbb{H}$ such that

$$J\psi = -\psi R.$$

If ψ is holomorphic, i.e. $D\psi = 0$, we can compute the Hopf field Q of D as follows:

$$\begin{aligned} Q\psi &= \frac{1}{2}(D\psi + JD(J\psi)) = -\frac{1}{2}J(\psi dR)'' \\ &= -\frac{1}{4}J(\psi dR - \psi R{*}dR) = \frac{1}{4}\psi(RdR + {*}dR). \end{aligned}$$

3.6. Pairings of holomorphic Bundles. In complex geometry, a holomorphic structure induces a holomorphic structure on the dual bundle and on tensor products. In holomorphic quaternionic geometry this does not work in the same way, but there are also some important related structures and bundles.

Definition. A *pairing* between complex quaternionic vector bundles (V_1, J) and (V_2, J) over M is a $\mathbb{R}$–linear map

$$(,)\colon V_1 \otimes_{\mathbb{R}} V_2 \to T^*M \otimes_{\mathbb{R}} \mathbb{H}$$

which is pointwise non–degenerate and satisfies

$$(\psi\lambda, \varphi\mu) = \bar{\lambda}(\psi, \varphi)\mu$$

and

$$*(\psi, \varphi) = (J\psi, \varphi) = (\psi, J\varphi)$$

for $\psi \in \Gamma(V_1)$, $\varphi \in \Gamma(V_2)$ and $\lambda,\ \mu\colon M \to \mathbb{H}$.

If (V_1, J) and (V_2, J) are paired by $(,)$ then (V_2, J) and (V_1, J) are paired by $\overline{(,)}$. If (L, J) is a complex quaternionic line bundle, then KL^{-1} with induced complex structure and L are paired by the evaluation: $(\omega, \psi) = \omega(\psi)$. Moreover we have the following proposition:

3.7. Proposition. *Let $\tilde{L}$ and L be paired complex quaternionic line bundles. Then $\Phi\colon \tilde{L} \to KL^{-1}$; $\varphi \mapsto (\varphi, .)$ is an isomorphism of complex quaternionic line bundles such that the pairing $(,)$ of $\tilde{L}$ and L corresponds to the evaluation pairing of KL^{-1} and L.*

Proof. First we show that Φ maps into KL^{-1}: For $p \in M$ and $\varphi \in \tilde{L}_p$, $\Phi(\varphi)$ is quaternionic linear on L_p and real linear on T_pM so $\Phi(\varphi) \in (T^*M \otimes_{\mathbb{R}} L^{-1})_p$. Since $*\Phi(\varphi) = \Phi(\varphi) \circ J$, we have by definition $\Phi(\varphi) \in KL^{-1}$. Moreover the map Φ is quaternionic linear by the definitions of a pairing and of the quaternionic structure on L^{-1} and since $(,)$ is non–degenerate, it must be an isomorphism. □

Definition. Let $(\tilde{L}, J)$ and (L, J) be paired complex quaternionic line bundles with holomorphic structures $\tilde{D}$ and D. The holomorphic structures are called *compatible* with respect to the pairing $(,)$ if for all sections $\psi \in \Gamma(\tilde{L})$ and $\varphi \in \Gamma(L)$ the equation

$$d(\psi, \varphi) = (\tilde{D}\psi \wedge \varphi) + (\psi \wedge D\varphi) \tag{3.7.1}$$

holds, where

$$(\tilde{D}\psi \wedge \varphi)(X,Y) = (\tilde{D}\psi(X), \varphi)(Y) - (\tilde{D}\psi(Y), \varphi)(X)$$

and

$$(\psi \wedge D\varphi)(X,Y) = (\psi, D\varphi(X))(Y) - (\psi, D\varphi(Y))(X).$$

3.8. Theorem. *If $(\tilde{L}, J)$ and (L, J) are paired complex quaternionic line bundles and D is a holomorphic structure on L, then there is an unique holomorphic structure $\tilde{D}$ on $\tilde{L}$, such that D and $\tilde{D}$ are compatible with respect to the pairing.*

Proof. Because of 3.7 it is enough to show that there exists exactly one holomorphic structure $\tilde{D}$ on KL^{-1} which is compatible to D with respect to the evaluation pairing $(,)$. For the existence, we define $\tilde{D}$ by

$$(\tilde{D}\omega, \psi) = d(\omega, \psi) - (\omega \wedge D\psi) \tag{3.8.1}$$

for $\omega \in \Gamma(KL^{-1})$ and $\psi \in \Gamma(L)$, where

$$(\omega \wedge D\psi)(X,Y) = -(\omega(X), D\psi(Y)) + (\omega(Y), D\psi(X)).$$

It is easy to verify that $\tilde{D}$ is well–defined and is in fact a holomorphic structure on KL^{-1}.

Now notice that the equation 3.7.1 for the evaluation pairing between KL^{-1} and L is exactly equation 3.8.1. This shows uniqueness. □

3.9. Connections and holomorphic Bundles. Let ∇ be a quaternionic connection on a complex quaternionic vector bundle (V, J), i.e. a linear connection on the (real) vector bundle V which satisfies $\nabla(\psi\lambda) = (\nabla\psi)\lambda$ for all $\lambda \in \mathbb{H}$. The decomposition

$$\nabla = \nabla' + \nabla''$$

into K–part and $\bar{K}$–part defines a holomorphic structure

$$D = \nabla'' = \frac{1}{2}(\nabla + J * \nabla)$$

on V. In this decomposition ∇' is an anti–holomorphic structure, i.e. a holomorphic structure on $(V, -J)$. We decompose ∇' and ∇'' into J–commuting and anti–commuting parts to obtain

$$\nabla = \nabla' + \nabla'' = (\partial + A) + (\bar{\partial} + Q). \tag{3.9.1}$$

Then ∂ and $\bar{\partial}$ are complex anti–holomorphic resp. complex holomorphic structures on (V, J) and

$$A = \frac{1}{2}(\nabla' + J\nabla' J) \in \Gamma(K\,\mathrm{End}_-(V))$$

and

$$Q = \frac{1}{2}(\nabla'' + J\nabla'' J) \in \Gamma(\bar{K}\,\mathrm{End}_-(V)).$$

Every quaternionic connection ∇ on a quaternionic vector bundle V induces a dual quaternionic connection $\nabla = \nabla^*$ on the dual bundle V^* by the product rule:

$$d(\omega, \psi) = (\nabla\omega, \psi) + (\omega, \nabla\psi)$$

for all $\omega \in \Gamma(V^*)$ and $\psi \in \Gamma(V)$. Therefore a connection on (V, J) induces holomorphic structures on (V, J) and on (V^*, J^*). We want to compare the decompositions of the dual connections:

3.10. Proposition. *Let (V, J) and (V^*, J) be dual complex quaternionic vector bundles with dual connections ∇ respectively ∇^*, and let $\nabla = (\partial + A) + (\bar\partial + Q)$ be the decomposition of ∇ by parts. Then*

$$\nabla^* = (\partial^* - Q^*) + (\bar\partial^* - A^*)$$

where ∂^* and $\bar\partial^*$ are the anti– respectively holomorphic structures on V^* induced by ∂ and $\bar\partial$.

Proof. First note that the operators ∂, $\bar\partial$, $(\partial + \bar\partial)$ and ∂^*, $\bar\partial^*$, $(\partial^* + \bar\partial^*)$ can be considered as anti–holomorphic structures resp. holomorphic structures resp. connections on the underlying bundles W and W^* and from the theory of complex holomorphic bundles we know that the statement is true for the complex case $Q = A = 0$. Hence we have

$$\begin{aligned} d(\omega, \psi) &= ((\partial^* + \bar\partial^*)\omega, \psi) + (\omega, (\partial + \bar\partial)\psi) \\ &= ((\partial^* - Q^* + \bar\partial^* - A^*)\omega, \psi) + (\omega, (\partial + A + \bar\partial + Q)\psi). \end{aligned}$$

This formula shows that in fact $\nabla^* = \partial^* - Q^* + \bar\partial^* - A^*$. It remains to compute the decomposition of ∇^*, but since we already know that $\partial^* + \bar\partial^*$ is the decomposition of the dual connection of $\partial + \bar\partial$, we only have to show that $Q^* \in \Gamma(K \operatorname{End}_-(V^*))$ and $A^* \in \Gamma(\bar K \operatorname{End}_-(V^*))$. For all $\omega \in \Gamma(V^*)$ and $\psi \in \Gamma(V)$ we compute

$$(*Q^*\omega, \psi) = (\omega, *Q\psi) = (\omega, -JQ\psi) = (-J\omega, Q\psi) = (-Q^*J\omega, \psi)$$

and similarly

$$(*A^*\omega, \psi) = (A^*J\omega, \psi).$$

Of course $JQ^* = -Q^*J$ and $JA^* = -AJ^*$ by the definition of the dual complex structure J so we conclude $*Q^* = -Q^*J = JQ^*$ and $*A^* = A^*J = -JA^*$ as claimed. □

Remark. This proposition shows why in general a holomorphic structure on a complex quaternionic bundle (V, J) has no canonical dual holomorphic structure on the dual bundle (V^*, J) : The dual structure (in an appropriate sense) of a holomorphic structure $\bar\partial + Q$ is $\bar\partial^* - Q^*$, but this is not a holomorphic structure since $Q^* \in \Gamma(K \operatorname{End}_-(V))$. We will come back to this in the next part.

If ∇ is a connection on the complex quaternionic line bundle (L, J) over a Riemann surface M and $\nabla = \nabla^*$ is its dual connection on L^{-1}, then the exterior differential $d^\nabla\colon \Omega^k(L^{-1}) \to \Omega^{k+1}(L^{-1})$ defines a holomorphic structure as follows. Using the identification $\bar K K \cong \Lambda^2 T^*M \otimes \mathbb{C}$ with $d\bar z \otimes_{\mathbb{C}} dz \mapsto d\bar z \wedge dz = -dz \wedge d\bar z$ we obtain

$$\Omega^2(L^{-1}) = \Gamma(\Lambda^2(T^*M) \otimes_{\mathbb{R}} L^{-1}) = \Gamma(\bar K K \otimes_{\mathbb{C}} L^{-1}) = \Gamma(\bar K K L^{-1}).$$

Then

$$D = d^\nabla\colon \Gamma(KL^{-1}) \to \Gamma(\bar K K L^{-1})$$

satisfies the Leibniz rule

$$d^\nabla(\omega\lambda) = (d^\nabla\omega)\lambda + (\omega d\lambda)''$$

for all $\omega \in \Gamma(KL^{-1})$ and $\lambda\colon M \to \mathbb{H}$. To see this, we assume that λ is $\mathbb{R}$-valued because ∇ is a quaternionic connection, and then we compute locally for $\omega = dz \otimes_{\mathbb{C}} \phi = \phi dx + J\phi dy$, where $z = x + iy$ is a conformal coordinate on M and $\phi \in \Gamma(L^{-1})$:

$$\begin{aligned}
d^\nabla(\omega\lambda) &= d^\nabla(\phi\lambda dx + J\phi\lambda dy)\\
&= \nabla(\phi) \wedge (\lambda dx) + \nabla(J\phi) \wedge (\lambda dy) + \phi d\lambda \wedge dx + J\phi d\lambda \wedge dy\\
&= d^\nabla(\omega)\lambda + (d\lambda \wedge dz) \otimes_{\mathbb{C}} \phi\\
&= d^\nabla(\omega)\lambda + \frac{1}{2}(d\lambda + i{*}d\lambda) \wedge (dx + idy) \otimes_{\mathbb{C}} \phi\\
&= d^\nabla(\omega)\lambda + \frac{1}{2}(d\lambda\phi dx + d\lambda J\phi dy + {*}d\lambda J\phi dx - {*}d\lambda\phi dy)\\
&= d^\nabla(\omega)\lambda + (\omega d\lambda)''
\end{aligned}$$

and therefore d^∇ defines a holomorphic structure on KL^{-1}. This holomorphic structure is compatible to the holomorphic structure on L induced by ∇ with respect to the pairing $(,)$ between KL^{-1} and L.

3.11. Lemma. *Let L be a complex quaternionic line bundle over a Riemann surface and ∇ a quaternionic connection on L. The holomorphic structures d^∇ and D on KL^{-1} and L induced by ∇ are compatible with respect to the natural pairing between KL^{-1} and L.*

PROOF. Locally we can write for each $\omega \in \Gamma(KL^{-1})$:

$$\omega = dz \otimes_{\mathbb{C}} \phi = \phi dx + J\phi dy,$$

where $z = x + iy$ is a conformal coordinate on M, and with $\psi \in \Gamma(L)$ we compute

$$\begin{aligned}
d(\omega,\psi) &= d(\phi(\psi)dx + J\phi(\psi)dy)\\
&= (\nabla\phi)(\psi) \wedge dx + (\nabla J\phi)(\psi) \wedge dy + \phi(\nabla\psi) \wedge dx + J\phi(\nabla\psi) \wedge dy\\
&= (\nabla\phi \wedge dx + \nabla(J\phi) \wedge dy)(\psi) - (dz \otimes_{\mathbb{C}} \phi \wedge \nabla\psi)\\
&= (d^\nabla\omega, \phi) + (\omega \wedge D\psi).
\end{aligned}$$

as claimed, compare with 3.8.

□

3.12. Mixed Structures.

Definition. A *mixed structure* on a complex quaternionic vector bundle (V, J) is a quaternionic linear map

$$\hat{D}\colon \Gamma(V) \to \Omega^1(V)$$

which satisfies the Leibniz rule

$$\hat{D}(\psi\lambda) = \hat{D}(\psi)\lambda + (\psi d\lambda)''$$

and

$${*}\hat{D} = -\hat{D}J.$$

As usual there is a decomposition of a mixed structure into K and $\bar{K}$ parts:

$$\hat{D} = \hat{D}' + +\hat{D}''.$$

Then we have

$$\hat{D}' = \frac{1}{2}(\hat{D} - *J\hat{D}) = \frac{1}{2}(\hat{D} + J\hat{D}J) = \hat{D}_-$$

and

$$\hat{D}'' = \frac{1}{2}(\hat{D} + *J\hat{D}) = \frac{1}{2}(\hat{D} - J\hat{D}J) = \hat{D}_+.$$

Now it is easy to see that $\hat{D}'' = \bar{\partial}$ is a (complex) holomorphic structure and that $\hat{D}' = A \in K\,\mathrm{End}_-(V)$.

For example, if $\nabla = \partial + A + \bar{\partial} + Q$ is a quaternionic connection on a complex quaternionic bundle (V, J), the operator $\hat{D} = \bar{\partial} + A$ is a mixed structure. In view of 3.10 the following lemma is not surprising.

3.13. Lemma. *Let (V, J) be a complex quaternionic vector bundle. Then, the product rule*

$$\frac{1}{2}(d < \alpha, \psi > + * d < \alpha, J\psi >) = < \hat{D}\alpha, \psi > + < \alpha, D\psi >$$

for $\alpha \in \Gamma(V^{-1})$ and $\psi \in \Gamma(V)$ gives a one–to–one correspondence between holomorphic structures $D = \bar{\partial} + Q$ on V and mixed structures $\hat{D} = \bar{\partial} - Q^*$ on V^{-1}.

A proof can be found in [**FLPP01**].

In the case of a holomorphic complex bundle V, i.e. $Q = 0$ and $D = \bar{\partial}$, the dual structure $\hat{D} = \bar{\partial}$ on V^{-1} is also a holomorphic structure.

CHAPTER II

Surfaces in S^4

In this chapter we study conformally immersed surfaces in the conformal 4–sphere using the theory discussed in Chapter I. The basic reference to the quaternionic approach to the theory of conformally immersed surfaces in S^4 is [**BFLPP02**].

4. Conformally Immersed Surfaces in $\mathbb{H}$

A mapping $f\colon M \to N$ between Riemannian Manifolds (M, g) and (N, h) is called *conformal* if there exists a smooth function $\lambda\colon M \to \mathbb{R}$ such that

$$e^{\lambda} g = f^* h. \tag{4.0.1}$$

We are primarily interested in mappings from a Riemann surface M to $\mathbb{R}^4 \cong \mathbb{H}$ or to S^4. On a Riemann surface is no canonical metric but 4.0.1 does only depend on the conformal type on the Riemannian Manifold and not on the special metric (of course the function λ changes when we change the metric conformally on M). On a Riemann surface M is a canonical conformal class of metrics: A Riemannian metric g is called compatible to the complex structure J if for all $p \in M$

$$J_p\colon (T_pM, g_p) \to (T_pM, g_p)$$

is an orthogonal linear mapping. In this case we have for any $\xi \in T_pM,\ \xi \neq 0$ an orthogonal basis $(\xi, J\xi)$ of T_pM, and the condition that a map $f\colon M \to N$ is conformal becomes that $(D_pf(\xi), D_pf(J\xi))$ forms an orthogonal basis of $\mathrm{im}(D_pf) \subset T_{f(p)}N$ for any $p \in M$ and $\xi \in T_pM,\ \xi \neq 0$. If N is a Riemann surface with complex structure $\tilde{J}$ the condition is $*df = \pm\tilde{J}df$, or equivalent that f is holomorphic respectively anti–holomorphic. If $N = \mathbb{C}$ and $\tilde{J}$ is the multiplication with i this is classical holomorphic or anti–holomorphic function theory. $\mathbb{H}$ has no canonical complex structure and moreover $\mathbb{H}$ is 4–dimensional. But because of the following fundamental lemma, we obtain a 'function theory with changing i.'

4.1. Lemma. *Let U be an oriented real 2–dimensional subspace of $\mathbb{H}$. There exist unique $N,\ R \in \mathbb{H}$ satisfying $N^2 = R^2 = -1$ and*

$$NU = U = UR,$$

such that for any nonzero $x \in U$ the $\mathbb{R}$–bases (x, Nx) and $(x, -xR)$ of U are positively oriented. Then the space U is given by

$$U = \{x \in \mathbb{H} | NxR = x\}.$$

Conversely, every pair of vectors $N,\ R \in \mathbb{H}$ satisfying $N^2 = R^2 = -1$ defines an oriented 2–plane by $U = \{x \in \mathbb{H} | NxR = x\}$.

Proof. First assume that $1 \in U$. Then there is an unique vector $a \in U$ such that $(1, a)$ is a positive oriented orthonormal basis of U. Of course we have $a^2 = -1$. Thus $N = a$ satisfies the conditions. To see the uniqueness of N note that $N^2 = -1$ implies that $N \in \operatorname{Im}(\mathbb{H})$ is an unit vector. Hence the only possibilities for N are $\pm a$, but since $(1, -a)$ is negative oriented we conclude that $N = a$ is unique. By the same arguments we see that $R = -a$ is also unique. Then $U = \{x \in \mathbb{H} | ax = xa\} = \{x \in \mathbb{H} | NxR = x\}$.

For arbitrary U, choose a nonzero $x \in U$ and consider the real 2–dimensional subspace $\tilde{U} = x^{-1}U$ with orientation given by the positive oriented basis $x^{-1}v_1, x^{-1}v_2$ of $\tilde{U}$, where v_1, v_2 is a positive oriented basis of U. Then $1 \in \tilde{U}$ and moreover N and R work for U if and only if $x^{-1}Nx$ and R work for $\tilde{U}$.

Conversely given N and R with $N^2 = R^2 = -1$. It is well–known, see [**BFLPP02**], that there is a $x \in \mathbb{H} \setminus \{0\}$, unique up to multiplication by a nonzero real number, such that $N = x(-R)x^{-1}$. Then it is easy to see that $U = \operatorname{span}(x, -xR)$ with orientation given by $x \wedge (-xR) > 0$. □

Remark. The vectors N and R are called the *left* and *right normal vector* of U, although they are in general not orthogonal to U. But in the case of $U \subset \mathbb{R}^3 \cong \operatorname{Im}(\mathbb{H})$ we have $N = R$ and this is the unit orthogonal vector of the oriented 2–plane U.

This lemma can be reformulated by using non–zero basis vectors of the appearing vector spaces:

4.2. Lemma. *Let V, W be 1–dimensional quaternionic vector spaces, and*

$$U \subset \operatorname{Hom}(V, W)$$

a 2–dimensional oriented real vector space. Then there exists a unique complex structure $J \in \operatorname{End}(V)$ such that

$$U = UJ$$

and J is compatible with the orientation, i.e. (F, FJ) is positive oriented in U for any nonzero $F \in U$. In this case there is a unique complex structure $\tilde{J} \in \operatorname{End}(W)$ such that

$$U = \{F \in \operatorname{Hom}(V, W) | \tilde{J}FJ = -F\},$$

$$U = \tilde{J}U.$$

$\tilde{J}$ is compatible with the orientation.

Definition. Let M be a Riemann surface. An immersion $f \colon M \to \mathbb{H}$ is called *conformal immersion* if there exist N, $R \colon M \to \mathbb{H}$ such that

$$*df = Ndf = -dfR \tag{4.2.1}$$

and $N^2 = R^2 = -1$.

N and R are unique and called the *left* and *right normal vector* of f.

4.3. Remark. Note that Lemma 4.1 shows that the existence of N or R satisfying the equation 4.2.1 is enough to guarantee the existence of the other normal vector R or N. Moreover, since f is assumed to be an immersion, $N^2 = R^2 = -1$ follows by 4.2.1.

If $f\colon M \to \operatorname{Im}(\mathbb{H}) \cong \mathbb{R}^3$ we have $N = R$ is the classical unit normal vector of f.

Note that the definition given here is the same as the classical definition in the setup of immersions from Riemann surfaces into $\mathbb{H}$.

We state two formulas for the second fundamental form and the mean curvature vector of f in terms of the normal vectors. The proofs can be found in [**BFLPP02**].

4.4. Proposition. *The second fundamental form* $\mathrm{II}(X,Y) = (Xdf(Y))^\perp$ *of an immersion* $f\colon M \to \mathbb{H}$ *of a Riemann surface is given by*

$$\mathrm{II}(X,Y) = \frac{1}{2}(*df(Y)dR(X) - dN(X) * df(Y)), \tag{4.4.1}$$

where N and R are the left and the right normal vector of f. The mean curvature vector $\mathcal{H} = \frac{1}{2}\operatorname{tr}\mathrm{II}$ *is given by*

$$\bar{\mathcal{H}}df = \frac{1}{2}(*dR + RdR), \quad df\bar{\mathcal{H}} = \frac{1}{2}(*dN + NdN). \tag{4.4.2}$$

5. The Quaternionic Projective Space

The natural space to do conformal geometry is not the Euclidean space but the conformal sphere S^n. To have full use of the quaternionic theory we describe how to identify the conformal 4–sphere S^4 with the quaternionic projective space $\mathbb{HP}^1$.

Consider the equivalence relation $\sim$ on $\mathbb{H}^{n+1} \setminus \{0\}$ given by

$$v \sim w\colon \Leftrightarrow v\mathbb{H} = w\mathbb{H},$$

where $v\mathbb{H} := \operatorname{span}(v)$. Then the *projective space* is defined by

$$\mathbb{HP}^n := (\mathbb{H}^{n+1} \setminus \{0\})/\sim .$$

We have the natural projection $\pi\colon \mathbb{H}^{n+1}\setminus\{0\} \to \mathbb{HP}^n;\ v \mapsto [v]$. We consider the quaternionic projective space with the quotient topology induced by π. Note that we can regard the quaternionic projective space as the space of quaternionic 1–dimensional subspaces l in $\mathbb{H}^{n+1}$.

The manifold structure is defined as for real or complex projective spaces: For any non–zero linear form $\alpha \in (\mathbb{H}^{n+1})^*$ the mapping

$$u_\alpha\colon \{\pi(x) \in \mathbb{HP}^n \mid \alpha(x) \neq 0\} \to \mathbb{H}^{n+1};\ \pi(x) \mapsto x(\alpha(x))^{-1}$$

is well–defined and maps the open set $U_\alpha := \{\pi(x) \in \mathbb{HP}^n \mid \alpha(x) \neq 0\}$ bijectively onto the affine hyperplane $\alpha = 1$, which is affine isomorphic to $\mathbb{H}^n$. The maps u_α are called *affine coordinates* and define a real analytic atlas for $\mathbb{HP}^n$. Of course the projection π is smooth.

If we choose a basis $e_1, .., e_{n+1}$ of $\mathbb{H}^{n+1}$ with dual basis $\alpha_1, .., \alpha_{n+1}$ we also denote the mapping

$$[e_1x_1 + .. + e_k + .. + e_{n+1}x_{n+1}] \in U_{\alpha_k} \mapsto (x_1, .., x_{k-1}, x_{k+1}, .., x_{n+1}) \in \mathbb{H}^n \tag{5.0.3}$$

by u_{α_k} and call it *Euclidean coordinate.*

The set $\{\pi(x)|\alpha(x) = 0\}$ is called hyperplane at infinity. In the special case $n = 1$, the hyperplane at infinity is just a single point and therefore

$\mathbb{H}P^1$ is the one–point–compactification of $\mathbb{R}^4$. But the same is true for S^4, so both must be diffeomorphic:

$$\mathbb{H}P^1 = S^4.$$

A diffeomorphism is given by choosing a north pole $N \in S^4$ and the point at infinity $\infty \in \mathbb{H}P^1$. Then via stereographic projection

$$S^4 \setminus \{N\} \cong \mathbb{R}^4 \cong \mathbb{H}P^1 \setminus \{\infty\},$$

and if we map N to ∞ everything becomes smooth. The exact proof works as for $S^2 \cong \mathbb{C}P^1$.

The *standard conformal structure* on S^4 is given by the equivalence class of the induced metric of the canonical embedding $S^4 \hookrightarrow \mathbb{R}^5$. Since $S^4 \cong \mathbb{H}P^1$ we get a conformal structure on $\mathbb{H}P^1$. Note that the Euclidean coordinates (for a basis α, β of $(\mathbb{H}^2)^*$))

$$u_\alpha \colon U_\alpha \to \mathbb{H}$$

are conformal maps from the set U_α, considered with the induced conformal structure by $U_\alpha \hookrightarrow \mathbb{H}P^1$, onto $\mathbb{H}$, see [**BFLPP02**] for a more detailed discussion. We have the following lemma:

5.1. Lemma. *A map $f \colon M \to \mathbb{H}P^1$ from a Riemann surface (or Riemannian manifold) M is conformal if and only if for any basis α_1, α_2 of $(\mathbb{H}^2)^*$ the maps $u_{\alpha_i} \circ f \colon f^{-1}(U_\alpha) \to \mathbb{H}$, $i = 1, 2$ are conformal.*

Proof. If f is conformal then for $i = 1, 2$ $u_{\alpha_i} \circ f$ is also conformal since the maps u_{α_i} are conformal. Conversely if both maps $u_{\alpha_i} \circ f$ are conformal, the maps $f_{|f^{-1}(U_{\alpha_i})}$ must be conformal. Since being conformal is a local property and the sets $f^{-1}(U_{\alpha_i})$ cover M we conclude that f is conformal. □

Example (The Tautological Bundle)**.** The *tautological bundle* Σ is a quaternionic line bundle over $\mathbb{H}P^n$:

$$\Sigma := \{([x], v) \in \mathbb{H}P^n \times \mathbb{H}^{n+1} | v = x\lambda,\ \lambda \in \mathbb{H}\}$$
$$\pi_\Sigma \colon \Sigma \to \mathbb{H}P^n,\ ([x], v) \mapsto [x].$$

Let $v_1, .., v_{n+1}$ be a basis of $\mathbb{H}^{n+1}$ and $\alpha_1, ..\alpha_{n+1}$ its dual basis. Then Σ is trivial over the sets U_{α_k} by

$$\Psi_k \colon \Sigma_{|U_{\alpha_k}} \to U_{\alpha_k} \times \mathbb{H};$$
$$([\sum_{i=1}^{n+1} v_i x_i], (\sum_{i=1}^{n+1} v_i x_i)\lambda) \mapsto ([\sum_{i=1}^{n+1} v_i x_i], x_k \lambda).$$

The transition functions are given by

$$\Psi_k \Psi_l^{-1} \colon (U_{\alpha_k} \cap U_{\alpha_l}) \times \mathbb{H} \to (U_{\alpha_k} \cap U_{\alpha_l}) \times \mathbb{H};$$
$$([\sum_{i=1}^{n+1} v_i x_i], \lambda) \mapsto ([\sum_{i=1}^{n+1} v_i x_i], x_k x_l^{-1} \lambda)$$

and therefore are pointwise $\mathbb{H}$–linear. This shows that the Tautological bundle is a quaternionic line bundle over $\mathbb{H}P^n$.

5.2. The Tangent Space of $\mathbb{H}\mathrm{P}^n$. Similar to the real or complex case, we can describe the tangent space at $l \in \mathbb{H}\mathrm{P}^n$ as follows:

$$T_l \, \mathbb{H}\mathrm{P}^n \cong \mathrm{Hom}(l, \mathbb{H}^{n+1}/l).$$

To see this we first mention that $\pi(x) = \pi(x\lambda)$ for all $x \in \mathbb{H}^{n+1} \setminus \{0\}$ and $\lambda \in \mathbb{H} \setminus \{0\}$ implies $d_x\pi(v) = d_{x\lambda}\pi(v\lambda)$, where $\pi \colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}\mathrm{P}^n$ is the projection. The differential $d_x\pi \colon \mathbb{H}^{n+1} \to T_{[x]} \, \mathbb{H}\mathrm{P}^n$ is surjective with $\ker d_x\pi = x\mathbb{H}$. Therefore linear algebra tells us that $\mathbb{H}^{n+1}/(x\mathbb{H})$ is isomorphic to $T_{[x]} \, \mathbb{H}\mathrm{P}^n$ via $d_x\pi$, but this isomorphism depends on the choice of x. But the map

$$\xi \in \mathrm{Hom}(x\mathbb{H}, \mathbb{H}^{n+1}/(x\mathbb{H})) \mapsto d_x\pi(\xi(x)) \in T_{[x]} \, \mathbb{H}\mathrm{P}^n$$

is well–defined, and because of $d_x\pi(v) = d_{x\lambda}\pi(v\lambda)$ it is also independent of the choice of x. Of course this map is also injective and because both spaces have the same dimension, it must be an isomorphism. The inverse is given by the mapping

$$d_x\pi(v) \in T_{[x]} \, \mathbb{H}\mathrm{P}^n \mapsto (x\lambda \mapsto v\lambda + x\mathbb{H}) \in \mathrm{Hom}(x\mathbb{H}, \mathbb{H}^{n+1}/x\mathbb{H}). \tag{5.2.1}$$

5.3. Lemma. *For a map $f = \pi \circ \tilde{f} \colon M \to \mathbb{H}\mathrm{P}^n$ with non–vanishing $\tilde{f} \colon M \to \mathbb{H}^{n+1} \setminus \{0\}$, the differential of f at a point $p \in M$ is given by:*

$$v \in T_pM \mapsto (\tilde{f}(p)\lambda \mapsto d_p\tilde{f}(v)\lambda + l) \in \mathrm{Hom}(l, \mathbb{H}^{n+1}/l) \cong T_l \, \mathbb{H}\mathrm{P}^n,$$

where $l = \tilde{f}(p)\mathbb{H} = f(p)$.

We denote the differential in this interpretation by δf :

$$\delta f(v)(\tilde{f}) = d\tilde{f}(v) + \tilde{f}\mathbb{H}.$$

Proof. The lemma follows directly by 5.2.1 and by the chain rule $d_pf(v) = d_{\tilde{f}(p)}\pi(d_p\tilde{f}(v))$. □

The tangential bundle is given by

$$T \, \mathbb{H}\mathrm{P}^n \cong \mathrm{Hom}(\Sigma, H/\Sigma),$$

where $H = \mathbb{H}\mathrm{P}^n \times \mathbb{H}^{n+1}$ and Σ is the tautological bundle over $\mathbb{H}\mathrm{P}^n$.

5.4. Moebius Transformations on $\mathbb{H}\mathrm{P}^1$. As in the real or complex case the group $\mathrm{GL}(2, \mathbb{H})$ acts on $\mathbb{H}\mathrm{P}^1$ transitively as a group of diffeomorphism by

$$G(v\mathbb{H}) = (G(v))\mathbb{H}$$

for $G \in \mathrm{GL}(2, \mathbb{H})$ and $v \in \mathbb{H}^2 \setminus \{0\}$. The kernel of this action is given by the set $\{r \,\mathrm{Id}\, | r \in \mathbb{R}^*\}$.

Proposition. *The group of orientation preserving conformal diffeomorphisms of $S^4 \cong \mathbb{H}\mathrm{P}^1$ is given by $\mathrm{GL}(2, \mathbb{H})/\{r \,\mathrm{Id}\, | r \in \mathbb{R}^*\}$ with the action described above.*

Proof. We only show that the action of $\mathrm{GL}(2, \mathbb{H})$ is in fact orientation preserving and conformal. The rest of the proof can be found in [**KP88**]. To see that the action is orientation preserving we just state that $\mathrm{GL}(2, \mathbb{H})/\{r \,\mathrm{Id}\, | r \in \mathbb{R}^*\}$ is a connected group.

Let $G \in \mathrm{GL}(2, \mathbb{H})$ and denote the action of G also by G. Let e_1, e_2 be the standard basis of $\mathbb{H}^2$. Then $v_1 = Ge_1,\ v_2 = Ge_2$ is another basis of $\mathbb{H}^2$ with dual basis $\alpha_1,\ \alpha_2$ and we obtain

$$u_{\alpha_2} \circ G([e_1 x + e_2]) = x$$

and

$$u_{\alpha_1} \circ G([e_1 + e_2 y]) = y.$$

Since the mappings $x \mapsto [e_1 x + e_2]$ and $y \mapsto [e_1 + e_2 y]$ and their inverse mappings are conformal, we conclude that the diffeomorphism $G \in \mathrm{DIFF}(\mathbb{H}\mathrm{P}^1)$ is indeed conformal, see 5.1. □

6. Conformally Immersed Surfaces in $\mathbb{H}P^1$

There are several ways to deal with conformal mappings to $S^4 \cong \mathbb{H}\mathrm{P}^1$. Here we use the quaternionic approach as in [**BFLPP02**].

To make full use of the quaternions, one should study some associated bundles instead of the conformal mapping. If $f\colon M \to \mathbb{H}\mathrm{P}^1$ is a map we get the bundle

$$L := f^*\Sigma,$$

which is a line sub–bundle of $H := \mathbb{H}\mathrm{P}^1 \times \mathbb{H}^2$ with fibers $L_x = f(x) \subset \mathbb{H}^2$. Conversely, every line sub–bundle L of H determines a map $f\colon M \to \mathbb{H}\mathrm{P}^1$ by $f(x) := L_x$. There is a natural way to obtain the differential of $f\colon M \to \mathbb{H}\mathrm{P}^1$ in the interpretation of 5.3 from the line bundle $f^*\Sigma$:

If $L \subset H = M \times \mathbb{H}^{n+1}$ is a line sub–bundle and $\pi_L\colon H \to H/L \in \Gamma(\mathrm{Hom}(H, H/L))$ is the projection we can define

$$\delta \in \Omega^1(\mathrm{Hom}(L, H/L))$$

in the following way: We consider a section $\psi \in \Gamma(L) \subset \Gamma(H)$ as a map $\psi\colon M \to \mathbb{H}^{n+1}$ such that $\psi(p) \in L_p$ for any $p \in M$. Then, for any $X \in T_pM$, we have $d\psi(X) \in H_p = \mathbb{H}^{n+1}$ and

$$\delta(X)(\psi) := \pi_L(d\psi(X)) \in (H/L)_p = \mathbb{H}^{n+1}/L_p.$$

Of course, δ is tensorial in X and because of

$$\delta(X)(\psi\lambda) = \pi_L(d\psi(X)\lambda + \psi d\lambda(X)) = \delta(X)(\psi)\lambda$$

for any $\lambda\colon M \to \mathbb{H}$ we see that δ is also tensorial in ψ, so δ is in fact an element of $\Omega^1(\mathrm{Hom}(L, H/L))$.

If $L = f^*\Sigma$, where $f\colon M \to \mathbb{H}\mathrm{P}^1$, comparison with Lemma 5.3 yields that $\delta \in \Omega(\mathrm{Hom}(L, H/L))$ can be regarded as the differential of the map f. The map f is immersed at $p \in M$ if and only if $\delta(T_pM) \subset \mathrm{Hom}(L_p, (H/L)_p)$ is a 2–dimensional real subspace.

6.1. Definition. Let $L \subset H = M \times \mathbb{H}^{n+1}$ be a line sub–bundle. The *derivative* of L is the $\mathrm{Hom}(L, H/L)$–valued 1–form

$$\delta \in \Omega^1(\mathrm{Hom}(L, H/L))$$

defined by

$$\delta(X)(\psi) = \pi_L(d\psi(X))$$

for any $X \in T_pM$ and $\psi \in \Gamma(L)$.

As a consequence of 4.2 we obtain:

6.2. Proposition. *Let $L = f^*\Sigma \subset H = M \times \mathbb{H}^2$ be an immersed oriented surface in $\mathbb{H}\mathrm{P}^1$ with derivative $\delta \in \Omega^1(\mathrm{Hom}(L, H/L))$. Then there exist unique complex structures J on L and $\tilde{J}$ on H/L such that for all $p \in M$*

$$\tilde{J}\delta(T_pM) = \delta(T_pM) = \delta(T_pM)J,$$
$$\tilde{J}\delta = \delta J$$

and J is compatible with the orientation induced by $\delta\colon T_pM \to \delta(T_pM)$.

6.3. Lemma. *Let $L \subset H = M \times H^{n+1}$ be a line sub–bundle over a Riemann surface. Then there exists a quaternionic linear $\beta\colon \mathbb{H}^{n+1} \to \mathbb{H}$ such that the restrictions to the fibers of L never vanish.*

Proof. The (smooth) vector bundle $L^\perp$ with fibers

$$L_p^\perp := \{\omega \in (\mathbb{H}^{n+1})^* \mid \omega_{|L_p} = 0\}$$

has a total space of real dimension $4n+2$, but $(\mathbb{H}^{n+1})^*$ has real dimension $4n+4$. Therefore the (smooth) map

$$\omega \in Ł^\perp \mapsto \omega \in (\mathbb{H}^{n+1})^*$$

is not onto, hence there exists a $\beta \in (\mathbb{H}^{n+1})^*$ such that for all $p \in M$:

$$\beta_{|L_p} \neq 0.$$

□

6.4. Remark. If $L = f^*\Sigma$, where $f\colon M \to \mathbb{H}\mathrm{P}^n$, this Lemma shows that it is always possible to find a basis $e_1, ..e_{n+1}$ of $\mathbb{H}^{n+1}$ with dual basis $\alpha_1, .., \alpha_{n+1}$ such that the image of f lies in the open set $U_{\alpha_{n+1}}$:

$$\mathrm{im}(f) \subset U_{\alpha_{n+1}} \cong \mathbb{H}^n$$

via the map $u_{\alpha_{n+1}}$, see 5.0.3. In the special case of maps into $\mathbb{H}\mathrm{P}^1$ one can always find a map $g\colon M \to \mathbb{H}$ such that $f = [e_1 g + e_2] \in \mathbb{H}\mathrm{P}^1$ and $\psi = e_1 g + e_2 \in \Gamma(L)$ is a nowhere vanishing section of L.

Now we are able to formulate (and prove) a condition in terms of the bundle $L := f^*\Sigma$ whether $f\colon M \to \mathbb{H}\mathrm{P}^1$ is conformal:

6.5. Proposition. *An immersion $f\colon M \to \mathbb{H}\mathrm{P}^1 \cong S^4$ of a Riemann surface M into the 4–sphere is conformal if and only if the complex structures J and $\tilde{J}$ on $L := f^*\Sigma$ and H/L given by Proposition 6.2 are compatible with the complex structure on M in the following sense:*

$$*\delta = \tilde{J}\delta = \delta J,$$

where δ is the derivative of L.

Remark. Because of 6.2 it is enough to know that one of the complex structures J or $\tilde{J}$ is compatible to the complex structure on M.

Proof. Choose a basis α, β of $(\mathbb{H}^2)^*$ with dual basis e_1, e_2 of $\mathbb{H}^2$ such that $\beta \in \Gamma(L^{-1})$ never vanishes. This is always possible by 6.3. Then $\mathrm{im}(f) \subset U_\beta$ and with $g := u_\beta \circ f\colon M \to \mathbb{H}$ we get $f = [e_1 g + e_2]\colon M \to \mathbb{H}\mathrm{P}^1$. Therefore f is conformal if and only if g is conformal.

If $g\colon M \to \mathbb{H}$ is conformal let $R\colon M \to \mathbb{H}$ be the right normal vector of g. We define the complex structure $J \in \mathrm{End}(L)$ by:

$$J\psi\lambda = -\psi R\lambda,$$

where $\psi = (e_1 g + e_2) \in \Gamma(L)$ and $\lambda\colon M \to \mathbb{H}$. Since ψ is a nowhere vanishing section and $\psi R \in \Gamma(L)$, we conclude that this defines a complex structure on L. Since

$$\begin{aligned}\delta(-\psi R) &= -\pi_L d((e_1 g + e_2)R) = -\pi_L(e_1 dgR + (e_1 g + e_2)dR)\\ &= -\pi_L(e_1 dgR) = \pi_L(e_1 * dg) = *\delta\psi\end{aligned}$$

we obtain

$$\delta J = *\delta$$

and moreover we see that $\delta(T_pM)J = \delta(T_pM)$ and that J is compatible with the orientation induced by δ. Therefore J must be the unique complex structure given by 6.2 and is compatible to the complex structure on M.

Conversely let J be a complex structure on L, then there exists a mapping $R\colon M \to \mathbb{H}$ such that $J\psi = \psi R$, where $\psi = [e_1 g + e_2]$. The same computation as above shows that R is the right normal vector of the mapping $g\colon M \to \mathbb{H}$, i.e. g and consequently f are conformal. □

Definition. A line sub–bundle $L \subset H = M \times \mathbb{H}^{n+1}$ over a Riemann surface M is called a *conformal* or *holomorphic curve* in $\mathbb{HP}^n$, if there exists a complex structure $J \in \mathrm{End}(L)$ on L such that

$$*\delta = \delta J.$$

L is called *immersed* if $\delta(T_pM) \subset \mathrm{Hom}(L_p, (H/L)_p)$ is a 2–dimensional subspace for all $p \in M$.

We will see later that a holomorphic curve $L = f^*\Sigma$ in $\mathbb{HP}^1$ is in fact a holomorphic quaternionic vector bundle, but this is a special property in the case of $n = 1$. In general L would not have a canonical holomorphic structure, but a canonical mixed structure:

6.6. Lemma. *Every holomorphic curve $L \subset H = M \times \mathbb{H}^{n+1}$ inherits a canonical mixed structure $\hat{D}$ defined by*

$$\hat{D} = \frac{1}{2}(d + *dJ)_{|L}.$$

Proof. The condition $*\delta = \delta J$ is equivalent to $\pi_L(d + *dJ)_{|L} = 0$. This is exactly the fact that $\hat{D} = \frac{1}{2}(d + *dJ)_{|L}$ maps $\Gamma(L)$ to $\Omega^1(L)$. Of course $\hat{D}$ is quaternionic linear. Moreover we have

$$*\hat{D} = \frac{1}{2}(*d - dJ)_{|L} = -\hat{D}J.$$

It remains to show that the Leibniz rule holds: For any $\psi \in \Gamma(L)$ and $\lambda\colon M \to \mathbb{H}$ we compute:

$$\begin{aligned}\hat{D}(\psi\lambda) &= \frac{1}{2}(d(\psi\lambda) + *d(J(\psi)\lambda))\\ &= \frac{1}{2}(d\psi + *d(J\psi))\lambda + \frac{1}{2}(\psi d\lambda + J(\psi) * d\lambda) = \hat{D}(\psi)\lambda + (\psi d\lambda)''.\end{aligned}$$

□

6.7. Theorem. *Let $L \subset H = M \times \mathbb{H}^{n+1}$ be a holomorphic curve with complex structure J and L^{-1} its dual bundle with induced complex structure $J = J^*$. The complex quaternionic bundle (L^{-1}, J) has a canonical holomorphic structure D characterized by the following fact:*

$$D\omega_L = 0,$$

for all $\omega \in (\mathbb{H}^{n+1})$, where $\omega_L \in \Gamma(L^{-1})$ denotes the restriction of ω to the fibers of L.

Moreover D is the dual (in the sense of Lemma 3.13) holomorphic structure on L^{-1} to the mixed structure $\hat{D}$ on L given by Lemma 6.6.

Proof. By Lemma 6.3 there exists a $\beta \in (\mathbb{H}^{n+1})^*$ such that the section $\beta_L \in \Gamma(L^{-1})$ never vanishes. Then we can define a unique holomorphic structure D on L^{-1} by

$$D\beta_L = 0.$$

First we show that this holomorphic structure D is the dual structure to $\hat{D}$. Any $\alpha \in \Gamma(L^{-1})$ is given by $\alpha = \beta_L f$ for some $f : M \to \mathbb{H}$, with this we compute for $\psi \in \Gamma(L) \subset \Gamma(H)$

$$\begin{aligned}
&\frac{1}{2}(d < \alpha, \psi > + * d < \alpha, J\psi >) = \frac{1}{2}(d < \beta f, \psi > + * d < \beta f, J\psi >) \\
&= \frac{1}{2}(d\bar{f} < \beta, \psi > + < \beta f, d\psi > + * d\bar{f} < \beta, J\psi > +* < \beta f, dJ\psi >) \\
&=< D(\beta f), \psi > + < \beta f, \hat{D}\psi >=< D\alpha, \psi > + < \alpha, \hat{D}\psi >
\end{aligned}$$

as claimed. Moreover this computation shows that $D\alpha_L = 0$ if $\alpha_L \in \Gamma(L^{-1})$ is the corresponding section to some $\alpha \in (\mathbb{H}^{n+1})^*$. □

6.8. Remark. This holomorphic structure is Moebius invariant in the following sense: If $G \in \mathrm{GL}(n+1, \mathbb{H})$ acts by $[v] \mapsto [Gv]$ on $\mathbb{HP}^n$, the line bundle $L \subset H = M \times \mathbb{H}^{n+1}$ is a conformal curve in $\mathbb{HP}^n$ if and only if $GL \subset H$ is a conformal curve in $\mathbb{HP}^n$. Then the bundles L^{-1} and $(GL)^{-1}$ with its canonical holomorphic structures are isomorphic as holomorphic quaternionic bundles by

$$\omega \in (GL)_x^{-1} \mapsto G^*\omega \in L_x^{-1}$$

where G^* is the dual operator to $G : L_x \subset \mathbb{H}^{n+1} \to GL_x \subset \mathbb{H}^{n+1}$. We will come back to this later.

6.9. The Mean Curvature Sphere. A 2–sphere in $\mathbb{HP}^1$ is a subset which corresponds to a real 2–plane in $\mathbb{H}$ under a suitable affine (or Euclidean) coordinate. An oriented 2–sphere is a 2–sphere, which is oriented as a manifold. An orientation can be given by an orientation of the above real 2–plane.

We consider the set

$$\mathcal{Z} = \{S \in \mathrm{End}(\mathbb{H}^2) | S^2 = -\mathrm{Id}\},$$

and for $S \in \mathcal{Z}$ we define

$$S' := \{l \in \mathbb{HP}^1 \, | Sl = l\}.$$

6.10. Proposition. *There is a one–to–one correspondence between oriented 2–spheres in* $\mathbb{HP}^1$ *and* $S \in \mathcal{Z}$ *given by the map*

$$S \in \mathcal{Z} \mapsto S',$$

where the orientation of S' *is given as follows: the tangent space at* $l \in S'$ *is given by*

$$\mathrm{Hom}_+(l, H^2/l) \subset \mathrm{Hom}(l, H^2/l) \cong T_l\,\mathbb{HP}^1,$$

where we consider the complex structures on l *and* $\mathbb{H}^2/l$ *given by* S. *Then post–composition by* S *on* $\mathrm{Hom}(l, \mathbb{H}^2/l)$ *is the rotation by* $\frac{\pi}{2}$ *in positive direction.*

A proof can be found in [**Boh03**].

If S is a sphere congruence, i.e. a mapping $S\colon M \to \mathcal{Z}$, we obtain a complex quaternionic vector bundle (H, S), where $H = M \times \mathbb{H}^2$. On H we have the trivial (quaternionic) connection $\nabla = d$, and we can decompose d as in 3.9.1:

$$d = d' + d'' = \partial + A + \bar\partial + Q,$$

where $A \in \Gamma(K\,\mathrm{End}_-(H))$ and $Q \in \Gamma(\bar K\,\mathrm{End}_-(H))$.

6.11. Theorem. *Let* $L \subset H = M \times \mathbb{H}^2$ *be an immersed holomorphic curve in* $\mathbb{HP}^1$. *Then there exists a unique complex structure* $S \in \mathrm{End}(H)$ *on* $H = M \times \mathbb{H}^2$ *such that*

$$\begin{aligned} SL &= L \\ *\delta &= \delta \circ S = S \circ \delta, \\ Q_{|L} &= 0 \ \ (\iff \ \mathrm{im}\, A \subset L). \end{aligned}$$

A proof is given in [**BFLPP02**].

Remark. S is a family of 2–spheres with the following properties: $SL = L$ means that the 2–sphere S'_p goes through L_p for all $p \in M$, while $*\delta = \delta \circ S = S \circ \delta$ implies that all spheres S'_p are tangent to L with the same orientation. $Q_{|L} = 0$ or equivalently $\mathrm{im}\, A \subset L$ expresses the fact that in affine coordinates the 2–spheres have the same mean curvature vector as the immersion, see [**BFLPP02**].

6.12. Definition. This unique family of 2–spheres given by 6.11 is called the *mean curvature sphere* of L.

The differential forms $A \in \Gamma(K\,\mathrm{End}_-(\mathbb{H})$ and $Q \in \Gamma(\bar K\,\mathrm{End}_-(H))$ are called the *Hopf fields* of L.

If $L \subset H$ is a holomorphic curve in $\mathbb{HP}^1$ and S is its mean curvature sphere we compute for any section $\psi \in \Gamma(L)$:

$$S \circ \pi_L(d\psi + S * d\psi) = S\delta\psi - *\delta\psi = 0,$$

where $\pi_L\colon H \to \mathbb{H}/L$. Thus any holomorphic curve L in $\mathbb{HP}^1$ has a natural holomorphic structure

$$D\colon \Gamma(L) \to \Gamma(\bar K L);\ \psi \mapsto (d\psi)'' = \frac{1}{2}(d\psi + S * d\psi),$$

where d is the trivial connection on $H = M \times \mathbb{H}^2$.

6.13. Example. If $G \in \mathrm{GL}(2,\mathbb{H})$ acts as a Moebius transformation on $\mathbb{H}\mathrm{P}^1$ and $L = f^*\Sigma \subset H = M \times \mathbb{H}^2$ is a holomorphic curve, the bundle $GL \subset H$ is the corresponding bundle to the conformal map $Gf\colon M \to \mathbb{H}\mathrm{P}^1$, i.e. $(Gf)^*\Sigma = GL$. Of course GL is also a holomorphic curve. We want to compute the mean curvature sphere and the Hopf fields of GL in terms of these on L : First we consider the derivatives δ_L and δ_{GL} of the bundles L and GL. For any 1–dimensional quaternionic subspace $U \subset \mathbb{H}^2$ we have an isomorphism

$$G\colon \mathrm{Hom}(U, \mathbb{H}^2/U) \tilde{\to} \mathrm{Hom}(GU, \mathbb{H}^2/(GU)), \quad \Phi \mapsto G \circ \Phi \circ G^{-1},$$

where we denote the mapping $v+U \in \mathbb{H}^2/U \mapsto Gv+(GU) \in \mathbb{H}^2/(GU)$ by G, too. They induce an isomorphism $G\colon \mathrm{Hom}(L, H/L) \to \mathrm{Hom}(GL, H/(GL))$ of bundles. Then δ_L and δ_{GL} correspond under this isomorphism since

$$\begin{aligned} G(\delta_L(X))(G\psi) &= G(\delta_L(X)(\psi)) = G \circ \pi_L(d\psi(X)) \\ &= \pi_{GL}(d(G\psi)(X)) = \delta_{GL}(X)(G\psi) \end{aligned}$$

for any $X \in T_pM$ and $\psi \in \Gamma(L)$, where $\pi_L\colon \mathbb{H}^2 \to \mathbb{H}^2/L$ and $\pi_{GL}\colon \mathbb{H}^2 \to \mathbb{H}^2/(GL)$ are the projections. Now it is easy to see that $S\colon M \to \mathcal{Z}$ is the mean curvature sphere of L if and only if $GSG^{-1}\colon M \to \mathcal{Z}$ is the mean curvature sphere of GL, and the Hopf fields of GL are given by GAG^{-1} and GQG^{-1}, where A and Q are the Hopf fields of L :

$$\begin{aligned} SL = L &\iff GSG^{-1}(GL) = GL \\ dSL \subset L &\iff d(GSG^{-1})(GL) \subset GL \end{aligned}$$

is obvious. Therefore we get

$$\begin{aligned} *\delta_{GL} &= G(*\delta_L) \\ &= G \circ \delta_L \circ S \circ G^{-1} = \delta_{GL} \circ GSG^{-1} \\ &= G \circ S \circ \delta_L \circ G^{-1} = GSG^{-1} \circ \delta_{GL}. \end{aligned}$$

The Hopf fields $\tilde{A}$ and $\tilde{Q}$ of GL are given by:

$$\begin{aligned} \tilde{A} &= \frac{1}{2}(d' + GSG^{-1}d'(GSG^{-1})) \\ &= \frac{1}{4}(d - GSG^{-1} * d + GSG^{-1}d(GSG^{-1}) + *d(GSG^{-1}) \\ &= G\frac{1}{4}(d - S * d + SdS + *dS)G^{-1} = GAG^{-1} \end{aligned}$$

and similar

$$\tilde{Q} = GQG^{-1}.$$

This yields

$$Q_L = 0 \iff \tilde{Q}_{GL} = 0$$

and therefore GSG^{-1} is in fact the mean curvature sphere of GL.

6.14. Example. Let $S \in \mathcal{Z}$. We consider the 2–sphere $S' \subset \mathbb{H}\mathrm{P}^1$ as a conformal immersion with corresponding line bundle L. Then the mean curvature sphere of L is the constant map $l \in S' \mapsto S \in \mathcal{Z}$. We have to check that the equations in 6.11 are satisfied: $SL = L$ holds by the definition of L, by constancy we have $dSL = \{0\} \subset L$ and moreover $Q = \frac{1}{4}(SdS - *dS) = 0$.

Assume that $L = [e_1 f + e_2]$ is a holomorphic curve in $\mathbb{HP}^1$, where $f\colon M \to \mathbb{H}$ is conformal and e_1, e_2 is a basis of $\mathbb{H}^2$. We want to compute the mean curvature sphere in Euclidean coordinates. We use the framing e_1, ψ of $\mathbb{H}^2$, where $\psi = e_1 f + e_2$. Because of $SL \subset L$ we obtain the following matrix representation relative to this frame:

$$S = \begin{pmatrix} N & 0 \\ -H & -R \end{pmatrix}. \tag{6.14.1}$$

With $S^2 = -\operatorname{Id}$ we see

$$N^2 = -1 = R^2 \quad \text{and} \quad RH = HN.$$

The functions N, R and H have the following geometric meaning: It is easy to compute

$$\begin{aligned} *\delta(\psi) &= \pi_L(e_1 * df), \\ \delta S(\psi) &= \pi_L(-e_1 df R) \\ S\delta(\psi) &= \pi_L(e_1 N df). \end{aligned}$$

Since ψ is non–vanishing and $e_1 \notin L$

$$*\delta = S\delta = \delta S$$

is equivalent to

$$*df = N df = -R df,$$

hence N and R are the left and right normal vectors of f, see 4.2.1. The condition that $Q_L = 0$, or equivalently $AH \subset L$, is equivalent to the fact that

$$H = -\bar{\mathcal{H}} N = -R\bar{\mathcal{H}},$$

where $\mathcal{H}$ is the mean curvature vector of f. This is proven in [**BFLPP02**].

6.15. Remark. This shows that the mean curvature vector of f at $p \in M$ is determined by S_p. Since S_p is also the mean curvature sphere of S'_p, both, f and S'_p have under the Euclidean coordinate given by the basis e_1, e_2 the same mean curvature vector at p.

CHAPTER III

Willmore Spheres in S^3

In this chapter we study the relationship between Willmore spheres in S^3 and minimal surfaces of genus zero with flat ends.

7. Willmore Surfaces

7.1. The Willmore functional. We have already seen in Chapter II that for a conformal immersion $f\colon M \to \mathbb{HP}^1$ there exist two associated complex quaternionic line bundles: L and L^{-1} together with their natural mixed respectively holomorphic structures. We define the Willmore functional of f in terms of these bundles:

Definition. The *Willmore functional* of an immersed holomorphic curve $L = f^*\Sigma$ in $\mathbb{HP}^1$ over a compact Riemann surface M is defined to be the Willmore energy of the holomorphic bundle L^{-1} with its natural holomorphic structure D :

$$\mathcal{W}(f) := W(L^{-1}, J^*, D).$$

We will also denote the Willmore functional by $\mathcal{W}(L)$.

The Willmore functional can also be computed in terms of the Hopf fields of the holomorphic curve in $\mathbb{HP}^1$ or in Euclidean quantities:

7.2. Lemma. *Let L be an immersed holomorphic curve in $\mathbb{HP}^1$ and $A \in \Gamma(K\,\mathrm{End}_-(H))$ its (K–part) Hopf field. Then the Willmore functional of L is given by*

$$\mathcal{W}(f) = 4\int < A \wedge *A > .$$

If $L = [e_1 g + e_2]$, where $g\colon M \to \mathbb{H}$ and e_1, e_2 is a basis of $\mathbb{H}^2$ then

$$2 < Q_D \wedge *Q_D >= 4 < A \wedge *A >= (|\mathcal{H}|^2 - K - K^\perp)\,\mathrm{dvol}_g$$

and consequently

$$\mathcal{W}(f) = \int_M (|\mathcal{H}|^2 - K - K^\perp)\,\mathrm{dvol}_g,$$

where Q_D is the Hopf field of the holomorphic bundle L^{-1}, $\mathcal{H}$ is the mean curvature vector of g, K and $K^\perp$ are the Gaussian and normal curvatures of g, and dvol_g is the volume form on M induced by the immersion g.

Proof. We will only prove the first statement, the proof of second part can be found in [**BFLPP02**]. It is enough to show that the appearing 2–forms coincide.

We compute for the K–part Hopf field A of the holomorphic curve L :

$$\begin{aligned} A_{|L} &= \frac{1}{2}(d' + Sd'S)_{|L} = \frac{1}{4}(d - S * d + SdS + *dS)_{|L} \\ &= \frac{1}{4}(d + *dS - S * (d + *dS))_{|L} = \frac{1}{4}(d + *dJ - *J(d + *dJ))_{|L} \\ &= \frac{1}{2}(\hat{D} - *J\hat{D}) = \hat{D}_-, \end{aligned}$$

where $\hat{D} = \frac{1}{2}(d + *dJ)_{|L}$ is the canonical mixed structure on L. Then by Lemma 3.13 and Theorem 6.7 we have

$$Q_D = -(A_{|L})^*.$$

Because of the properties of the (real) trace and because $\operatorname{im} A \subset L$ we then obtain locally

$$\begin{aligned} &2 < Q_D \wedge *Q_D > (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = \frac{2}{4}\operatorname{tr}_{\mathbb{R}}(-2Q_D(\frac{\partial}{\partial x}) \circ Q_D(\frac{\partial}{\partial x})) \\ &= \frac{1}{2}\operatorname{tr}_{\mathbb{R}}(-2A_{|L}(\frac{\partial}{\partial x}) \circ A_{|L}(\frac{\partial}{\partial x})) = \frac{4}{8}\operatorname{tr}_{\mathbb{R}}(-2A(\frac{\partial}{\partial x}) \circ A(\frac{\partial}{\partial x})) \\ &= 4 < A \wedge *A > (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}), \end{aligned}$$

for any conformal coordinate $z = x + iy$ on M. By definition of the Willmore energy of L^{-1} the proof is complete. □

7.3. Proposition. *The Willmore functional is Moebius invariant, i.e. for an immersed holomorphic curve $f\colon M \to \mathbb{H}\mathrm{P}^1$ and for $G \in \mathrm{GL}(2, \mathbb{H})$ we have $\mathcal{W}(f) = \mathcal{W}(Gf)$. In fact the Willmore integrand $< A \wedge *A >$ is also Moebius invariant.*

Proof. In 6.13 we have computed that the (K–part) Hopf field $\tilde{A}$ of Gf is given by GAG^{-1}, where A is the (K–part) Hopf field of f. Because of the Ad–invariance of the (real) trace we obtain

$$< GAG^{-1} \wedge *GAG^{-1} >=< GA \wedge *AG^{-1} >=< A \wedge *A > .$$

Comparison with the previous Lemma 7.2 shows that the 2–forms and the functionals coincide. □

7.4. Willmore Surfaces. Consider a (smooth) variation L_t of a holomorphic curve $L = L_0$ in $\mathbb{H}\mathrm{P}^1$ over a Riemann surface M with complex structure J on M. In general, the curve $L_t \subset H = M \times \mathbb{H}^2$ is not conformal for $t \neq 0$. But if L is an immersed holomorphic curve and t is small enough, L_t is an immersion, too, and therefore defines a complex structure J_t on M. Of course we have $J_t = J$ if and only if L_t is a holomorphic curve over the Riemann surface M.

Since any almost complex 2–dimensional manifold, i.e. a 2–manifold with complex structure, is in fact a complex manifold, we get a family M_t of Riemann surfaces (with underlying real manifold M), such that L_t is a holomorphic curve in $\mathbb{H}\mathrm{P}^1$ over M_t. Therefore the Willmore functional is well–defined for any immersion into $\mathbb{H}\mathrm{P}^1$ and we can ask if a given immersion is a critical value of the Willmore functional with respect to all variations. This gives rise for the following definition:

Definition. A *Willmore surface* is an immersed holomorphic curve $L = f^*\Sigma$ in $\mathbb{HP}^1$ over a compact Riemann surface M such that f is a critical point of the Willmore functional with respect to all variations.

If the image of the immersion f lies in $\mathbb{R}^3 \subset S^3 \subset \mathbb{HP}^1$, the normal curvature $K^\perp$ vanishes, and since $\int K\,\mathrm{dvol}_f$ is a topological invariant, one immediately finds that Willmore surfaces in the 3–space are precisely the critical values for the functional $\int H^2\,\mathrm{dvol}_f$. This is the classical setup of Blaschke and others.

Of course, one can ask for critical values of the Willmore functional with respect to variations L_t, such that every L_t is a holomorphic curve with respect to the complex structure of the Riemann surface M. Such Surfaces are called *constrained Willmore surfaces.*

The condition that a surface is Willmore is given in the following Theorem.

7.5. Theorem. *Let L be an immersed holomorphic curve in $\mathbb{HP}^1$ over a compact Riemann surface M. Then L is Willmore if and only if*

$$d * Q = 0 \quad (\iff d * A = 0),$$

where A and Q are the Hopf fields of L.

A proof is given in [**BFLPP02**]. They also have determined the Euler–Lagrange equation of the Willmore functional in terms of Euclidean quantities: If $L = [e_1 f + e_2]$ for a conformal immersion $f\colon M \to \mathbb{H}$ and a basis e_1, e_2 of $\mathbb{H}^2$, they showed that the condition that L is Willmore is given by the equation

$$d(R * dH + \frac{1}{2}H(NdN - *dN)) = 0, \tag{7.5.1}$$

where N and R are the left and right normal vectors of f and $H = -\bar{\mathcal{H}}N$ for the mean curvature vector $\mathcal{H}$ of f.

The last equation shows that every minimal surface satisfies the Willmore condition, but there are no compact minimal surfaces. We will see in the rest of this chapter, how Willmore surfaces can arise as minimal surfaces and moreover that every Willmore sphere is minimal in suitable affine coordinates.

8. Minimal Surfaces with Flat Ends

In this section we study minimal surfaces which extend to immersed compact Willmore surfaces in $\mathbb{HP}^1$.

8.1. Definition. Let M be a compact Riemann surface and $p_1, .., p_n \in M$. A minimal surface $f\colon M \setminus \{p_1, .., p_n\} \to \mathbb{R}^4 \cong \mathbb{H}$ is called (immersed) *minimal surface with flat ends* at $p_1, .., p_n$, if there exist an immersed holomorphic curve L in $\mathbb{HP}^1$ over M and a basis e_1, e_2 of $\mathbb{H}^2$ such that,

$$L_{|M\setminus\{p_1,..,p_n\}} = [e_1 f + e_2].$$

Clearly, the holomorphic curve L is unique up to Moebius transformations. Moreover all mean curvature spheres S_p of L at $p \in M$ pass through $[e_1] \in \mathbb{HP}^1$, compare with 6.15. This is a useful condition to decide whether a conformal surface L is with respect to a suitable affine coordinate a minimal surface.

8.2. Theorem. *Let L be an immersed holomorphic curve in $\mathbb{H}\mathrm{P}^1$ over a compact Riemann surface M. If there exists a point $\infty = [e_1] \in \mathbb{H}\mathrm{P}^1$ such that all mean curvature spheres S_p of L pass through ∞, i.e. $(S_p e_1)\mathbb{H} = e_1\mathbb{H}$ for all $p \in M$, then there exist a basis e_1, e_2 of $\mathbb{H}^2$, finitely many points $p_1, .., p_n \in M$ and a minimal surface with flat ends $f\colon M \setminus \{p_1, .., p_n\} \to \mathbb{H}$ such that $L_{|M\setminus\{p_1,..,p_n\}} = [e_1 f + e_2]$.*

Proof. Since L is immersed and M is compact, there are only finitely many points $p_1, .., p_n \in M$ such that $L_{p_k} = \infty$. Choose a basis α, β of $(\mathbb{H}^2)^*$ such that $\ker\beta = \infty$ and its dual basis e_1, e_2 of $\mathbb{H}^2$. Then we have $\infty = [e_1] \in \mathbb{H}\mathrm{P}^1$ and L is given in the Euclidean coordinate u_β by a conformal map $f\colon M \setminus \{p_1, .., p_n\} \to \mathbb{H}$. The mean curvature vector $\mathcal{H}$ of f at a point $p \in M$ is determined by the mean curvature sphere S_p of L at p, see Remark 6.15. Since all mean curvature spheres S_p pass through ∞, they are in the Euclidean coordinate u_β 2–planes in $\mathbb{H}$, and therefore the mean curvature vector of f vanishes, i.e. f is minimal. f has flat ends by definition. □

8.3. Lemma. *Let M be a compact Riemann surface and $p_1, .., p_n \in M$. Let $f\colon M \setminus \{p_1, .., p_n\} \to \mathbb{H}$ be an immersed minimal surface such that there exist an open set $U \subset M$ with $p_1, .., p_n \in U$ and a conformal immersion $g\colon U \to \mathbb{H}$ such that $fg = 1$ on $U \setminus \{p_1, .., p_n\}$. Then f has flat ends at $p_1, .., p_n$.*

Proof. Let e_1, e_2 be a basis of $\mathbb{H}^2$ with dual basis α, β. We define $F\colon M \to \mathbb{H}\mathrm{P}^1$ by $F(p) = e_1 f(p) + e_2$ for $p \notin \{p_1, ..p_n\}$ and $F(p_l) = e_2$. Then $u_\alpha \circ F = g$ and $u_\beta \circ F = f$, and 5.1 implies that F is conformal. Thus $L := F^*\Sigma$ is an immersed holomorphic curve in $\mathbb{H}\mathrm{P}^1$ such that $L_{|M\setminus\{p_1,..,p_n\}} = [e_1 f + e_2]$.

□

8.4. Minimal Surfaces in S^3. We are mostly interested in surfaces lying in S^3 and want to use the quaternionic approach to study them. So we have to deal with conformal maps $f\colon M \to \mathbb{H}\mathrm{P}^1$ such that the image of f lies in a real 3–plane under a suitable affine coordinate.

8.5. Lemma. *If L is given as in 8.2 and lies in a 3–sphere, then we can choose e_1, e_2 in such a way that $f\colon M \setminus \{p_1, .., p_n\} \to \mathrm{Im}(H)$ is a minimal surface with flat ends and $L_{|M\setminus\{p_1,..,p_n\}} = [e_1 f + e_2]$.*

Proof. We denote the basis and the map given by 8.2 by $\tilde{e}_1, \tilde{e}_2$ and $\tilde{f}$ and define $\tilde{M} := M \setminus \{p_1, .., p_n\}$. First we show that we can assume that $\mathrm{im}(f)$ lies in a real 3–dimensional subspace of $\mathbb{H}$. Note that f lies in an affine 3–dimensional subspace of $\mathbb{H}$, since L pass through ∞. If $q \in \tilde{M}$ then $\tilde{e}_1, (\tilde{e}_1\tilde{f}(q) + \tilde{e}_2)$ is a new basis of $\mathbb{H}^2$ and $L_{|\tilde{M}} = [\tilde{e}_1(\tilde{f} - \tilde{f}(q)) + (\tilde{e}_1\tilde{f}(q) + \tilde{e}_2)]$. Of course $\tilde{f} - \tilde{f}(q)\colon \tilde{M} \to \mathbb{H}$ is also minimal and maps into a 3–dimensional real subspaces of $\mathbb{H}$. Thus, without loss of generality, we can assume that $\mathrm{im}(\tilde{f})$ lies in a 3–dimensional real subspace $U \subset \mathbb{H}$. Let $\mu \in \mathbb{H}$ be a unit normal vector of $U \subset \mathbb{H} \cong \mathbb{R}^4$ with its standard inner product $<,>$. Then the map $\bar{\mu}\tilde{f}\colon M \to \mathbb{H}$ is also minimal and with $e_1 := \tilde{e}_1\bar{\mu}^{-1}, e_2 := \tilde{e}_2$ we obtain

$$L_{|\tilde{M}} = [e_1 f + e_2].$$

Because of

$$\mathrm{Re}(f) = \mathrm{Re}(\bar{\mu}\tilde{f}) =< \mu, \tilde{f} >= 0$$

f maps into the imaginary quaternions as claimed and has flat ends by definition. □

As we have mentioned in 4.3, in the case of $\mathrm{Im}(\mathbb{H})$–valued conformal maps f the right and the left normal of f coincide: $N = R$ and moreover N is the classical unit normal vector of f.

8.6. Lemma. *Let M be a compact Riemann surface and let $f\colon M \setminus \{p_1, .., p_n\} \to \mathrm{Im}(\mathbb{H})$ be a minimal surface with flat ends. Then there exists a continuous extension $N\colon M \to S^2 \subset \mathrm{Im}(\mathbb{H})$ of the (left) normal vector N of f.*

Proof. We show, that for any $p_l \in \{p_1, .., p_n\}$ there is a $N(p_l) \in S^2 \subset \mathrm{Im}(\mathbb{H})$, such that N becomes continuous. Let L and e_1, e_2 be as in 8.1 and U be a small neighborhood of p_l, such that $f_{|U} \neq 0$. Then $L_{|U} = [e_1 + e_2 g]$, where $g = f^{-1}\colon U \to \mathrm{Im}(\mathbb{H})$ is conformal. There is the following relation between the normal vectors N_f of f and N_g of g : $N_f = g^{-1}N_g g$, which follows directly by the definition of the left normal vector and the fact that $df = dg^{-1} = -g^{-1}dgg^{-1}$. Applying an Euclidean rotation, we can assume that $N_g(p_l) = \mathbb{k} \in \mathrm{Im}(\mathbb{H})$. Therefore and because g is conformal, we find a local holomorphic coordinate $z = x + iy$ centered at p_l, such that $g(z) = x\mathbb{i} + y\mathbb{j} + h(z)$, where h is $\mathrm{Im}(\mathbb{H})$–valued and $h \in o(|z|)$. Since N_g is continuous on U, $N_g(z) = \mathbb{k} + r(z)$, where r is $\mathrm{Im}(\mathbb{H})$–valued and $\lim_{z\to 0} r(z) = 0$. For $z \neq 0$ we obtain:

$$\begin{aligned} N_f(z) = g^{-1}(z)N_g(z)g(z) &= \frac{(-x\mathbb{i} - y\mathbb{j} - h(z))(\mathbb{k} + r(z))(x\mathbb{i} + y\mathbb{j} + h(z))}{\| g(z) \|^2} \\ &= -\frac{(x^2+y^2)\mathbb{k} + (x\mathbb{i} + y\mathbb{j})r(z)g(z) + h(z)(\mathbb{k} + r(z))g(z)}{\| g(z) \|^2}. \end{aligned}$$

Since for $z \neq 0$ $\lim_{z\to 0} \frac{x^2+y^2}{\|g(z)\|^2} = 1$, we conclude $\lim_{z\to 0} N_f(z) = -\mathbb{k}$. Thus we can extend N continuously on M. □

8.7. Remark. For further computations it turns out to be useful to rotate f such that for the extension $N\colon M \to S^2$ and every $p_l \in \{p_1, .., p_n\}$ we have

$$N(p_l) \neq \pm\mathbb{i} \in S^2 \subset \mathrm{Im}(\mathbb{H}).$$

This can be done as follows: We can find a $\mu \in \mathbb{H}$, such that for all $l = 1, .., n$: $\mu N_f(p_l)\mu^{-1} \neq \pm\mathbb{i}$. Then $L_{|\tilde{M}} = [e_1\mu^{-1}(\mu f \mu^{-1}) + e_2\mu^{-1}]$, and $\mu f \mu^{-1}$ and its normal vector have the required properties.

8.8. The Weierstrass Representation. In this part we develop the global Weierstrass representation of conformal surfaces in $\mathbb{H}$ with special attention to minimal surfaces in $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$. This generalization of the classical Weierstrass representation of minimal surface in $\mathbb{R}^3$ (or later conformal surfaces in $\mathbb{R}^3$) is due to Pedit and Pinkall, see [**PP98**].

8.9. Theorem. *Let M be a Riemann surface and $f\colon M \to \mathbb{H}$ a conformal immersion. Then there are paired holomorphic bundles KL^{-1} and L and holomorphic sections $\varphi \in \Gamma(KL^{-1})$ and $\psi \in \Gamma(L)$ such that*

$$df = (\varphi, \psi),$$

where $(,)$ denotes the pairing between KL^{-1} and L. The appearing holomorphic bundles and sections are uniquely determined by f up to isomorphisms.

The representation $df = (\varphi, \psi)$ is called the Weierstrass representation *of f.*

Proof. First we show the existence of the representation. Let e_1, e_2 be a basis of $\mathbb{H}^2$ and $\alpha, \beta \in (\mathbb{H}^2)^*$ its dual basis. Consider the line bundle $L = [e_1 f + e_2] \subset M \times \mathbb{H}^2$. Then we can regard α and β as sections $\alpha_L, \beta_L \in L^{-1}$. Since β_L is nowhere vanishing we can define a quaternionic connection ∇ on L^{-1} by

$$\nabla \beta_L = 0.$$

This connection ∇ defines holomorphic structures d^∇ and D on KL^{-1} respectively L, which are compatible with respect to the pairing $(,)$ as we have seen in 3.11. Moreover the section $\alpha_L = \beta \bar{f} \in \Gamma(L^{-1})$ is holomorphic with respect to the holomorphic structure $D = \nabla''$ on L^{-1}, since this is in fact the canonical holomorphic structure given by 6.7. Hence $\varphi := \nabla \alpha_L = \beta_L d\bar{f} \in \Gamma(KL^{-1})$ and

$$d^\nabla(\varphi) = (\nabla \beta_L) d\bar{f} + \beta_L d^2 \bar{f} = 0$$

shows that φ is holomorphic in KL^{-1}. The section $\psi = (e_1 f + e_2) \in \Gamma(L)$ is also a holomorphic section because of $\beta(\psi) = 1$ and because of the definition of D. Then we have

$$(\varphi, \psi) = (\beta_L d\bar{f}, \psi) = df$$

as claimed. Note that the sections ψ and φ are nowhere vanishing.

Let $\tilde{\varphi} \in \Gamma(K\tilde{L}^{-1})$ and $\tilde{\psi} \in \Gamma(\tilde{L})$ be holomorphic sections of paired holomorphic line bundles $K\tilde{L}^{-1}$ and $\tilde{L}$ which satisfy $df = (\tilde{\varphi}, \tilde{\psi})$. Since f is an immersion, $\tilde{\psi}$ is nowhere vanishing and therefore the mapping

$$\Phi\colon L \to \tilde{L};\ \psi\lambda \mapsto \tilde{\psi}\lambda$$

is an isomorphism of quaternionic line bundles. We have to show that it is also an isomorphism of holomorphic quaternionic bundles. Let $R\colon M \to \mathbb{H}$ be the right normal vector of f, then we have

$$(\varphi, J\psi) = {*df} = df R = (\varphi, \psi R)$$

and

$$(\tilde{\varphi}, \tilde{J}\tilde{\psi}) = {*df} = df R = (\tilde{\varphi}, \tilde{\psi} R).$$

This yields $J\psi = \psi R$ and $\tilde{J}\tilde{\psi} = \tilde{\psi} R$ and since Φ is quaternionic linear we obtain

$$\Phi(J\psi) = \Phi(\psi R) = \Phi(\psi) R = \tilde{\psi} R = \tilde{J}\tilde{\psi},$$

which implies that L and $\tilde{L}$ are isomorphic as complex quaternionic bundles via Φ. Since $\psi \in \Gamma(L)$ and $\tilde{\psi} \in \Gamma(\tilde{L})$ are holomorphic sections and nowhere vanishing, they define the holomorphic structures D and $\tilde{D}$ on the line bundles L and $\tilde{L}$ uniquely and we obtain $\tilde{D} \circ \Phi = \Phi \circ D$. This shows that L and $\tilde{L}$ are isomorphic as holomorphic quaternionic bundles and ψ and $\tilde{\psi}$ coincide with respect to this isomorphism. Similarly it can be shown that KL^{-1} and φ are also unique up to isomorphism. $\square$

We want to study the Weierstrass representation of minimal surfaces with flat ends in $\mathbb{R}^3$.

8.10. Lemma. *Let M be a compact Riemann surface, $p_1, .., p_n \in M$ and $f \colon M \setminus \{p_1, .., p_n\} \to \operatorname{Im}(\mathbb{H})$ be an immersed minimal surface with flat ends. There exist a (complex) holomorphic quaternionic line bundle L over M, which is paired with itself by $(,)$, and a holomorphic section $\psi \in \Gamma(L_{|M\setminus\{p_1,..,p_n\}})$ with $df = (\psi, \psi)$. The holomorphic bundle L and the holomorphic section ψ are unique up to isomorphism.*

Proof. Let L be as in 8.1, i.e. $L_{|\tilde{M}} = [e_1 f + e_2]$ where $\tilde{M} := M \setminus \{p_1, .., p_n\}$ and e_1, e_2 be a basis of $\mathbb{H}^2$. We first show that L and KL^{-1} inherit holomorphic structures such that the restrictions $L_{|\tilde{M}}$ and $KL^{-1}_{|\tilde{M}}$ are the holomorphic bundles given by 8.9. Let $\alpha, \beta \in (\mathbb{H}^2)^*$ be the dual basis of e_1, e_2 and ∇ be the quaternionic connection on $L^{-1}_{|\tilde{M}}$ defined by $\nabla \beta_{L_{|\tilde{M}}} = 0$ with decomposition $\nabla = \partial + A + \bar{\partial} + Q$. Since the holomorphic structure $D = \bar{\partial} + Q$ on L^{-1} is the canonical holomorphic structure on the dual bundle to the holomorphic curve L, the operator $\bar{\partial} \colon \Gamma(L^{-1}) \to \Gamma(\bar{K}L^{-1})$ is defined globally on M. The holomorphic structure on $L_{|M}$ induced by ∇ is given by $D = \bar{\partial}^* - A^*$, see 3.10. Since $\psi = (e_1 f + e_2) \in \Gamma(L_{|M})$ is a holomorphic section and f is minimal, 3.5 and 4.4.2 imply that $-A^* = 0$. Therefore $D = \bar{\partial}^* \colon \Gamma(L) \to \Gamma(\bar{K}L)$ is also defined globally on M and is indeed the double of a complex holomorphic structure of the underlying complex bundle. Since d^∇ is compatible to D with respect to $(,)$ on $\tilde{M}$, it can also be extended to $d^\nabla \colon \Gamma(KL^{-1}) \to \Gamma(\bar{K}KL^{-1})$ globally on M.

Note that $-\overline{(,)}$ is a pairing between L and KL^{-1}, and since

$$-\overline{(\varphi, \psi)} = -d\bar{f} = df = (\varphi, \psi),$$

the uniqueness statement of 8.9 yields that the holomorphic bundles $KL^{-1}_{|\tilde{M}}$ and $L_{|\tilde{M}}$ are isomorphic such that φ and ψ coincide under this isomorphism. Denote this isomorphism by $\tilde{\Phi} \colon KL^{-1}_{|\tilde{M}} \to L_{|\tilde{M}}$.

It remains to show that $\tilde{\Phi}$ can be extended to a global isomorphism

$$\Phi \colon KL^{-1} \to L$$

of holomorphic quaternionic line bundles. Let $g := f^{-1}$. Then we have

$$\beta_L d\bar{f} f^{-1} = \beta_L f f^{-1} d(-f) f^{-1} = -\beta_L (-f) f^{-1} df f^{-1} = -\alpha_L dg,$$

since $\bar{f} = -f$ and $dg = -f^{-1} df f^{-1}$. This implies that

$$\tilde{\Phi}(\alpha_L dg) = \tilde{\Phi}(-\varphi f^{-1}) = -\psi f^{-1} = -(e_1 + e_2 g).$$

Note that for the smooth extension of g by $g(p_l) = 0$ we get $\alpha_L dg(p_l) \neq 0$ since $L(p_l) = [e_1]$ and $dg(p_l) \neq 0$. Therefore we can define $\Phi(\alpha_L dg(p_l)) = -(e_1 + e_2 g)(p_l) = -e_1 \in L(p_l)$ for all $p_l \in \{p_1, .., p_n\}$ and $\Phi_{|\tilde{M}} = \tilde{\Phi}$ to obtain a smooth isomorphism $\Phi\colon KL^{-1} \to L$. Because $\tilde{\Phi}$ is an isomorphism of holomorphic bundles, continuity implies that Φ commutes with the complex and holomorphic structures on KL^{-1} and L, i.e. $J \circ \Phi = \Phi \circ J$ and $D \circ \Phi = \Phi \circ d^\nabla$, and therefore Φ is also an isomorphism of holomorphic bundles. □

As we have seen in the proof of 8.10, the bundle L inherits a holomorphic structure $D = \bar{\partial}$, i.e. L is the double of a complex holomorphic bundle S :

$$L = S \oplus S.$$

It turns out that the bundle S is a *spin bundle* of the Riemann surface M, i.e. $S \otimes S = K$ as complex holomorphic bundles, where the holomorphic structure on K is given by exterior differentiation, see [**GriHa**] for details.

8.11. Lemma. *The holomorphic bundle L given by 8.10 is the double of a spin bundle S of the compact Riemann surface M.*

Proof. The bundles KL^{-1} and L are isomorphic as complex quaternionic bundles. We have seen in 1.4 and 1.5 that the decomposition into the $\pm\mathbb{i}$–eigenspaces of KL^{-1} is given by

$$KL^{-1} = KS^{-1} \oplus KS^{-1}$$

and since $L = S \oplus S$ we obtain

$$KS^{-1} \cong S \quad (\iff K \cong S \otimes S)$$

as complex line bundles. We claim that this is an isomorphism of complex holomorphic line bundles. To see this consider the (complex) holomorphic structure $\bar{\partial}_1 := d^\nabla_{|KS^{-1}}$ induced by d^∇ and the (complex) holomorphic structure $\bar{\partial}_2$ on KS^{-1} induced by d on K and $\bar{\partial}_{|S^{-1}}$ on S^{-1} via product rule, where $\bar{\partial}$ is the complex holomorphic structure on $L^{-1} = S^{-1} \oplus S^{-1}$ defined in the proof of 8.10 by $\nabla = \partial + A + \bar{\partial} + Q = \partial + \bar{\partial} + Q$. It remains to show that $\bar{\partial}_1 = \bar{\partial}_2$. Since $\tilde{M} := M \setminus \{p_1, .., p_n\}$ is a dense open subset of M it is enough to show $\bar{\partial}_1(\omega) = \bar{\partial}_2(\omega)$ for sections $\omega \in \Gamma(\tilde{M}, KS^{-1}) \subset \Gamma(\tilde{M}, KL^{-1})$. Locally we can write $\omega = dz \otimes_{\mathbb{C}} \phi = \phi dx + J\phi dy$, where $z = x + iy$ is a holomorphic coordinate on M and $\phi \in \Gamma(S^{-1})$, to obtain:

$$\begin{aligned}
\bar{\partial}_1 \omega &= (\nabla\phi) \wedge dx + (\nabla J\phi) \wedge dy \\
&= (\partial + \bar{\partial} + Q)(\phi) \wedge dx + (\partial + \bar{\partial} + Q)(J\phi) \wedge dy \\
&= (\bar{\partial}\phi) \wedge dx + J(\bar{\partial}\phi) \wedge dy \\
&= \bar{\partial}_2(\omega).
\end{aligned}$$

□

Now we are able to prove the existence of the *spinor representation* of minimal surfaces (with flat ends), which is a coordinate–free reformulation of the classical Weierstrass representation of minimal surfaces, see [**KuSch95**].

8.12. Theorem. *Let M be a compact Riemann surface and $f\colon \tilde{M} = M\backslash\{p_1,..,p_n\} \to \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ be a minimal surface with flat ends such that $N(p_l) \neq \pm\mathbb{i}$. There exist a complex holomorphic spinor bundle S over M and two holomorphic sections $s_1, s_2 \in \Gamma(\tilde{M}, S)$ such that*

$$df = \mathbb{j}s_1^2 + \mathbb{j}s_1s_2\mathbb{j} + s_1s_2 + s_2^2\mathbb{j}, \tag{8.12.1}$$

where we consider the 1–forms s_is_j as $\mathbb{C} = \mathrm{span}(1,\mathbb{i}) \subset \mathbb{H}$ valued.

PROOF. The holomorphic section $\psi \in \Gamma(\tilde{M}, L)$ given by 8.10 satisfies $df = (\psi, \psi)$. Because of 8.11 we get two holomorphic sections s_1, s_2 in a spinor bundle S, such that $L = S \oplus S$ and $\psi = s_1 \oplus s_2$. If we consider $s_1 \oplus s_2$ as a section of the holomorphic bundle $KL^{-1} = KS^{-1} \oplus KS^{-1}$, which is isomorphic to L, equation 1.4.1 shows that

$$df = (\psi,\psi) = (s_1 \oplus s_2, s_1 \oplus s_2) = \mathbb{j}s_1^2 + \mathbb{j}s_1s_2\mathbb{j} + s_1s_2 + s_2^2\mathbb{j}.$$

□

8.13. Properties of the spinors. Next we study the analytical properties of the spinors $s_1, s_2 \in \Gamma(M \setminus \{p_1,..,p_n\}, S)$ of a minimal surface with flat ends in $\mathbb{R}^3$.

8.14. Lemma. *The spinors s_1, s_2 given by 8.12 are meromorphic sections in S with poles at the p_l's.*

PROOF. Let $N = R$ be the continuous (right=left) normal vector of f with $N(p_l) \neq \pm\mathbb{i}$. The complex structure J on L operates on $\psi = (e_1 f + e_2)$ by $J\psi = J(e_1f + e_2) = -(e_1f + e_2)N$. Therefore the decomposition of $\psi \in \Gamma(\tilde{M}, L) = \Gamma(\tilde{M}, S \oplus S)$ into $\pm\mathbb{i}$–eigenspaces of J is given by

$$\psi = \frac{1}{2}(\psi - J\psi\mathbb{i}) \oplus \frac{1}{2}(\psi + J\psi\mathbb{i}) = \psi\frac{1}{2}(1 + N\mathbb{i}) \oplus \psi\frac{1}{2}(1 - N\mathbb{i}).$$

Let $U \subset M$ be a small neighborhood of p_l such that there exists a holomorphic section $s \in \Gamma(\hat{L}_{|U}) \subset \Gamma(L_{|U}) \subset \Gamma(U \times \mathbb{H}^2)$ without zeros. We can choose U small enough such that for the section $s \in \Gamma(U \times \mathbb{H}^2)$, the standard norm $\|\ .\ \|$ of $\mathbb{H}^2 = \mathbb{R}^8$ and some constant $C > 0$ the estimation $\| s \| \leq C$ holds. Now s_1 is given by $s_1 = gs$ for a function $g\colon U \to \mathbb{C} \cup \{\infty\}$. Since s_1 and s are holomorphic on $U\backslash\{p_l\}$, g is holomorphic with singularity at p_l. We have to show that this singularity is not essential. Note that

$$\| s_1 \| = \| s(\mathrm{Re}(g)) + Js(\mathrm{Im}(g)) \| = |g|\, \| s \| \leq C|g|. \tag{8.14.1}$$

Since $\| \psi \| \longrightarrow \infty$ for $p \to p_l$ and $\| 1 + Ni \| \geq c$ in a small neighborhood of p_l for some constant $c > 0$, we see that $\| s_1 \| \longrightarrow \infty$ for $p \to p_l$. Therefore 8.14.1 yields $|g| \longrightarrow \infty$ for $p \to p_l$ and by the theorem of Casorati–Weierstrass g is meromorphic. By the definition of meromorphic sections in holomorphic (complex) line bundles, s_1 is meromorphic. By the same argument, s_2 is also a meromorphic section. □

Remark. We used the existence of a local holomorphic non–vanishing section in the complex holomorphic line bundle S with $K = S \otimes S$. If $z\colon U \subset M \to \mathbb{C}$ is a holomorphic coordinate, then a section $s = \sqrt{dz} \in \Gamma(U, S)$ with $\sqrt{dz} \otimes \sqrt{dz} = dz$ is a local holomorphic nowhere vanishing section.

8.15. Lemma. *The spinors s_1, s_2 given by 8.12 have poles of order 1 exactly at $p_1, .., p_n$.*

PROOF. Let $z = x + iy$ be a conformal coordinate centered in $p_l \in M$ and let $g = f^{-1}$. First we show

$$\frac{1}{C}\frac{1}{|z|} \leq |f(z)| \leq \frac{1}{c}\frac{1}{|z|} \tag{8.15.1}$$

for some constants $0 < c < C$ and small z. Assume that $N_f(p_l) = -\mathbb{k}$ and therefore $N_g(p_l) = \mathbb{k}$, compare with the proof of 8.6. By changing the conformal coordinate we can assume that $dg(\frac{\partial}{\partial x}_{|0}) = \mathbb{i}$. Since $N_g(p_l) = \mathbb{k}$, the differential of g is given by $dg = \mathbb{i}dx + \mathbb{j}dy$, thus we have $g(x+iy) = x\mathbb{i} + y\mathbb{j} + o_1(x+iy)$ for some $o_1(z) \in o(|z|)$. So we can find constants $0 < \tilde{c} < 1 < \tilde{C}$ such that in a small neighborhood of p_l the estimation $\tilde{c}|z| \leq |g(z)| \leq \tilde{C}|z|$ holds. Because $|ab| = |a||b|$ for any $a, b \in H$, we obtain 8.15.1 in the special case of $N(p_l) = \mathbb{k}$ and $dg(\frac{\partial}{\partial x}_{|0}) = \mathbb{i}$. Since any Euclidean rotation does not change this equation and a variation of the conformal coordinate z only changes the constants c and C, 8.15.1 holds in general.

Formula 8.12.1 and a direct computation result in

(8.15.2)

$$\begin{aligned} f(z) &= f(z_0) + \int_{z_0}^{z} df \\ &= f(z_0) + \mathrm{Re}(-2i\int_{z_0}^{z} s_1 s_2)\mathbb{i} + \mathrm{Re}(\int_{z_0}^{z}(s_1^2 + s_2^2))\mathbb{j} + \mathrm{Re}(-i\int_{z_0}^{z}(s_2^2 - s_1^2))\mathbb{k}. \end{aligned}$$

Note that there is a local holomorphic non–vanishing section $\sqrt{dz}$ in S such that $dz = \sqrt{dz}^2$. With 8.14 we obtain the following Laurent expansion of the spinors at $z = 0$:

$$s_1 = (a_n z^n + ...)\sqrt{dz} \quad \text{and} \quad s_2 = (b_m z^m + ...)\sqrt{dz},$$

where $a_n b_m \neq 0$ and $n, m \in \mathbb{Z}$. Using 8.15.2 we get:

$$\begin{aligned} f(z) = f(z_0) &+ \mathrm{Re}(-2i(a_n b_m z^{n+m+1} + ..))\mathbb{i} \\ &+ \mathrm{Re}(a_n^2 z^{2n+1} + b_m^2 z^{2m+1} + ..)\mathbb{j} \\ &+ \mathrm{Re}(-i(b_m^2 z^{2m+1} - a_n^2 z^{2n+1} + ..))\mathbb{k}. \end{aligned} \tag{8.15.3}$$

There is a $\hat{z} \neq 0$ near $z = 0$, such that $0 \neq A := -2ia_n b_m \hat{z}^{n+m+1} \in \mathbb{R}^+$. Hence we obtain for $t \in \mathbb{R}$ sufficient small

$$|f(t\hat{z})| \geq \frac{A}{2}|t|^{n+m+1} \geq \tilde{C}|t\hat{z}|^{n+m+1}$$

for some constant $\tilde{C} > 0$. Since $|f(z)| \leq \frac{1}{c}|z|^{-1}$ by 8.15.1 we get $n+m+1 \geq -1$. We already know, that the spinors have poles at p_l, i.e. $n \leq -1$ and $m \leq -1$. Thus the only possibility is $n = m = -1$ and the poles of the spinors are of order 1. □

We identify the conformal 2–sphere $S^2 \subset \mathrm{Im}(\mathbb{H})$ with $\mathbb{CP}^1$ in the usual way: we use the stereographic projection with north pole $\mathbb{i}$, to obtain

$$S^2 \cong \mathbb{C} \cup \{\mathbb{i}\} \cong \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1 .$$

Then we have the following

8.16. Proposition. *The normal vector* $N\colon M \to \mathbb{C}\mathrm{P}^1 \cong S^2 \subset \mathrm{Im}(\mathbb{H})$ *of a minimal surface* $f\colon M\backslash\{p_1,..,p_n\} \to \mathrm{Im}(\mathbb{H})$ *with flat ends is a holomorphic mapping. If* s_1, s_2 *are the meromorphic spinors of* f*, then the normal vector is given by the holomorphic mapping* $N = -is_2/s_1\colon M \to \mathbb{C}\mathrm{P}^1$ *.*

Proof. The inverse of the stereographic projection is given by

$$\Phi\colon \mathbb{C} \hookrightarrow S^2,\ (x+iy) \mapsto \frac{1}{x^2+y^2+1}((x^2+y^2-1)\mathbb{i} + 2x\mathbb{j} + 2y\mathbb{k}).$$

Let $p \in M \setminus \{p_1,..,p_n\}$. If $s_1(p) \neq 0$ we can find a holomorphic coordinate $z = x + iy$ centered in p such that $s_1(p) = \sqrt{dz}$ and $s_2(p) = (a+ib)\sqrt{dz}$. Then by 8.12.1 we get

$$df(\frac{\partial}{\partial x}_{|p}) = 2b\mathbb{i} + (1+a^2-b^2)\mathbb{j} + 2ab\mathbb{k}$$

and

$$df(\frac{\partial}{\partial y}_{|p}) = 2a\mathbb{i} - 2ab\mathbb{j} + (a^2-b^2-1)\mathbb{k}.$$

Because f maps to $\mathrm{Im}(\mathbb{H})$, the left normal vector N is the classical Euclidean normal vector and we compute

$$N(p) = \frac{1}{a^2+b^2+1}((a^2+b^2-1)\mathbb{i} + 2b\mathbb{j} - 2a\mathbb{k}) = \Phi(b-ia) = \Phi(-is_2(p)/s_1(p)).$$

Since N is continuous this equation also holds at the points $p_1,..,p_n$. At points $p \in M$ with $s_1(p) = 0$ we have $N(p) = \mathbb{i} \,\hat{=}\, \infty = s_2(p)/s_1(p)$. Since s_1 and s_2 are meromorphic sections, its quotient mapping is a holomorphic mapping into $\mathbb{C}\mathrm{P}^1$, showing that N is holomorphic. $\square$

8.17. Corollary. *Let* M *be a compact Riemann surface of genus* g*,* $f\colon M \setminus \{p_1,..,p_n\} \to \mathrm{Im}(\mathbb{H})$ *be an immersed minimal surface with flat ends,* K *its Gaussian curvature and* dvol_f *the induced volume form on* $M \setminus \{p_1,..,p_n\}$*. Then*

$$\int_{M\setminus\{p_1,..,p_n\}} K\,\mathrm{dvol}_f = -4\pi(n+g-1).$$

Proof. We can assume that $N(p_l) \neq \pm\mathbb{i}$, where $N\colon M \to \mathbb{C}\mathrm{P}^1 \cong S^2$ is the (holomorphic) normal vector of f. Since f is minimal we have $K \leq 0$. Therefore we can think of $-K\,\mathrm{dvol}_f$ as the volume form of N. Since $N\colon M \to S^2 \cong \mathbb{C}\mathrm{P}^1$ is holomorphic and M is compact, every value $z \in \mathbb{C}\mathrm{P}^1$ is taken $k \in \mathbb{N}$ times, where k is the number of sheets of the (holomorphic) covering N. It remains to show that $k = n+g-1$. By 8.15 the spinors have poles of order 1 at $p_1,..,p_n$, and therefore the divisors of s_1 and s_2 have the form

$$(s_1) = \sum_{i=1}^{I} n_i q_i - \sum_{l=1}^{n} p_l$$

and

$$(s_2) = \sum_{j=1}^{J} m_j \tilde{q}_j - \sum_{l=1}^{n} p_l,$$

where $q_i, \tilde{q}_j \in M \setminus \{p_1, .., p_n\}$ and $n_i, m_i \in \mathbb{N}$. A complex spinor bundle S over a Riemann surface of genus g has degree $g-1$, so we obtain

$$g - 1 = \deg(s_2) = \sum_{j=1}^{J} m_j - n.$$

Since f is immersed, we have $q_i \neq \tilde{q}_j$, hence s_2/s_1 has zeros of order l exactly at the points where s_2 has a zero of order l. Then the number k of zeros of $-is_2/s_1$ with multiplicity is given by

$$k = \sum_{j=1}^{J} m_j = n + g - 1$$

as claimed.

□

8.18. Lemma. *Let $f\colon M \setminus \{p_1, .., p_l\} \to \mathbb{R}^3$ be a minimal surface with flat ends at the $p_l \in M$. For every p_l, there exist a neighborhood $U_l \subset M$ and two parallel affine 2–planes in $\mathbb{R}^3$, such that f restricted to U_l lies between these planes.*

Furthermore, f is asymptotic to a suitable affine 2–plane at each end.

Proof. First we show that there is a 2–plane such that the distance of f in a neighborhood of p_l to the plane is bounded. Let $g = f^{-1}$. As in the previous Lemma, we can assume that for a conformal coordinate z centered in p_l, g is locally given by $g(x+iy) = x\mathbb{i} + y\mathbb{j} + o_1(z)\mathbb{i} + o_2(z)\mathbb{j} + o_3(z)\mathbb{k}$, where $o_1, o_2, o_3 \in o(|z|)$ are $\mathbb{R}$–valued. Note that o_3 is in fact an element of $O(|z|^2)$ since f is smooth. Hence locally f is given by

$$\begin{aligned} f(z) &= \overline{g(z)}(g(z)\overline{g(z)})^{-1} \\ &= \frac{-(x\mathbb{i} + y\mathbb{j} + o_1(z)\mathbb{i} + o_2(z)\mathbb{j}) - o_3(z)\mathbb{k}}{(x + o_1(z))^2 + (y + o_2(z))^2 + (o_3(z))^2}. \end{aligned}$$

Thus the (directed) distance between $f(z)$ and the $\mathbb{i} - \mathbb{j}$–plane is given by $d(z) = \frac{o_3(z)}{h(z)}$, where $h(z) = (x+o_1(z))^2 + (y+o_2(z))^2 + (o_3(z))^2$. In a sufficient small neighborhood of 0, the estimation $h(z) \geq \frac{1}{2}|z|^2$ holds. Because $o_3 \in O(|z|^2)$ we obtain that the distance satisfies $|d(z)| \leq C$ in a neighborhood of $z = 0$ for some constant $C > 0$.

To see that f is asymptotic to an affine 2–plane it remains to show that the (directed) distance $d(z)$ extends continuously at $z = 0$. But $d(z)$ is the $\mathbb{k}$–coordinate function of f, and since f is minimal it must be harmonic. Because $d(z)$ is bounded near $z = 0$ there must be a continuous extension at $z = 0$. □

Remark. This lemma is the reason why these surfaces are called minimal surfaces with flat or planar ends. Of course the first part of the previous lemma is also true for conformal surfaces with flat ends.

Let M be Riemann surface. Consider a holomorphic function $h = u + iv\colon M \to \mathbb{C}$. Because of the Cauchy–Riemann equations we compute using a holomorphic coordinate $z = x + iy$:

$$2\partial u = (du - i * du) = (u_x - iu_y)(dx + idy) = dh.$$

This and 8.15.2 yield

(8.18.1) $$2\partial f = e_1 \otimes (-2is_1s_2) + e_2 \otimes (s_1^2 + s_2^2) + e_3 \otimes (-i(s_2^2 - s_1^2)),$$

where e_1, e_2, e_3 is the standard basis of $\mathbb{C}^3$. Since f has no periods on M, we know that the residues of the 1–forms $-2is_1s_2$, $s_1^2 + s_2^2$ and $-i(s_2^2 - s_1^2)$ at p_l lie in $i\mathbb{R}$. The next Lemma shows, that the periods vanish.

8.19. Lemma. *Let $f\colon M \setminus \{p_1, .., p_n\} \to \mathrm{Im}(\mathbb{H})$ be a minimal surface with flat ends such that $N(p_l) \neq \pm\mathbb{i}$, and let s_1, s_2 be its spinors given by 8.12. Then the Abelian differentials s_is_j $(i, j \in \{1, 2\})$ on M have no residues. This is equivalent to the fact that the $\mathbb{C}^3$–valued meromorphic 1–form ∂f has no residues.*

Proof. We work in a local holomorphic coordinate z centered at p_l. Then we can write locally $s_1(z) = (a_{-1}z^{-1} + a_0 + a_1z + \dots)\sqrt{dz}$ and $s_2(z) = (b_{-1}z^{-1} + b_0 + b_1z + \dots)\sqrt{dz}$ with $a_{-1}b_{-1} \neq 0$. By formula 8.15.3 the $\mathbb{k}$–component of f is locally given by

$$c + \mathrm{Re}(-i((b_{-1}^2 - a_{-1}^2)z^{-1} + (b_{-1}b_0 - a_{-1}a_0)\ln(z) + \dots)).$$

First assume that $N(p_l) = \mathbb{k}$. Thus by lemma 8.18 the $\mathbb{k}$–component of f is bounded in a neighborhood of p_l. If $b_{-1}^2 - a_{-1}^2 \neq 0$ or $b_{-1}b_0 - a_{-1}a_0 \neq 0$, then the $\mathbb{k}$–component could not be bounded, so we get $b_{-1}^2 = a_{-1}^2$ and $a_{-1}a_0 = b_{-1}b_0$. Using this and the fact that the residues at p_l of $s_1^2 + s_2^2$ and $-2is_1s_2$ are imaginary, we compute

$$0 = \mathrm{Re}(res_{p_l}(s_1^2 + s_2^2)) = \mathrm{Re}(2a_{-1}a_0)$$
$$0 = \mathrm{Re}(res_{p_l}(-2is_1s_2)) = \mathrm{Re}(-4ia_{-1}a_0) = 2\,\mathrm{Im}(2a_{-1}a_0).$$

We obtain $a_{-1}a_0 = 0$. Since $a_{-1}b_{-1} \neq 0$ we get $a_0 = b_0 = 0$. With this and 8.18.1 we see that $res_{p_l}(\partial f) = (0, 0, 0)$.

The general case follows easily: Rotation of the minimal surface f changes $res_{p_l}(\partial f)$ according to this rotation. Hence we have also in the case of $N(p_l) \neq \mathbb{k}$, that $res_{p_l}(\partial f) = (0, 0, 0)$. Using this, it is a straightforward computation that $a_0 = b_0 = 0$ and therefore s_1^2, s_2^2 and s_1s_2 have no residues at p_l.

□

9. Willmore Spheres as Minimal Surfaces

In this section we work out the relationship between minimal surfaces with flat ends and immersed Willmore surfaces, especially in the case of genus zero surfaces.

9.1. Proposition. *Let M be a compact Riemann surface and $p_1, .., p_n \in M$. Let $f\colon M \setminus \{p_1, .., p_n\} \to \mathbb{R}^4 \cong \mathbb{H}$ be an immersed minimal surface with flat ends at $p_1, .., p_n$ and L be an immersed holomorphic curve as in 8.1, i.e. $L_{|M\setminus\{p_1,..,p_n\}} = [e_1f + e_2]$ for a basis e_1, e_2 of $\mathbb{H}^2$. Then L is a compact Willmore surface.*

Proof. The Euler–Lagrange equation $d{*}Q = 0$ of the Willmore functional is on $M \setminus \{p_1, .., p_n\}$ equivalent to

$$d(R * dH + \frac{1}{2}H(NdN - *dN)) = 0,$$

where N and R are the normal vectors of f, $\mathcal{H}$ is the mean curvature vector of f and $H = -\bar{\mathcal{H}}N$. Since f is minimal, the second equation holds on $M \setminus \{p_1, .., p_n\}$ and because of continuity $d * Q = 0$ is satisfied globally on M. □

In the case of compact genus zero surfaces in the conformal 3–space S^3, the converse is also true. This was first proven by Bryant [**Br84**]. A generalization of this for Willmore spheres in S^4 is due to Montiel [**Mo00**]. A proof, which is based on the quaternionic theory, can be found in [**BFLPP02**].

9.2. Theorem. *Every immersed Willmore Sphere in S^3 is a minimal surface under a suitable affine coordinate.*

9.3. Remark. The proof given in [**BFLPP02**] shows the existence of a point $\infty \in \mathbb{HP}^1$, such that all mean curve spheres S_p pass through ∞. Because of 8.2 we can use the results of the last section.

9.4. Lemma. *Let L be an immersed Willmore sphere in $S^3 \subset \mathbb{HP}^1$. Then the point $\infty \in \mathbb{HP}^1$ given by 9.3 is unique or L is the totally umbilic 2–sphere, i.e. an affine 2–plane under suitable affine coordinates.*

Proof. Let $\infty \neq P \in \mathbb{HP}^1$ be two points such that all mean curvature spheres S_p of L pass through ∞ and P. Let f, $e_1, e_2 \in \mathbb{H}^2$ with $[e_1] = \infty$ be as in 8.2, such that f lies in $\mathrm{Im}(\mathbb{H})$. By translation we can assume that $P = 0 \in \mathrm{Im}(\mathbb{H})$ under the Euclidean coordinate defined by the basis e_1, e_2. Thus also f^{-1} must be minimal. Using 6.14.1 and the fact that f is minimal, i.e. $H = 0$, and lies in $\mathrm{Im}(\mathbb{H})$, i.e. $N = R$, we obtain the following matrix representation of S relative to the frame $e_2, (e_1 + e_2 f^{-1})$:

$$S = \begin{pmatrix} f^{-1}Nf & 0 \\ -Nf - fN & -f^{-1}Nf \end{pmatrix}.$$

Thus f^{-1} is minimal if and only if

$$-Nf - fN = 0 \iff NfN = f \iff <N, f>= 0,$$

see 4.1 and the remark below. But the only minimal surface f which satisfies $<N, f>= 0$ is a 2–plane. □

Definition. A meromorphic map $F\colon M \to \mathbb{C}^3$ has null tangents if $(dF, dF) = 0$, where $(,) = dz_1 \otimes dz_1 + dz_2 \otimes dz_2 + dz_3 \otimes dz_3$ is the standard complex inner product on $\mathbb{C}^3$.

9.5. Lemma. *Let $f\colon \mathbb{CP}^1 \setminus \{p_1, .., p_n\} \to \mathbb{R}^3$ be an immersed minimal surface with flat ends at $p_1, .., p_n$. Then there exists a meromorphic map $F\colon \mathbb{CP}^1 \to \mathbb{C}^3$ with simple poles at $p_1, .., p_n$ such that $\mathrm{Re}(F) = f$. Moreover F is an immersion with null tangents.*

Proof. Let $z\colon \mathbb{CP}^1 \setminus \{p_l\} \to \mathbb{C}$ be a holomorphic chart. Let γ be a closed curve in $\mathbb{C}$ such that γ meets no poles of ∂f. Since ∂f has no residues $\int_\gamma \partial f = 0$, thus ∂f is exact on $\mathbb{CP}^1 \setminus \{p_1, .., p_n\}$. Let F be a solution of $dF = 2\partial f$. Because $df = 2\,\mathrm{Re}(\partial f) = \mathrm{Re}(dF) = d\,\mathrm{Re}(F)$, we can assume that $f = \mathrm{Re}(F)$. By Lemma 8.19, ∂f has poles of order 2 exactly at the

p_l's. Therefore F has poles of order 1 at the p_l's. Since f is an immersion, F must also be an immersion and has null tangents because

$$(dF, dF) = 4(\partial f, \partial f) = 4((-2is_1s_2)^2 + (s_1^2 + s_2^2)^2 + (-i(s_2^2 - s_1^2))^2) = 0.$$

□

The converse is in fact true for arbitrary (compact) Riemann surfaces:

9.6. Lemma. *Given a meromorphic immersion* $F\colon M \to \mathbb{C}^3$ *with poles of order 1 at* $p_1, .., p_n$ *and with null tangents. Then* $f = \mathrm{Re}(F)$ *is an immersed minimal surface with flat ends at the* p_l*'s.*

Proof. First we show that f is minimal. In a local holomorphic chart $z = x + iy$ we compute

$$0 = (dF, dF) = 4(\partial f, \partial f) = (< f_x, f_x > - < f_y, f_y > -2i < f_x, f_y >)dz \otimes dz,$$

where $<, >$ denotes the standard inner product of $\mathbb{R}^3$. Therefore we obtain $< f_x, f_x >=< f_y, f_y >$ and $< f_x, f_y >= 0$. Locally, the differential of F is given by $dF = (f_x - if_y)dz$. Since F is immersed and $||f_x|| = ||f_y||$ we conclude $||f_x|| = ||f_y|| \neq 0$. Hence f is an immersion and conformal, compare with 4.3. Since $f = \mathrm{Re}(F)$, f is also harmonic. This implies that f is minimal.

It remains to show that f has flat ends. This is a local property. We work in a local holomorphic coordinate $z = x+iy$ centered at $p_l \in \{p_1, .., p_n\}$. Let $F(z) = z^{-1}v_{-1} + v_0 + zv_1 + ..$ be the Laurent expansion of F at p_l. First we assume that $v_0 = 0$ and $v_{-1} = (1, i, 0)$. This yields

$$f(z) = \mathrm{Re}(F(z)) = (\frac{x}{x^2 + y^2} + O_1(z))\mathbb{i} + (\frac{-y}{x^2 + y^2} + O_2(z))\mathbb{j} + O_3(z)\mathbb{k},$$

where $z = x + iy \neq 0$, $O_1, O_2, O_3 \in O(|z|)$ and where we identify $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$. Now we can compute f^{-1} locally

$$\begin{aligned} f^{-1}(z) &= \frac{\overline{f(z)}}{f(z)\overline{f(z)}} \\ &= \frac{(\frac{-x}{x^2+y^2} - O_1(z))\mathbb{i} + (\frac{y}{x^2+y^2} - O_2(z))\mathbb{j} - O_3(z)\mathbb{k}}{(\frac{x}{x^2+y^2} + O_1(z))^2 + (\frac{-y}{x^2+y^2} + O_2(z))^2 + (O_3(z))^2} \\ &= \frac{(-x - O_1(z)(x^2 + y^2))\mathbb{i} + (y - O_2(z)(x^2 + y^2))\mathbb{j} + O_3(z)(x^2 + y^2)\mathbb{k}}{1 + 2xO_1(z) - 2yO_2(z) + ((O_1(z))^2 + (O_2(z))^2 + (O_3(z))^2)(x^2 + y^2)}. \end{aligned}$$

This formula shows that f^{-1} extents continuously to $z = 0$. We have seen in the proof of 8.6 that f^{-1} is conformal on the open set $\{p \in M \mid p \neq p_1, .., p_n;\ f(p) \neq 0\}$. Moreover f^{-1} is differentiable at p_l because f^{-1} is the quotient of differentiable functions, where the denominator is, in a neighborhood of $z = 0$, non–vanishing. Since $O_1, O_2, O_3 \in O(|z|)$, we compute

$$\frac{\partial f^{-1}}{\partial x}(0) = -\mathbb{i}, \quad \frac{\partial f^{-1}}{\partial y}(0) = \mathbb{j}.$$

Therefore $df^{-1}(0) = -\mathbb{i}dx + \mathbb{j}dy$ and $*df^{-1} = -\mathbb{k}df^{-1}(0)$. This shows us that $N_{f^{-1}}(p_l) = -\mathbb{k}$. Because f^{-1} is $\mathrm{Im}(\mathbb{H})$–valued, $N_{f^{-1}}$ is the classical unit normal vector, which is differentiable. This yields that f^{-1} is conformal in a neighborhood of p_l with normal vector $N_{f^{-1}}$. By 8.3 f has a flat end at p_l.

If $v_0 \neq 0$ and $v_{-1} \neq (1, i, 0)$ we argue as follows: First we consider the meromorphic map with null tangents $\tilde{F} := F - v_0$. Then the bundles $[e_1 \operatorname{Re}(F) + e_2]$ and $[e_1 \operatorname{Re}(\tilde{F}) + e_2]$ on $M \setminus \{p_1, .., p_n\}$ differ only by a Moebius transformation on $\mathbb{HP}^1$. Thus $\operatorname{Re}(F)$ has flat ends if and only if $\operatorname{Re}(F - v_0)$ has flat ends. Therefore we can assume $v_0 = 0$ without loss of generality. Since locally

$$0 = (dF, dF) = (z^{-4}(v_{-1}, v_{-1}) + z^{-2}(..) + ..)dz \otimes dz,$$

we conclude $(v_{-1}, v_{-1}) = 0$. Decompose $v_{-1} = \vec{a} + i\vec{b} \in \mathbb{R}^3 + i\mathbb{R}^3$. Then $(v_{-1}, v_{-1}) = 0$ translates to $< \vec{a}, \vec{b} >= 0$, $||\vec{a}|| = ||\vec{b}|| = \lambda \in \mathbb{R} \setminus \{0\}$. Therefore we can find an $A \in SO(3, \mathbb{R}) \subset SO(3, \mathbb{C})$, such that $A\vec{a} = (\lambda, 0, 0)$, $A\vec{b} = (0, \lambda, 0)$ or equivalently $Av_{-1} = (\lambda, i\lambda, 0)$. For any $v \in \mathbb{C}^3$ and $A \in SO(3, \mathbb{R})$ we have $\operatorname{Re}(\frac{1}{\lambda}Av) = \frac{1}{\lambda}A\operatorname{Re}(v)$. But again, the bundles $[e_1 f + e_2]$ and $[e_1 \operatorname{Re}(\frac{1}{\lambda}A(F)) + e_2]$ differ only by a Moebius transformation on $\mathbb{HP}^1$ and the general case follows. □

The following theorem allows us to use powerful methods of algebraic geometry. We will come back to this in the next chapter.

9.7. Theorem. *There is a one–to–one correspondence between immersed Willmore spheres L in S^3 with $\mathcal{W}(L) = 4\pi(n-1)$ and immersed minimal surfaces $f\colon \mathbb{CP}^1 \setminus \{p_1, .., p_n\} \to \mathbb{R}^3$ with flat ends at $p_1, .., p_n$.*

There exists an immersed minimal surface $f\colon \mathbb{CP}^1 \setminus \{p_1, .., p_n\} \to \operatorname{Im}(\mathbb{H})$ with flat ends at $p_1, .., p_n$ if and only if there exists a meromorphic map $F\colon \mathbb{CP}^1 \setminus \{p_1, .., p_n\} \to \mathbb{C}^3$ with null tangents and with simple poles at $p_1, .., p_n$ such that $f = \operatorname{Re}(F)$.

The image of F lies in an affine complex 2–plane if and only if f is a flat plane.

Proof. The second statement follows directly by 9.5 and 9.6. Since the real part of an affine complex 2–plane in $\mathbb{C}^3$ is an affine real 2–plane, F lies in a complex 2–plane if and only f lies in a real 2–plane. But the only minimal surface f with flat ends which lies in a 2–plane is the 2–plane itself, which has exactly one flat end.

In view of 9.1 and 9.2 we already know that there is a correspondence between immersed Willmore spheres and immersed minimal surface of genus zero with flat ends. By 9.4 this correspondence is one–to–one up to Moebius transformations on S^3 and similarity transformations on $\mathbb{R}^3$. It remains to show that the Willmore functional of the corresponding immersed Willmore sphere L of a minimal surface f with n flat ends is $\mathcal{W}(L) = 4\pi(n-1)$. Since L lies in S^3 and f is minimal, 7.2 and 7.3 imply $\mathcal{W}(L) = -\int_{M \setminus \{p_1, .., p_n\}} K \operatorname{dvol}_f$, where K is the Gaussian curvature of f and dvol_f is the volume form on $M \setminus \{p_1, .., p_n\}$ induced by f. The genus of $\mathbb{CP}^1$ is 0 and with 8.17

$$\mathcal{W}(L) = -\int_{M \setminus \{p_1, .., p_n\}} K \operatorname{dvol}_f = 4\pi(n-1),$$

as claimed. □

CHAPTER IV

Classification

In this chapter we show that the spaces $\mathcal{M}_n$ of immersed Willmore spheres L with Willmore functional $\mathcal{W}(L) = 4\pi(n-1)$ are empty for $n \in \{2,3,5,7\}$. Moreover we give examples of Willmore spheres $L \in \mathcal{M}_n$ for $n \notin \{2,3,5,7\}$ and we determine the space $\mathcal{M}_4$.

10. Curves in Q^3 and $\mathbb{C}\mathrm{P}^3$

We study special algebraic curves in the spaces Q^3 and $\mathbb{C}\mathrm{P}^3$. The results presented in this section are useful to classify Willmore spheres. We follow the way of Bryant in [**Br88**].

10.1. Null Curves in Q^3. We define Q^3 as follows: Let W be a complex vector space of dimension 5 and $<,>$ be a non–degenerate bilinear form on W with corresponding quadratic form q.

Definition. The complex 3–quadric $Q^3 \subset PW$ is defined as the space of q null lines in W.

Q^3 is a compact complex manifold of dimension 3. By the theorem of Sylvester we can choose for every null vector $\xi \in W$, i.e. $<\xi,\xi>= 0$, a basis $e_0, .., e_4$ with $[e_0] = [\xi]$ such that

(10.1.1) $$<,>= -e_0^* \otimes e_4^* - e_4^* \otimes e_0^* + e_1^* \otimes e_1^* + e_2^* \otimes e_2^* + e_3^* \otimes e_3^*.$$

Then we have $\xi^\perp = \mathrm{span}\{e_0, e_1, e_2, e_3\}$ and

$$\begin{aligned} Q^3 \cap P\xi^\perp =& \{[(1, z_1, z_2, z_3, 0)] \in PW \mid z_1^2 + z_2^2 + z_3^2 = 0\} \\ &\cup \{[(0, z_1, z_2, z_3, 0)] \in PW \mid z_1^2 + z_2^2 + z_3^2 = 0\}. \end{aligned}$$

Hence

$$Q^3 \backslash P\xi^\perp = \{[(\frac{1}{2}(z_1^2 + z_2^2 + z_3^2), z_1, z_2, z_3, 1)] \mid z_1, z_2, z_3 \in \mathbb{C}\},$$

where we use the short notation $(z_0, z_1, z_2, z_3, z_4) = z_0 e_0 + z_1 e_1 + z_2 e_2 + z_3 e_3 + z_4 e_4$. Thus we obtain a (holomorphic) diffeomorphism

(10.1.2) $$\pi_p \colon Q^3 \backslash P\xi^\perp \to \mathbb{C}^3$$

with inverse

$$\iota_p \colon \mathbb{C}^3 \to Q^3 \backslash P\xi^\perp; (z_1, z_2, z_3) \mapsto [(\frac{1}{2}(z_1^2 + z_2^2 + z_3^2), z_1, z_2, z_3, 1)].$$

Definition. The maps $\pi_{[\xi]} \colon Q^3 \setminus P\xi^\perp \to \mathbb{C}^3$ are called *stereographic projections* at $p = [\xi] \in Q^3$ (defined by the basis $e_0, .., e_4$).

They depend on the choice of $e_0 \in p \subset W$, e_4 and on $e_1, .., e_3$. We consider $\mathbb{C}^3$ with its standard complex inner product

$$(,) = dz_1 \otimes dz_1 + dz_2 \otimes dz_2 + dz_3 \otimes dz_3.$$

If we define

$$\varphi_p \colon \mathbb{C}^3 \to W;\ (z_1, z_2, z_3) \mapsto (\frac{1}{2}(z_1^2 + z_2^2 + z_3^2), z_1, z_2, z_3, 1),$$

then we have for any $v, w \in \mathbb{C}^3$ and for any $p \in Q^3$:

$$(v, w) = 0 \iff < d\varphi_p(v), d\varphi_p(w) > = 0. \tag{10.1.3}$$

Moreover we have

10.2. Lemma. *The stereographic projections at a point $p \in Q^3$ are uniquely determined up to multiplication by non–zero complex numbers, the action of* $\mathrm{SO}(3,\mathbb{C})$ *on $\mathbb{C}^3$ and translation by $v \in \mathbb{C}^3$.*

PROOF. Let $p \in Q^3$. We fix a basis $e_0, .., e_4$ of W such that $e_0 \in p \subset W$ and $<,>= -e_0^* \otimes e_4^* - e_4^* \otimes e_0^* + e_1^* \otimes e_1^* + e_2^* \otimes e_2^* + e_3^* \otimes e_3^*$, and the corresponding stereographic projection π_p. Let $\tilde{e}_0, .., \tilde{e}_4$ be another basis of W with the same properties, i.e. $\tilde{e}_0 \in p$ and $<,>= -\tilde{e}_0^* \otimes \tilde{e}_4^* - \tilde{e}_4^* \otimes \tilde{e}_0^* + \tilde{e}_1^* \otimes \tilde{e}_1^* + \tilde{e}_2^* \otimes \tilde{e}_2^* + \tilde{e}_3^* \otimes \tilde{e}_3^*$. This and $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \in e_0^\perp = \mathrm{span}(e_0, .., e_3)$ imply that there exist complex numbers $a \neq 0, v_1, v_2, v_3$ and $A \in \mathrm{SO}(3,\mathbb{C})$ such that

$$\begin{aligned}
\tilde{e}_0 &= ae_0 \\
\tilde{e}_1 &= Ae_1 + v_1(ae_0) \\
\tilde{e}_2 &= Ae_2 + v_2(ae_0) \\
\tilde{e}_3 &= Ae_3 + v_3(ae_0) \\
\tilde{e}_4 &= \frac{1}{a}e_4 + \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)(ae_0) + v_1 Ae_1 + v_2 Ae_2 + v_3 Ae_3.
\end{aligned} \tag{10.2.1}$$

Then for the stereographic projection $\tilde{\pi}_p$ at p defined by the basis $\tilde{e}_0, ..\tilde{e}_4$ we obtain

$$\tilde{\pi}_p = \frac{1}{a} A^{-1} \circ \pi_p - (v_1, v_2, v_3).$$

□

Definition. Let M be a Riemann surface. A curve $\gamma \colon M \to Q^3 \subset PW$ is called null curve, if there exist for any $p \in M$ an open set U_p and a holomorphic lift $\hat{\gamma} \colon U_p \to W$ of γ, i.e. $[\hat{\gamma}] = \gamma_{|U_p}$, such that $< \hat{\gamma}', \hat{\gamma}' >= 0$.

A curve $\gamma \colon M \to Q^3 \subset PW$ is called non–planar if it does not lie in a 2–plane.

10.3. Remark. If γ is a null curve in Q^3, then we obtain $< \hat{\gamma}', \hat{\gamma}' >= 0$ for any lift $\hat{\gamma}$ of γ to W.

A curve $\gamma \colon M \to Q^3$ is non–planar if and only if

$$\hat{\gamma} \wedge \hat{\gamma}' \wedge \hat{\gamma}'' \wedge \hat{\gamma}''' \neq 0$$

almost everywhere for local holomorphic lifts $\hat{\gamma}$ of γ.

The following lemma is fundamental for our study of Willmore spheres as algebraic curves.

10.4. Lemma. *Let M be a compact Riemann surface and $F\colon M\setminus \mathrm{supp}(D)\to\mathbb{C}^3$ be a meromorphic null immersion with simple poles along a divisor D. Then for any $[\xi]=p\in Q^3$,*

$$\gamma := \iota_p\circ F\colon M\to Q^3$$

is a holomorphic null immersion of degree $d=\deg D$. Conversely given a holomorphic null immersion $\gamma\colon M\to Q^3$, then there is a null vector $\xi\in W\setminus\{0\}$ such that γ meets $P\xi^\perp\cap Q^3$ transversely in a divisor D. Then

$$F:=\pi_{[\xi]}\circ\gamma\colon M\setminus\mathrm{supp}(D)\to\mathbb{C}^3$$

is a meromorphic null immersion with simple poles along D.

Proof. It is clear from definition and 10.1.3 that besides the points $p_l\in\mathrm{supp}(D)$ the curve γ is an immersed null curve in Q^3. We have to show that γ extends at p_l as an immersion. Let z be a local conformal coordinate centered at $p_l\in\mathrm{supp}(D)$ and F be locally given by $F(z)=z^{-1}v_{-1}+v_0+zv_1+..$, where $v_i\in\mathbb{C}^3$. Since F is a null immersion we have $(v_{-1},v_{-1})=0$. Then

$$\hat\gamma(z):=z\varphi_p\circ F(z)=((v_{-1},v_0)+..,zF_1(z),zF_2(z),zF_3(z),z)$$

does not vanish at $z=0$ since F has a pole at p_l. Moreover $\hat\gamma(0)$ and $\hat\gamma'(0)$ are linearly independent. This implies that if we define $\gamma(p_l):=\hat\gamma(0)$, γ becomes an immersion into Q^3. Smoothness implies that γ is a null curve. It intersects the hyperplane $Pe_0^\perp$ transversal precisely in the points $p_l\in\mathrm{supp}(D)$, i.e. the degree d of γ is $d=\deg D$.

Conversely, let γ be given and $\hat\gamma(z)$ be a local holomorphic lift to W, where z is a conformal coordinate centered in $p_l\in\mathrm{supp}(D)$. We will see in Lemma 10.16 that there always exists a $\xi\in W\setminus\{0\}$ such that γ meets $P\xi^\perp$ transversely. Let $e_0,..,e_4$ be a basis of W with $[e_0]=[\xi]$, such that $<,>$ is given as in 10.1.1. Since γ meets $P\xi^\perp$ transversely we can assume that $\hat\gamma(z)=(a(z),b(z),c(z),d(z),z)$. Because γ is null, we see that $(b(0),c(0),d(0))\neq 0$. For $z\neq 0$ we have $zF(z)=(b(z),c(z),d(z))$, so we conclude F has a pole of order 1 at p_l. Moreover equation 10.1.3 implies that F has null tangents. ☐

10.5. Lemma. *The null map $F\colon M\setminus\mathrm{supp}(D)\to\mathbb{C}^3$ lies in a complex 2–dimensional affine subspace $A\subset\mathbb{C}^3$ if and only if the null curve $\gamma=\iota_{[\xi]}\circ F\colon M\to Q^3$ is planar, i.e. $\hat\gamma\wedge\hat\gamma'\wedge\hat\gamma''\wedge\hat\gamma'''=0$ for local homomorphic lifts $\hat\gamma$.*

Proof. If $F=(F_1,F_2,F_3)$ lies in the complex 2–dimensional affine space A, we obtain $F'\wedge F''\wedge F'''=0$. There exist $x,y\in\mathbb{C}$ such that $xF'+yF''=F'''$. A (local) holomorphic lift is given by $\hat\gamma(z):=\frac12(F,F)e_0+F_1e_1+F_2e_2+f_3e_3+e_4$. Then $\hat\gamma'=(F',F)e_0+F_1'e_1+F_2'e_2+F_3'e_3$, $\hat\gamma''=(F'',F)e_0+F_1''e_1+F_2''e_2+F_3''e_3$ and $\hat\gamma'''=(F''',F)e_0+F_1'''e_1+F_2'''e_2+F_3'''e_3=x\hat\gamma'+y\hat\gamma''$ are linearly dependent and therefore $\hat\gamma\wedge\hat\gamma'\wedge\hat\gamma''\wedge\hat\gamma'''=0$.

Conversely, let $\gamma=\iota_{[\xi]}\circ F$ be a null curve, and assume that F does not lie in any complex 2–dimensional affine subspaces of $\mathbb{C}^3$ and therefore $F'\wedge F''\wedge F'''\neq 0$. This implies that $\hat\gamma'\wedge\hat\gamma''\wedge\hat\gamma'''\neq 0$, where $\hat\gamma=\varphi_{[\xi]}\circ F=\frac12(F,F)e_0+F_1e_1+F_2e_2+F_3e_3+e_4$. Since $\hat\gamma',\ \hat\gamma'',\ \hat\gamma'''\in\mathrm{span}(e_0,e_1,e_2,e_3)$ we obtain $\hat\gamma\wedge\hat\gamma'\wedge\hat\gamma''\wedge\hat\gamma'''\neq 0$ as required. ☐

10.6. The Action of $\mathrm{SL}(2,\mathbb{C})$ **on** Q^3**.** Consider the 5–dimensional complex vector space of homogeneous polynomials of degree 4 on $\mathbb{C}^2$:

$$W := \{P \in \mathbb{C}[X,Y] \mid P \in \mathrm{span}(X^4, X^3Y, X^2Y^2, XY^3, Y^4)\}.$$

Let e_1, e_2 be the standard basis of $\mathbb{C}^2$. We use the short notations $(x,y) := xe_1 + ye_2 \in \mathbb{C}^2$ and $[x,y] := [xe_1 + ye_2] \in \mathbb{C}\mathrm{P}^1$. It is well–known that for every non–zero homogeneous polynomial P on $\mathbb{C}^2$ of degree n there exist, unique up to permutation, points $p_i = [x_i, y_i] \in \mathbb{C}\mathrm{P}^1$, $i = 1,..n$, and a number $a \in \mathbb{C} \setminus \{0\}$ such that

$$P = a \prod_{i=1}^{n} (X - t_i Y),$$

where $t_i := x_i/y_i$ and $(X - \infty Y) := Y$. If we write a homogeneous polynomial in the form $P = a \prod (X - t_i Y)^{n_i}$ such that $t_i \neq t_j$, a point $p_i = [x_i, y_i] \in \mathbb{C}\mathrm{P}^1$ with $t_i = x_i/y_i$ is called *zero of order* n_i of P. A zero $p = [x,y]$ is characterized by the fact that for every $(x,y) \in p \subset \mathbb{C}^2$ we have $P(x,y) = 0$. Given n points $p_1, .., p_n \in \mathbb{C}\mathrm{P}^1$ they determine, up to multiple, a homogeneous polynomial by $P = a \prod (X - t_i Y)$. Thus we can think of PW as the space of unordered 4–tuples $(p_1, .., p_4)$ in $\mathbb{C}\mathrm{P}^1$.

10.7. Example. The *rational normal curve* γ_4 in PW is given by $p \in \mathbb{C}\mathrm{P}^1 \mapsto (p,p,p,p)$, i.e. every point $p \in \mathbb{C}\mathrm{P}^1$ is mapped to the class of polynomials in W whose only zero (4–times) is p. Then a holomorphic lift $\hat{\gamma}(t)$ is given in terms of the Euclidean coordinate $t = x/y$: $\mathbb{C}\mathrm{P}^1 \setminus \{\infty\} \to \mathbb{C}$ by

$$t \mapsto (X - tY)^4 = X^4 - 4tX^3Y + 6t^2X^2Y^2 - 4t^3XY^3 + t^4Y^4 \in W.$$

Of course γ_4 is non–planar and of degree 4. Moreover γ_4 has no ramification points, i.e. it is an immersion. We want to find a non–degenerate bilinear form on W such that γ_4 is a null curve.

10.8. Lemma. *There is, unique up to multiplication, a non–degenerate bilinear form* $<,>$ *on* W*, such that* γ_4 *is a null curve in* $Q^3 \subset PW$*.*

Proof. We have to show that the values of $<,>$ on the basis X^4, X^3Y, X^2Y^2, XY^3, Y^4 are determined uniquely (up to the same factor) by the null conditions. If γ_4 is a null curve in $Q^3 \subset PW$ we have the following set of equations:

$$\begin{aligned}
0 \equiv\, &< \hat{\gamma}_4(t), \hat{\gamma}_4(t) > = < X^4, X^4 > -8t < X^4, X^3Y > \\
&+ t^2(12 < X^4, X^2Y^2 > +16 < X^3Y, X^3Y >) \\
&+ t^3(-8 < X^4, XY^3 > -48 < X^3Y, X^2Y^2 >) \\
&+ t^4(2 < X^4, Y^4 > +32 < X^3Y, XY^3 > +36 < X^2Y^2, X^2Y^2 >) \\
&+ t^5(-8 < X^3Y, Y^4 > -48 < X^2Y^2, XY^3 >) \\
&+ t^6(12 < X^2Y^2, Y^4 > +16 < XY^3, XY^3 >) \\
&+ t^7(-8 < XY^3, Y^4 >) + t^8 < Y^4, Y^4 >
\end{aligned}$$

and

$$\begin{aligned} 0 \equiv &< \hat{\gamma}_4'(t), \hat{\gamma}_4'(t) >= 16(< X^3Y, X^3Y > -6t < X^3Y, X^2Y^2 > \\ &+ t^2(6 < X^3Y, XY^3 > +9 < X^2Y^2, X^2Y^2 >) \\ &+ t^3(-2 < X^3Y, Y^4 > -18 < X^2Y^2, XY^3 >) \\ &+ t^4(6 < X^2Y^2, Y^4 > +9 < XY^3, XY^3 >) \\ &+ t^5(-6 < XY^3, Y^4 >) + t^6 < Y^4, Y^4 > . \end{aligned}$$

These equations directly imply

$$\begin{aligned} &0 =< X^4, X^4 >=< X^4, X^3Y >=< X^4, X^2Y^2 >=< X^4, XY^3 > \\ &0 =< X^3Y, X^3Y >=< X^3Y, X^2Y^2 >=< X^3Y, Y^4 > \\ &0 =< X^2Y^2, XY^3 >=< X^2Y^2, Y^4 > \\ &0 =< XY^3, XY^3 >=< XY^3, Y^4 >=< Y^4, Y^4 > \end{aligned}$$

and

$$\begin{aligned} &0 = 6 < X^3Y, XY^3 > +9 < X^2Y^2, X^2Y^2 > \\ &0 = 2 < X^4, Y^4 > +32 < X^3Y, XY^3 > +36 < X^2Y^2, X^2Y^2 > . \end{aligned}$$

Thus, in the dual basis $\alpha_0, ..\alpha_4$ of $X^4, .., Y^4$, the corresponding quadric form q of $<,>$ must be (up to multiplication by non–zero complex numbers) given by

$$q = 12\alpha_0\alpha_4 - 3\alpha_1\alpha_3 + \alpha_2\alpha_2. \tag{10.8.1}$$

Of course the bilinear form $<,>$ is non–degenerate. □

An element $A \in \mathrm{SL}(2,\mathbb{C})$ acts on PW as follows: If $P \in W \setminus \{0\}$ has the zeros $p_1, .., p_4$, then $A[P] \in PW$ is given by the zeros $Ap_1, .., Ap_n$, where $SL(2,\mathbb{C})$ acts on $\mathbb{C}\mathrm{P}^1$ in the usual way.

10.9. Proposition. *Consider the space W with its non–degenerate bilinear form $<,>$ given by the quadric form 10.8.1. There exists a lie group monomorphism*

$$\rho\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(W) \subset \mathrm{GL}(W)$$

such that the induced action of $\mathrm{SL}(2,\mathbb{C})$ *on PW is as described above.*

Proof. First we define a homomorphism $\rho\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(W)$ by $\rho(A)(P) := P \circ A^{-1} \in W$. Since $\rho(A \circ B)(P) = \rho(A) \circ \rho(B)(P)$, ρ is in fact a homomorphism. It is obvious that ρ is smooth and injective. Moreover $p \in \mathbb{C}\mathrm{P}^1$ is a zero of $P \in W$ if and only if $Ap \in \mathbb{C}\mathrm{P}^1$ is a zero of $\rho(A)(P)$. Since the set of polynomials $P \in W$ with 4 distinct zeros is dense in W, the induced action of $\mathrm{SL}(2,\mathbb{C})$ on PW is as described above.

It remains to show that ρ is $\mathrm{SO}(W)$–valued, where we consider W with the non–degenerate bilinear form $<,>$ given by the quadric form q of 10.8.1. First note that γ_4 is a null curve in $Q^3(<,>) \subset PW$, where $Q^3(<,>)$ is the quadric induced by $<,>$ on W, if and only if $A^{-1}\gamma_4$ is a null curve in $Q^3(\rho(A)^* <,>) \subset PW$ for $A \in \mathrm{SL}(2,\mathbb{C})$. But γ_4 is invariant under the action of $\mathrm{SL}(2,\mathbb{C})$, thus 10.8 implies that there exists a $\mu_A \in \mathbb{C} \setminus \{0\}$ such that $\rho(A)^* <,>= \mu_A <,>$. We have $q(X^2Y^2) = 1$. Moreover we compute for a set of generators of $\mathrm{SL}(2,\mathbb{C})$:

$$q(\rho(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})(X^2Y^2)) = q(X^2Y^2) = 1$$
$$q(\rho(\begin{pmatrix} \sqrt{a} & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix})(X^2Y^2)) = q(X^2Y^2) = 1$$
$$q(\rho(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix})(X^2Y^2)) = q(b^2Y^4 - 2bXY^3 + X^2Y^2) = 1,$$

where $a, b \in \mathbb{C}$, $a \neq 0$ are arbitrary. Since ρ is a group homomorphism we obtain $\mu_A = 1$ for any $A \in \mathrm{SL}(2,\mathbb{C})$, which shows that $\rho(A) \in \mathrm{SO}(W) \subset \mathrm{GL}(W)$. □

10.10. The Klein Correspondence. Let V be a complex vector space of dimension 4 and let $\Omega \in \Lambda^2(V^*)$ be non–degenerate. It is easy to see that there is a basis $e_1, .., e_4$ of V with dual basis $\alpha_1, .., \alpha_4$ of V^*, such that $\Omega = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$. Therefore $\Omega \wedge \Omega = 2\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4 \neq 0$. The 5–dimensional subspace $W\colon = \{\omega \in \Lambda^2(V) \mid \Omega(\omega) = 0\}$ has a canonical non–degenerate bilinear form given by $< \omega_1, \omega_2 >:= \frac{1}{2}\Omega \wedge \Omega(\omega_1 \wedge \omega_2)$ with corresponding quadric form q. Thus we can think of Q^3 as the space of q–null lines in W.

Definition. A 2–plane $A \subset V$ is called *Ω–null* if $\Omega(v_1 \wedge v_2) = 0$ for a basis v_1, v_2 of A.

Contact lines in PV are the lines of the form PA, where $A \subset V$ is a Ω–null 2–plane.

Null lines $L \subset Q^3 \subset PW$ are lines of the form $L = PU$, where U is a 2–planes in W consisting entirely of q–null 2–vectors.

10.11. Lemma. *There is a one–to–one correspondence between points of Q^3, i.e. q–null lines in W, and contact lines in PV, i.e. Ω–null 2–planes in V.*

Proof. Let PA be a contact line in PV, where A is a Ω–null 2–plane. Let v_1, v_2 be a basis of A. Then $\Omega(v_1 \wedge v_2) = 0$ and $[A] := [v_1 \wedge v_2] \in PW$ is independent of the choice of v_1, v_2. Moreover

$$< v_1 \wedge v_2, v_1 \wedge v_2 >= \frac{1}{2}\Omega \wedge \Omega(v_1 \wedge v_2 \wedge v_1 \wedge v_2) = 0$$

and therefore $[A]$ lies in Q^3 by definition of Q^3.

The converse is also true: Let $p = [\zeta] \in Q^3$ for a non–zero q–null $\zeta \in W \subset \Lambda^2(V)$, i.e. $\Omega(\zeta) = 0$ and $< \zeta, \zeta >= \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4(\zeta \wedge \zeta) = 0$. Since $\alpha_1, .., \alpha_4$ is a basis of V^*, $\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$ is a nonzero element of the 1–dimensional space $\Lambda^4(V^*) \cong (\Lambda^4(V))^*$ and therefore $\zeta \wedge \zeta = 0$. Since V is 4–dimensional there exist suitable $w_1, .., w_4 \in V$ such that $\zeta = w_1 \wedge w_2 + w_3 \wedge w_4$. Then $\zeta \wedge \zeta = 0$ implies that $w_1, .., w_4$ are linearly dependent and we see that ζ must be of the form $\zeta = v_1 \wedge v_2$ for suitable $v_1, v_2 \in V$. Moreover $\zeta \in W$ implies that $\Omega(v_1 \wedge v_2) = 0$ and this does mean that the plane $A \subset V$ spanned by v_1 and v_2 is a Ω–null 2–plane, so we obtain a contact line PA, which does not depend on the choice of ζ and v_1, v_2. □

10.12. Lemma. *There is a one–to–one correspondence between points in PV and null lines in Q^3.*

Proof. Let $[\xi] = p \in PV$. Then the space $\xi \wedge V \subset \Lambda^2(V)$ is a complex 3–dimensional subspace. Because Ω is non–degenerate, $\xi \wedge V$ does not lie in W and therefore $\hat{L}_\xi := (\xi \wedge V) \cap W$ has complex dimension 2. Any element in $(\xi \wedge V) \cap W$ is of the form $\xi \wedge v$ for suitable $v \in V$, such that $\Omega(\xi \wedge v) = 0$. By definition $L_\xi := P((\xi \wedge V) \cap W)$ is a null line.

Conversely, let $L \subset Q^3$ be a null line. Thus L is the projectivation of a 2–plane $\hat{L} \subset W$ which entirely consists of q–null 2–vectors. As we have seen in the proof of 10.11 this means that all these 2–vectors are decomposable. Let $a_1 \wedge a_2, b_1 \wedge b_2$ be a basis of $\hat{L}$ for suitable $a_1, a_2, b_1, b_2 \in V$. Then $a_1 \wedge a_2 + b_1 \wedge b_2 \in \hat{L}$ is also q–null. This implies $a_1 \wedge a_2 \wedge b_1 \wedge b_2 = 0$, and therefore a_1, a_2, b_1, b_2 are linearly dependent. Thus there exists a non–zero $\xi \in \operatorname{span}\{a_1, a_2\} \cap \operatorname{span}\{b_1, b_2\}$ and we obtain $a_1 \wedge a_2, b_1 \wedge b_2 \in (\xi \wedge V) \cap W$. Since $a_1 \wedge a_2, b_1 \wedge b_2$ is a basis of $\hat{L}$ and $\hat{L}_\xi$ is also 2–dimensional, we conclude $L = L_\xi$. □

10.13. Associated Curves. A detailed discussion of the geometry of curves in projective spaces and associated curves can be found in [**GriHa**].

Let M be a Riemann surface. If $v\colon M \to \mathbb{C}^{n+1}$ is a non–zero holomorphic map, the zeros of v are isolated and we can define a map $f\colon M \to \mathbb{C}P^n$ as follows: if $v(p) \neq 0$ then $f(p) := [v(p)] \in \mathbb{C}P^n$, if v has a zero of order k at p and z is a conformal coordinate centered in p then $f(z) := [z^{-k}v(z)]$. Then f is a well–defined holomorphic curve in $\mathbb{C}P^n$.

Now assume that $f\colon M \to \mathbb{C}P^n$ is a holomorphic non–degenerate curve and $v = v(z)$ is a local holomorphic lift of f. Then for $k \leq n$

$$v(z) \wedge v'(z) \wedge ... \wedge v^{(k)}(z)$$

is a holomorphic mapping with isolated zeros. Besides these zeros we define for $k < n$ a holomorphic curve locally by

$$z \mapsto [v(z) \wedge v'(z) \wedge ... \wedge v^{(k)}(z)] \in P(\Lambda^{k+1}\mathbb{C}^{n+1})$$

and at the zeros as described above. It turns out that this is independent of the local coordinate z and of the choice of the lift v, and therefore we obtain a globally defined holomorphic curve on M, the *kth associated curve* of f :

$$f_k\colon M \to P(\Lambda^{k+1}\mathbb{C}^{n+1}).$$

Clearly, the 0th associated curve f_0 is f itself.

The *normal form* of a non–degenerate curve $f\colon M \to \mathbb{C}P^n$ at $p \in M$ is given as follows: Let z be a conformal coordinate centered at p, and $v = v(z)$ be a local holomorphic lift of f, then we find a basis $e_0, .., e_n$ of $\mathbb{C}^{n+1}$ such that v is given by:

(10.13.1)

$$v(z) = (1 + ..)e_0 + (z^{1+\alpha_1} + ..)e_1 + (z^{2+\alpha_1+\alpha_2} + ..)e_2 + .. + (z^{n+\alpha_1+..+\alpha_n} + ..)e_n$$

for suitable $\alpha_1, .., \alpha_n \in \mathbb{N}$. To see this define $e_0 := v(0)$ and e_i recursively as follows: Since f is non–degenerate, there is for $i \leq n$ a smallest integer $l \in \mathbb{N}$ such that $v^{(l)}(0) = \frac{\partial^l v}{\partial z^l}(0) \notin \operatorname{span}(e_0, .., e_{i-1})$. Then $e_i := v^{(l)}(0)$.

The numbers α_i are independent of the choice of the coordinate z and the lift v. The *ramification index* of f at p is defined as $\beta(p) := \alpha_1$. It is zero if and only if f is immersed at p. It turns out that the ramification index $\beta_k(p)$ of the kth associated curve f_k at p is given by α_{k+1}. The set $\{p \in M \mid \beta(p) \neq 0\}$ is discrete and therefore we can define for compact Riemann surfaces M the *total ramification index* $\beta = \beta_0$ of f as $\Sigma_p \beta(p)$. Clearly, the total ramification index is zero if and only if the curve is immersed.

The extrinsic data of a non–degenerate curve f and its associated curves are related by the *Plücker formulas*:

10.14. Theorem. *Let $f\colon M \to \mathbb{C}\mathrm{P}^n$ be a non–degenerate holomorphic curve over a compact Riemann surface M of genus g. For $k = 0, .., n$ let d_k and β_k be the degree respectively the total ramification index of the kth associated curve f_k. Then we have*

$$d_{k-1} - 2d_k + d_{k+1} = 2g - 2 - \beta_k$$

for $k = 0, .., n-1$, where $d_{-1} = d_n = 0$.

A proof is given in [**GriHa**].

10.15. Tangent Lines and Tangent Surfaces. Let U be a complex vector space. We define the *tangent line* $T_p(f)$ to a holomorphic (non–linear) curve $f\colon M \to PU$ at a point $p \in M$ as follows: Every non–zero decomposable 2–vector $v \wedge w \in \Lambda^2(U)$ defines the line $P\,\mathrm{span}(v, w) \subset PU$. This does not depend on the choice of $v \wedge w \in [v \wedge w]$. Then the tangent line to the curve f at p is given by $f_1(p)$, where $f_1\colon M \to P\Lambda^2(U)$ is the first associated curve of f. The *tangent surface* of a curve $f\colon M \to PU$ is the space $\cup_{p\in M} T_p(f)$.

Example. If $\gamma\colon M \to Q^3 \subset PW$ is a non–linear null curve, then the tangent line to γ at any point $p \in M$ is a null line. To see this let z be a conformal coordinate centered in $p \in M$ and $\hat\gamma = \hat\gamma(z)$ be a holomorphic lift. Assume that p is no branch point of γ, i.e. $(\hat\gamma \wedge \hat\gamma')(0) \neq 0$. Then the tangent line to $\gamma(M)$ at p is given by $P\,\mathrm{span}(\hat\gamma(0), \hat\gamma'(0))$. Since

$$< \hat\gamma(0), \hat\gamma(0) >=< \hat\gamma(0), \hat\gamma'(0) >=< \hat\gamma'(0), \hat\gamma'(0) >= 0,$$

the plane consist entirely of q–null vectors, i.e. the tangent line at p is a null line. The set of branch points is discrete and therefore continuity implies that the tangent line at these points is also a null line.

10.16. Lemma. *Let $\gamma\colon M \to Q^3 \subset PW$ be a non–linear immersed null curve. Then γ meets the hyperplane $P\xi^\perp$ transversely for a q–null $\xi \in W \setminus \{0\}$ if and only if $[\xi]$ does not lie on the tangent surface of γ. Consequently there exists a nonzero $\xi \in W$ with $q(\xi) = 0$ such that γ and $P\xi^\perp$ intersect transversely.*

PROOF. Let z be a conformal coordinate centered in p and $\hat\gamma = \hat\gamma(z)$ be a local holomorphic lift of γ. Then γ meets $P\xi^\perp$ in p if and only if $\hat\gamma(0) \in \xi^\perp$. In this case they intersect transversely in p if and only if $\hat\gamma'(0) \notin \xi^\perp$. First assume that γ meets $P\xi^\perp$ not transversely. Then we can choose p, z and $\hat\gamma(z)$ as above and we obtain $\hat\gamma(0), \hat\gamma'(0) \in \xi^\perp$. Since γ is a null curve we get

$$< \hat\gamma(0), \hat\gamma'(0) >=< \hat\gamma(0), \hat\gamma(0) >=< \hat\gamma'(0), \hat\gamma'(0) >= 0,$$

thus $\hat{\gamma}(0), \hat{\gamma}'(0), \xi \in \hat{\gamma}(0)^\perp \cap \hat{\gamma}'(0)^\perp \cap \xi^\perp$. But $\hat{\gamma}(0)$ and $\hat{\gamma}'(0)$ are linearly independent because γ is immersed. Using Linear Algebra we then obtain $\xi \in \operatorname{span}(\hat{\gamma}(0), \hat{\gamma}'(0))$, which does exactly mean that $[\xi]$ lies on the tangent line to γ at p, i.e. $[\xi]$ is an element of the tangent surface of γ.

Conversely, let $[\xi]$ lie on the tangent surface of γ for a q–null $\xi \in W \setminus \{0\}$. Then there exist a $p \in M$, a conformal coordinate z centered in p and a holomorphic lift $\hat{\gamma}(z)$ such that $\xi \in \operatorname{span}(\hat{\gamma}(0), \hat{\gamma}'(0))$. Since γ is a null curve this implies that $< \hat{\gamma}(0), \xi >=< \hat{\gamma}'(0), \xi >= 0$. Thus $\gamma(p) \in P\xi^\perp$, and γ and $P\xi^\perp$ does not intersect transversely at p.

The tangent surface of γ is (at its smooth points) complex 2–dimensional. Therefore there exists a point $p \in Q^3$ which does not lie on the tangent surface. Then for non–zero $\xi \in W$ with $[\xi] = p$, the hyperplane $P\xi^\perp$ and the curve γ intersect transversely. □

10.17. Contact Curves in $\mathbb{C}P^3$. Let V and $\Omega \in \Lambda^2(V^*)$ be given as in 10.10, i.e. V is a complex 4–dimensional vector space and Ω is non-degenerate.

Definition. A *contact* or *complex curve* $\lambda \colon M \to PV$ is a holomorphic curve such that the tangent line to λ at every point $p \in M$ is a contact line.

The *Lie transform* $\gamma_\lambda \colon M \to Q^3$ of a contact curve γ is defined as its first associated curve.

If $\hat{\lambda}(z) \colon U \subset M \to V$ is a local holomorphic lift of a curve λ, where $z \colon U \to \mathbb{C}$ is a conformal coordinate on M, then λ is a contact curve (on U) if and only if $\Omega(\hat{\lambda}(z) \wedge \hat{\lambda}'(z)) = 0$.

By definition of a contact curve and 10.11, the first associated curve of a contact curve is indeed a curve in $Q^3 \subset PW \subset P(\Lambda^2(V))$.

10.18. Theorem. *Let $\lambda \colon M \to PV$ be a non–degenerate contact curve, then its Lie transform $\gamma_\lambda \colon M \to Q^3$ is a non–planar null curve.*

If $\gamma \colon M \to Q^3$ is an immersed non–planar null curve, then $\gamma = \gamma_\lambda$ for a (unique) non–degenerate contact curve $\lambda \colon M \to PV$.

Proof. Let $\lambda \colon M \to PV$ be a non–degenerate contact curve and z be a conformal coordinate centered in $p \in M$ such that $\hat{\lambda} = \hat{\lambda}(z)$ is a local holomorphic lift. First assume that p is no ramification point of λ. Then locally the Lie transform is given by $\gamma_\lambda = [\hat{\lambda} \wedge \frac{d\hat{\lambda}}{dz}] = [\hat{\lambda} \wedge \hat{\lambda}']$. A local lift of γ_λ is $\hat{\lambda} \wedge \hat{\lambda}'$ and its derivative is $\hat{\lambda} \wedge \hat{\lambda}''$. Since $\hat{\lambda} \wedge \hat{\lambda}''$ is decomposable, it is q–null, so by definition γ_λ is a null curve besides the (isolated) ramification points of λ, but smoothness implies that also at these points the null condition is satisfied.

Conversely let $\gamma \colon M \to Q^3 \subset PW$ be an immersed non–planar null curve, then the tangent line to $\gamma(M)$ is a null line at every point $p \in M$ and therefore the correspondence of 10.12 defines a curve $\lambda \colon M \to PV$. First we show that λ is holomorphic. Let $p \in M$ and z be a local holomorphic coordinate centered in p. Since γ is immersed, we obtain $(\hat{\gamma} \wedge \hat{\gamma}')(z) \neq 0$ for a local holomorphic lift $\hat{\gamma} = \hat{\gamma}(z)$ and every z. Then there is a basis $e_0, .., e_3$ of V such that $\hat{\gamma}(0) = e_0 \wedge e_1$ and $\hat{\gamma}'(0) = e_0 \wedge e_2$. The following holomorphic conditions

$$\begin{aligned}
&\hat\lambda(z) = e_0 + a(z)e_1 + b(z)e_2 + c(z)e_3\\
&\hat\lambda(z) \wedge \hat\gamma(z) = 0\\
&\hat\lambda(z) \wedge \hat\gamma'(z) = 0
\end{aligned}$$

determine a unique holomorphic map $\hat\lambda(z)\colon U \to V$ at least locally, which is a local lift of γ, and therefore γ is holomorphic. Next we show that $\gamma = \gamma_\lambda$. If λ would be constant, we could choose a lift $\hat\lambda(z) = \lambda_0$. Then $\lambda_0 \wedge \hat\gamma(z) = 0$ for all z implies that $\hat\gamma(z)$ would lie in the 2–dimensional subspace $\hat L_{\lambda_0}$, compare with the proof of 10.12, which is a contradiction to our assumption that γ is non–planar. Therefore $\hat\lambda(z) \wedge \hat\lambda'(z) \neq 0$ at all but a discrete set of values of z. Differentiating the equation $\hat\lambda(z) \wedge \hat\gamma(z) = 0$, we obtain $\hat\lambda'(z) \wedge \hat\gamma(z) = 0$. Therefore there exists a holomorphic function $f(z)$ such that

$$\hat\lambda(z) \wedge \hat\lambda'(z) = f(z)\hat\gamma(z)$$

showing that $\gamma = \gamma_\lambda$.

To see that $\gamma = \gamma_\lambda$ is non–planar if and only if λ is non–degenerate we compute the derivatives of the lift $\hat\gamma = \hat\lambda\wedge\hat\lambda' : \hat\gamma' = \hat\lambda\wedge\hat\lambda''$, $\hat\gamma'' = \hat\lambda'\wedge\hat\lambda''+\hat\lambda\wedge\hat\lambda'''$ and $\hat\gamma'''' = 2\hat\lambda' \wedge \hat\lambda''' + \hat\lambda \wedge \hat\lambda''''$. Then we obtain

$$\begin{aligned}
&\hat\gamma \wedge \hat\gamma' \wedge \hat\gamma'' \wedge \hat\gamma'''\\
&= (\hat\lambda \wedge \hat\lambda') \bigwedge (\hat\lambda \wedge \hat\lambda'') \bigwedge (\hat\lambda' \wedge \hat\lambda'' + \hat\lambda \wedge \hat\lambda''') \bigwedge (2\hat\lambda' \wedge \hat\lambda''' + \hat\lambda \wedge \hat\lambda'''')\\
&= (\hat\lambda \wedge \hat\lambda') \bigwedge (\hat\lambda \wedge \hat\lambda'') \bigwedge (\hat\lambda' \wedge \hat\lambda'') \bigwedge (2\hat\lambda' \wedge \hat\lambda''')\\
&+ (\hat\lambda \wedge \hat\lambda') \bigwedge (\hat\lambda \wedge \hat\lambda'') \bigwedge (\hat\lambda \wedge \hat\lambda''') \bigwedge (2\hat\lambda' \wedge \hat\lambda''')\\
&+ (\hat\lambda \wedge \hat\lambda') \bigwedge (\hat\lambda \wedge \hat\lambda'') \bigwedge (\hat\lambda' \wedge \hat\lambda'') \bigwedge (\hat\lambda \wedge \hat\lambda''''),
\end{aligned}$$

where $\bigwedge$ denotes the wedge–product of the vector space $\Lambda^2(V)$. Now it is easy to see that the last sum vanishes if and only if

$$\hat\lambda \wedge \hat\lambda' \wedge \hat\lambda'' \wedge \hat\lambda''' = 0.$$

Comparison with 10.3 shows that γ_λ is non–planar if and only if λ is non–degenerate.

□

10.19. Lemma. *Let $\lambda\colon M \to PV$ be a non–degenerate contact curve. Then there is a projective isomorphism $\Phi\colon P\Lambda^3(V) \to PV$ such that the second associated curve $\lambda_2\colon M \to P\Lambda^3(V) \cong PV$ and λ coincide under this isomorphism.*

Proof. Let $e_1, .., e_4$ be a basis of V and $\alpha_1, ..\alpha_4$ its dual basis, such that $\Omega = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$ for the non–degenerate $\Omega \in \bigwedge^2(V^*)$. We define the map

$$\hat\Phi\colon \Lambda^3(V) \to V; \eta \mapsto x,$$

where x is determined by $\Omega(v \wedge x)e_1 \wedge e_2 \wedge e_3 \wedge e_4 = v \wedge \eta$ for all $v \in V$. This is well–defined because Ω is non–degenerate, and $\hat\Phi$ induces an isomorphism $\Phi\colon P\Lambda^3(V) \to PV$. In fact Φ does only depend on Ω and not on the choice of the basis $e_1, .., e_4$. Moreover λ and its second associated curve λ_2 coincide under this identification. To see this let $p \in M$ and $z\colon U_p \to \mathbb{C}$ be a local holomorphic coordinate centered in p and let $\hat\lambda = \hat\lambda(z)\colon U_p \to V$ be a local holomorphic lift of λ. Then a local lift of λ_2 is given by $\hat\lambda \wedge \hat\lambda' \wedge \hat\lambda''$ besides a discrete set of values. Thus

$$\lambda(z) = \Phi(\lambda_2(z))$$

if and only if for all $v \in V$

$$v \wedge \hat\lambda(z) \wedge \hat\lambda(z)' \wedge \hat\lambda(z)'' = 0 \Longleftrightarrow \Omega(v \wedge \hat\lambda(z)) = 0. \tag{10.19.1}$$

The contact condition $\Omega(\hat\lambda \wedge \hat\lambda') = 0$ implies $\Omega(\hat\lambda \wedge \hat\lambda'') = 0$. Therefore besides the points where $\hat\lambda \wedge \hat\lambda' \wedge \hat\lambda'' = 0$ the equivalence 10.19.1 is obviously true. By continuity λ and λ_2 are the equal. □

11. Moduli Spaces

In this section we want to study the moduli spaces $\mathcal{M}_n$ of Willmore spheres $f\colon \mathbb{CP}^1 \to S^3 \subset \mathbb{HP}^1$ with Willmore functional $4\pi(n-1)$. We use the methods of the last section.

11.1. Lemma. *Let $\lambda\colon M \to \mathbb{CP}^3$ be a non–degenerate contact curve over a compact Riemann surface M of genus g and $\gamma\colon M \to Q^3$ its Lie transform. The degrees d_λ and d_γ and the total ramification indices β_λ and β_γ of the curves are related by*

$$\begin{aligned} -2d_\lambda + d_\gamma &= 2g - 2 - \beta_\lambda \\ 2d_\lambda - 2d_\gamma &= 2g - 2 - \beta_\gamma. \end{aligned} \tag{11.1.1}$$

Consequently, there is a one–to–one correspondence between non–degenerate contact curves $\lambda\colon \mathbb{CP}^1 \to \mathbb{CP}^3$ of degree $d_\lambda = n-1$ with total ramification index $\beta_\lambda = n-4$ and immersed non–planar null curves $\gamma\colon \mathbb{CP}^1 \to Q^3$ of degree $d_\gamma = n$.

Proof. We have seen in 10.19 that λ and its second associated curve λ_2 are the same with respect to a suitable projective isomorphism. Therefore both curves have the same extrinsic data, i.e. $d_\lambda = d_2$ and $\beta_\lambda = \beta_2$, where d_2 and β_2 are the degree and the total ramification index of λ_2. Then the Plücker formulas 10.14 for $k = 0$ and $k = 1$ become

$$-2d_\lambda + d_\gamma = 2g - 2 - \beta_\lambda$$

and

$$2d_\lambda - 2d_\gamma = 2g - 2 - \beta_\gamma,$$

while the equation for $k = 2$ is the same as for $k = 0$.

By 10.18 every non–planar immersed null curve γ is the Lie transform of a non–degenerate contact curve λ. In the case of curves over $\mathbb{CP}^1$, i.e. $g = 0$, the second equation of 11.1.1 shows us that if γ is immersed, i.e. $\beta_\gamma = 0$, then $d_\lambda = d_\gamma - 1$. Hence the first equation of 11.1.1 turns into $\beta_\lambda = d_\gamma - 4$. Conversely, if $d_\lambda = n-1$ and $\beta_\lambda = n-4$ the first equation of 11.1.1 yields that $d_\gamma = n$ while the second equation shows us that $\beta_\gamma = 0$, thus the (non–planar) Lie transform γ of λ is immersed. □

11.2. Lemma. *There does not exist an immersed non–planar null curve $\lambda\colon \mathbb{CP}^1 \to Q^3$ of degree $2, 3$ or 5.*

Proof. By Lemma 11.1 every non–planar immersed null curve of degree 2 or 3 would correspond to a non–degenerate contact curve of degree 1 respectively 2 with total ramification index -2 respectively -1. But the total ramification index must be non–negative, thus there exist no immersed null curves of degree 2 or 3.

In the case of degree 5, we would have a contact curve λ of degree 4 with total ramification index 1. We choose a holomorphic coordinate $z\colon \mathbb{C}P^1 \setminus \{\infty\} \to \mathbb{C}$ such that the only ramification point is $z = 0$. Then there is a (meromorphic) lift $f\colon \mathbb{C}P^1 \to \mathbb{C}^4$ of λ such that f is a polynomial in z and nowhere vanishing. The polynomial f has degree 4 because the degree of λ is 4. After a change of coordinates of $\mathbb{C}^4$, we obtain the normalform of f at $z = 0$:

$$f(z) = (1 + .., z^{1+\beta_0(0)} + .., z^{2+\beta_0(0)+\beta_1(0)} + .., z^{3+\beta_1(0)+\beta_1(0)+\beta_2(0)} + ..),$$

where $\beta_k(0)$ is the ramification index of the kth associated curve at $z = 0$, see 10.13.1. But λ and its second associated curve λ_2, which are the same by 10.19, ramify at $z = 0$, therefore we get $\beta_0(0) = \beta_2(0) = 1$, and since γ is immersed: $\beta_1(0) = 0$. Therefore the polynomial f would have at least degree $5 > 4$. Thus there does not exist an immersed non–planar null curve of degree 5. □

11.3. Lemma. *There does not exist an immersed non–planar null curve $\gamma\colon \mathbb{C}P^1 \to Q^3$ of degree 7.*

Proof. Suppose such a γ exists. The corresponding non–degenerate complex curve λ would be of degree 6 and would have total ramification index 3 by 11.1. There are three possible ramification divisors: a triple point, a double point and one single point, and three single points. The first and the second case are impossible by the same reason as in 11.2 for the contact curve of degree 4.

It remains to show that there is no contact curve of degree 6 with three single ramification points. We can choose a conformal coordinate $z\colon \mathbb{C}P^1 \setminus \{\infty\} \to \mathbb{C}$ such that these ramification points are $z = 0$, $z = 1$ and ∞. Then there exists a meromorphic lift $f\colon \mathbb{C}P^1 \to \mathbb{C}^4$ of λ, which is a nowhere vanishing polynomial of degree 6 in z. f can be written in the form $f(z) = v_0 + zv_1 + .. + z^6 v_6$ for suitable $v_0, .., v_6 \in \mathbb{C}^4$, where $v_0 \neq 0$ and $v_6 \neq 0$. Since λ ramifies at 0 and ∞ we have $v_0 \wedge v_1 = 0$ and $v_5 \wedge v_6 = 0$. Hence we can write f in the form

$$f(z) = v_0(1 + az) + v_2 z^2 + v_3 z^3 + v_4 z^4 + v_6(bz^5 + z^6).$$

Since λ is a non–degenerate curve, v_0, v_2, v_3, v_4 and v_6 span $\mathbb{C}^4$.

First we assume that $v_0 \wedge v_6 = 0$. Thus there is a basis $e_1, ..e_4$ of $\mathbb{C}^4$ such that $f(z) = e_1(a + bz + cz^5 + dz^6) + e_2 z^2 + e_3 z^3 + e_4 z^4$ with $ad \neq 0$. Using $0 \equiv \Omega(f(z) \wedge f'(z))$ we obtain

$$\Omega(e_1 \wedge e_1) = \Omega(e_1 \wedge e_2) = \Omega(e_1 \wedge e_3) = \Omega(e_1 \wedge e_4) = 0.$$

This is a contradiction because Ω is non–degenerate and $e_1, .., e_4$ is a basis of $\mathbb{C}^4$.

Thus we can assume that v_0 and v_6 are linearly independent. Since λ is non–degenerate we have $\text{span}(v_0, v_2, v_3, v_4, v_6) = \mathbb{C}^4$ and we have to study three possible cases: $\mathrm{span}\{v_0, v_3, v_4, v_6\} = \mathbb{C}^4$, $\mathrm{span}\{v_0, v_2, v_3, v_6\} = \mathbb{C}^4$ and $\mathrm{span}\{v_0, v_2, v_4, v_6\} = \mathbb{C}^4$.

First we assume that v_0, v_3, v_4, v_6 is a basis of $\mathbb{C}^4$ and we obtain (in this coordinate system):

$$f(z) = (1 + az + bz^2, cz^2 + z^3, dz^2 + z^4, ez^2 + gz^5 + z^6)$$

and

$$f'(z) = (a + 2bz, 2cz + 3z^2, 2dz + 4z^3, 2ez + 5gz^4 + 6z^5).$$

Since λ is a contact curve we have

(11.3.1)

$$\begin{aligned} 0 \equiv \ \Omega(f(z) \wedge f'(z)) = (2z^9 + gz^8 + 4dz^7 + 3dgz^6 - 2ez^5)\ \Omega(v_4 \wedge v_6) \\ +(3z^8 + (4c + 2g)z^7 + 3cgz^6 - ez^4)\ \Omega(v_3 \wedge v_6) \\ +(4bz^7 + (3bg + 5a)z^6 + (6 + 4ag)z^5 + 5gz^4 + aez^2 + 2ez)\ \Omega(v_0 \wedge v_6) \\ +(z^6 + 2cz^5 - dz^4)\ \Omega(v_3 \wedge v_4) \\ +(2bz^5 + 3az^4 + 4z^3 + adz^2 + 2dz)\ \Omega(v_0 \wedge v_4) \\ +(bz^4 + 2az^3 + (3 + ac)z^2 + 2cz)\ \Omega(v_0 \wedge v_3). \end{aligned}$$

This formula yields $\Omega(v_3 \wedge v_6) = \Omega(v_4 \wedge v_6) = 0$. If we can show that also $\Omega(v_0 \wedge v_6) = 0$ we would have a contradiction because Ω is non–degenerate and we have assumed that v_0, v_3, v_4, v_6 is a basis of $\mathbb{C}^4$. If $b \neq 0$ then clearly $\Omega(v_0 \wedge v_6) = 0$. Thus we can work with $b = 0$ and 11.3.1 turns into:

(11.3.2)

$$\begin{aligned} 0 \equiv \Omega(f(z) \wedge f'(z)) = (5az^6 + (6 + 4ag)z^5 + 5gz^4 + aez^2 + 2ez)\Omega(v_0 \wedge v_6) \\ +(z^6 + 2cz^5 - dz^4)\ \Omega(v_3 \wedge v_4) \\ +(3az^4 + 4z^3 + adz^2 + 2dz)\ \Omega(v_0 \wedge v_4) \\ +(2az^3 + (3 + ac)z^2 + 2cz)\ \Omega(v_0 \wedge v_3). \end{aligned}$$

Let us assume that $\Omega(v_0 \wedge v_6) \neq 0$. If $a = 0$, we obtain $\Omega(v_3 \wedge v_4) = 0$, which implies $\Omega(v_0 \wedge v_6) = 0$. Therefore we can assume $a \neq 0$.
Define x_1, x_2, x_3 by $\Omega(v_3 \wedge v_4) = x_1\Omega(v_0 \wedge v_6)$, $\Omega(v_0 \wedge v_4) = x_2\Omega(v_0 \wedge v_6)$ and $\Omega(v_0 \wedge v_3) = x_3\Omega(v_0 \wedge v_6)$. So 11.3.2 turns into the following system of algebraic equations:

$$0 = 5a + x_1 \tag{11.3.3}$$

$$0 = 6 + 4ag + 2cx_1 \tag{11.3.4}$$

$$0 = 5g - dx_1 + 3ax_2 \tag{11.3.5}$$

$$0 = 4x_2 + 2ax_3 \tag{11.3.6}$$

$$0 = ae + adx_2 + (3 + ac)x_3 \tag{11.3.7}$$

$$0 = 2e + 2dx_2 + 2cx_3. \tag{11.3.8}$$

Since $a \neq 0$ 11.3.3 and 11.3.6 yield $x_1 = -5a \neq 0$ resp. $x_3 = -\frac{2x_2}{a}$ and then 11.3.7 turns into:

$$ae = x_2(\frac{6}{a} + 2c - ad),$$

but 11.3.8 turns into

$$ae = x_2(2c - ad).$$

The last two equations can only be satisfied simultaneously in the case $x_2 = 0$ and then also $x_3 = 0$, $e = 0$.
We also need the fact that $z = 1$ is a ramification point, i.e. $f(1) = x_4 f'(1)$, for suitable $x_4 \in \mathbb{C}^*$, since $f(1) \neq 0$, $f'(1) \neq 0$. Since $b = e = 0$ we obtain

$$(1 + a, c + 1, d + 1, g + 1) = x_4(a, 2c + 3, 2d + 4, 5g + 6), \tag{11.3.9}$$

and with $a \neq 0$ we get $x_4 = \frac{a+1}{a} \neq 0$. Then 11.3.9 turns into

$$\begin{aligned} 1 + c &= \frac{a+1}{a}(2c + 3) \\ 1 + d &= \frac{a+1}{a}(2d + 4) \\ 1 + g &= \frac{a+1}{a}(5g + 6). \end{aligned}$$

Using $a \neq 0$ we compute

$$c = -\frac{3 + 2a}{2 + a}, \quad d = -\frac{4 + 3a}{2 + a}, \quad g = -\frac{6 + 5a}{5 + 4a},$$

especially $a \neq -2$ and $a \neq -\frac{5}{4}$. Then 11.3.4 turns into $a^3 + 3a^2 + 3a + 1 = 0$, but the only solution is $a = -1$ and this cannot occur since $f(1) \neq 0$.

The second case $\text{span}(v_0, v_2, v_3, v_6) = \mathbb{C}^4$ is equivalent to the first case $\text{span}(v_0, v_3, v_4, v_6) = \mathbb{C}^4$. To see this we can work with the conformal coordinate $\omega = 1/z$ and with the nowhere vanishing polynomial $\omega^6 f(1/\omega)$ in ω of degree 6, which is also a meromorphic lift of $\lambda = \lambda(\omega)$, and we obtain the same set of equations as for the first case. Hence this case cannot occur.

We have only to deal with the third case $\text{span}(v_0, v_2, v_4, v_6) = \mathbb{C}^4$ if the first and the second do not appear, i.e. $\text{span}(v_0, v_2, v_3, v_6) \neq \mathbb{C}^4$ and $\text{span}(v_0, v_3, v_4, v_6) \neq \mathbb{C}^4$. Then $v_3 \in \text{span}(v_0, v_2, v_6) \cap \text{span}(v_0, v_4, v_6) = \text{span}(v_0, v_6)$ and f is given in the form

$$f(z) = (1 + az + bz^3)v_0 + z^2 v_2 + z^4 v_4 + (cz^3 + dz^5 + z^6)v_6$$

for suitable $a, b, c, d \in \mathbb{C}$. Then the contact condition is

(11.3.10)

$$\begin{aligned} 0 \equiv \ \Omega(f(z) \wedge f'(z)) &= (2z^9 + dz^8 - cz^6)\ \Omega(v_4 \wedge v_6) \\ &+(3bz^8 + 2bdz^7 + 5az^6 + (4ad + 6)z^5 + 5dz^4 + 2acz^3 + 3cz^2)\ \Omega(v_0 \wedge v_6) \\ &+(4z^7 + 3dz^6 + cz^4)\ \Omega(v_2 \wedge v_6) \\ &+(bz^6 + 3az^4 + 4z^3)\ \Omega(v_0 \wedge v_4) \\ &+2z^5\ \Omega(v_2 \wedge v_4) \\ &+(2z + az^2 - bz^4)\ \Omega(v_0 \wedge v_2). \end{aligned}$$

Therefore $\Omega(v_4 \wedge v_6) = \Omega(v_0 \wedge v_2) = 0$. If $b \neq 0$, 11.3.10 would also imply $\Omega(v_0 \wedge v_6) = 0$ and consequently $\Omega(v_2 \wedge v_6) = 0$, which is a contradiction to our assumptions that Ω is non–degenerate and that v_0, v_2, v_4, v_6 is a basis of $\mathbb{C}^4$. If $b = 0$, 11.3.10 directly implies $\Omega(v_2 \wedge v_6) = 0$. Then also $a = 0$ and the set of equations 11.3.10 turn into

$$0 \equiv (6z^5 + 5dz^4 + 3cz^2)\Omega(v_0 \wedge v_6) + 4z^3\Omega(v_0 \wedge v_4) + 2z^5\Omega(v_2 \wedge v_4)$$

and we also obtain $c = d = 0$. Therefore the curve must be of the form

$$\lambda(z) = [v_0 + z^2 v_2 + z^4 v_4 + z^6 v_6].$$

But this curve has no ramification point at $z = 1$, which shows that the last case cannot occur. Therefore there does not exist a non–degenerate contact curve in $\mathbb{C}\mathrm{P}^3$ of degree 6 with total ramification index 3, and consequently there does not exist an immersed non–planar null curve in Q^3 of degree 7.

□

As a consequence we obtain:

11.4. Theorem. *There does not exist an immersed Willmore sphere with Willmore functional* $4\pi(n-1)$ *for* $n \in \{2, 3, 5, 7\}$.

PROOF. If there would exist an immersed Willmore sphere L with Willmore functional $\mathcal{W}(L) = 4\pi(n-1)$, $n \in \mathbb{N} \setminus \{1\}$, then there would exist a non–planar immersed null curve γ of degree $d_\gamma = n$, see 9.7, 10.4 and 10.5. By 11.2 and 11.3 such curves do not exist for $n \in \{2, 3, 5, 7\}$. □

11.5. The Moduli Space $\mathcal{M}_4$. In the rest of this section we study the space $\mathcal{M}_4$ of Willmore sphere with Willmore functional 12π. We first need some lemmas.

11.6. Lemma. *Let* $\lambda\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^3$ *be a non–degenerate immersed contact curve of degree 3 and* $z\colon \mathbb{C}\mathrm{P}^1 \setminus \{\infty\} \to \mathbb{C}$ *be a conformal coordinate. Then there exists a basis* $e_0, .., e_3$ *of* $\mathbb{C}^4$ *such that* $\Omega = e_0^* \wedge e_3^* + e_1^* \wedge e_2^*$ *and*

$$\hat{\lambda}(z) := -\frac{1}{3}e_0 + ze_1 + z^2 e_2 + z^3 e_3$$

is a holomorphic lift of λ.

PROOF. Since λ is non–degenerate there exists a basis $v_0, .., v_3$ of $\mathbb{C}^4$ such that a holomorphic lift of λ is $f(z) = v_0 + zv_1 + z^2 v_2 + z^3 v_3$. Then the contact condition is

$$\begin{aligned} 0 \equiv &\Omega(f(z) \wedge f'(z)) = \Omega(v_0 \wedge v_1) + z\Omega(v_0 \wedge v_2) \\ &+ z^2(3\Omega(v_0 \wedge v_3) + \Omega(v_1 \wedge v_2)) + 2z^3\Omega(v_1 \wedge v_3) + z^4\Omega(v_2 \wedge v_3). \end{aligned}$$

Since Ω is non–degenerate there exists a $c \in \mathbb{C} \setminus \{0\}$ such that $\Omega(v_1 \wedge v_2) = c$. Then $\Omega(v_0 \wedge v_3) = -c/3$ and

$$e_0 := -\frac{3}{\sqrt{c}}v_0,\ e_1 := \frac{1}{\sqrt{c}}v_1,\ e_2 := \frac{1}{\sqrt{c}}v_2,\ e_3 := \frac{1}{\sqrt{c}}v_3$$

is a basis with $\Omega = e_0^* \wedge e_3^* + e_1^* \wedge e_2^*$. Hence

$$\hat{\lambda}(z) := \frac{1}{\sqrt{c}}f(z) = -\frac{1}{3}e_0 + ze_1 + z^2 e_2 + z^3 e_3$$

is a holomorphic lift as required. It is obvious that λ is immersed. □

11.7. Lemma. *Let* $\gamma, \tilde{\gamma}\colon \mathbb{C}\mathrm{P}^1 \to Q^3 \subset PW$ *be non–planar immersed null curves of degree 4. Then there exists a* $A \in \mathrm{SO}(W)$ *such that* $A\gamma = \tilde{\gamma}$. *Consequently, up to the action of* $\mathrm{SO}(W)$, *the only non–planar immersed null curve of degree 4 is the rational normal curve.*

Proof. Let $z\colon \mathbb{CP}^1 \setminus \{\infty\} \to \mathbb{C}$ be a conformal coordinate. By Lemma 11.1 there exist non–degenerate immersed contact curves of degree 3, λ and $\tilde{\lambda}$, such that γ and $\tilde{\gamma}$ are its Lie transform. Let $e_0, ..e_3$ and $v_0, .., v_3$ be bases of $\mathbb{C}^4$ such that $\Omega = e_0^* \wedge e_3^* + e_1^* \wedge e_2^* = v_0^* \wedge v_3^* + v_1^* \wedge v_2^*$ and such that holomorphic lifts of λ and $\tilde{\lambda}$ are given by $f(z) := -\frac{1}{3}e_0 + ze_1 + z^2 e_2 + z^3 e_3$ respectively $\tilde{f}(z) := -\frac{1}{3}v_0 + zv_1 + z^2 v_2 + z^3 v_3$, see 11.6. We define the linear isomorphism $\varphi\colon \mathbb{C}^4 \to \mathbb{C}^4$ by $\varphi(e_i) := v_i$. Then $\varphi^*\Omega = \Omega$ and $W = \ker \Omega \subset \Lambda^2(\mathbb{C}^4)$ is invariant under the induced isomorphism $\Lambda^2(\varphi)\colon \Lambda^2(\mathbb{C}^4) \to \Lambda^2(\mathbb{C}^4)$. With $A := \Lambda^2(\varphi)_{|W}$, the computation

$$\begin{aligned} < \eta, \xi > &= \frac{1}{2}\Omega \wedge \Omega(\eta \wedge \xi) = \frac{1}{2}(\varphi^*\Omega) \wedge (\varphi^*\Omega)(\eta \wedge \xi) \\ &= \frac{1}{2}\Omega \wedge \Omega(A(\eta) \wedge A(\xi)) =< A(\eta), A(\xi) > \end{aligned}$$

for arbitrary $\eta, \xi \in W$ shows us that $A \in \mathrm{SO}(W)$. Since $\lambda, \tilde{\lambda}$ are immersed we have in PW for all values $z \in \mathbb{CP}^1 \setminus \{\infty\}$:

$$A\gamma(z) = [A(f(z) \wedge f'(z))] = [\varphi(f(z)) \wedge \varphi(f'(z))] = [\tilde{f}(z) \wedge \tilde{f}'(z)] = \tilde{\gamma}(z).$$

By continuity $A\gamma = \tilde{\gamma}$ as claimed.

We have already seen that the rational normal curve is an immersed non–planar null curve in $Q^3 \subset PW$, where we consider the space W of homogeneous polynomials of degree 4 on $\mathbb{C}^2$ with the bilinear–form given by 10.8. □

By 10.4 and 10.16 every meromorphic immersion $F\colon \mathbb{CP}^1 \to \mathbb{C}^3$ with null tangents, where we consider $\mathbb{C}^3$ with its standard bilinear form $(,)$, and 4 poles of first order comes via stereographic projection from an immersed null curve $\gamma\colon \mathbb{CP}^1 \to Q^3$ of degree 4, for example from the rational normal curve γ_4. The following Theorem shows that there is up to transformations on $\mathbb{CP}^1$ and $\mathbb{C}^3$ only one such null immersion F.

11.8. Theorem. *Let $F_1, F_2\colon \mathbb{CP}^1 \to \mathbb{C}^3$ be meromorphic immersions with null tangents and 4 simple poles at $p_1, .., p_4$ respectively $q_1, .., q_4$. Then the points $\{p_1, .., p_4\}$ and $\{q_1, .., q_4\}$ have double ratio $e^{\pm i\frac{\pi}{3}}$. There exist a Moebius transformation $G \in \mathrm{SL}(2, \mathbb{C})$, a transformation $A \in \mathrm{SO}(3, \mathbb{C})$, a complex number $a \neq 0$ and a vector $v \in \mathbb{C}^3$ such that*

$$F_1 = aA \circ F_2 \circ G + v.$$

Proof. We work in the space W of homogeneous polynomials of degree 4 on $\mathbb{C}^2$ with the non–degenerate bilinear–form $<, >$ given by the quadric form q, see 10.8.1. First we need to understand the quadric Q^3 in terms of unordered 4–tuples $p_1, .., p_4 \in \mathbb{CP}^1$. The rational normal curve γ_4 is a null curve in Q^3, thus the tangent surface of γ_4 is a subset of Q^3. But the tangent surface is exactly the set of unordered 4–tuples (p, p, p, q) with arbitrary $p, q \in \mathbb{CP}^1$. Thus polynomials $P \in W \setminus \{0\}$ with a zero $p \in \mathbb{CP}^1$ of order 4, or with a zero $p \in \mathbb{CP}^1$ of order 3 and a zero $\tilde{p} \in \mathbb{CP}^1$ are q–null. Let $p_1 \in \mathbb{CP}^1$ be a zero of order 2 of a polynomial $P \in W \setminus \{0\}$ and p_2, p_3 be its other zeros. By Lemma 10.9 we can assume that $p_1 = \infty$, thus $P = aX^2Y^2 + bXY^3 + cY^4$ for $a, b, c \in \mathbb{C},\ a \neq 0$. Then $q(P) = a^2 \neq 0$ and $[P] \notin Q^3$. Now assume that $P \in W \setminus \{0\}$ has 4 distinct zeros $p_1, ..p_4 \in \mathbb{CP}^1$. Again by 10.9 we can assume that $p_4 = \infty$. Then $p_1 = [z_1, 1],\ p_2 = [z_2, 1]\ p_3 = [z_3, 1]$ for suitable $z_1, z_2, z_3 \in \mathbb{C}$ and we find a $a \neq 0$ such that

$$P = a(X - z_1Y)(X - z_2Y)(X - z_3Y)Y.$$

Then we compute

$$q(P) = a^2((z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2).$$

Using that z_1, z_2, z_3 are distinct and $D := (z_1 - z_2)/(z_1 - z_3)$ we obtain

$$q(P) = 0 \quad \Longleftrightarrow \quad D^2 - D + 1 = 0 \quad \Longleftrightarrow \quad D = e^{\pm i\frac{\pi}{3}}.$$

But D is the double ratio of the points p_4, p_1, p_3, p_2 and this is invariant under the action of $\mathrm{SL}(2, \mathbb{C})$. Thus an unordered 4–tuple $(p_1, .., p_4)$ of points in $\mathbb{C}\mathrm{P}^1$ is an element of Q^3 if and only if at least 3 points are the same or the points are distinct and the double ratio of $p_1, .., p_4$ is $e^{\pm i\frac{\pi}{3}}$.

Since a meromorphic null immersion $F\colon\ \mathbb{C}\mathrm{P}^1 \to \mathbb{C}^3$ with 4 simple poles cannot lie in a complex 2–dimensional affine subspace, 10.4, 10.5 and 10.16 imply that such a null immersion F is given by a stereographic projection of a non–planar immersed null curve γ of degree 4 at a point $p = [\xi] \in Q^3$ such that γ and $P\xi^\perp$ intersect transversely. Note that then for any $A \in \mathrm{SO}(W)$,

$$\pi_p \circ \gamma = \pi_{Ap} \circ A\gamma,$$

where $\pi_p\colon Q^3 \setminus\ P\xi^\perp \to \mathbb{C}^3$ and $\pi_{Ap}\colon Q^3 \setminus\ P(A\xi)^\perp \to \mathbb{C}^3$ are stereographic projection at p respectively Ap, defined by bases of the form 10.1.1 $e_0, .., e_4$ respectively $Ae_0, .., Ae_4$, see 10.1.2. Thus by 10.16 and 11.7 we only have to consider stereographic projections of the rational normal curve γ_4 exactly at the points $p \in Q^3$ which do not lie on the tangent surface of γ_4. But the set of points in Q^3 which do not lie on the tangent surface of γ_4 (or more general $\gamma_4 \circ G$ for a Moebius transformation $G \in \mathrm{SL}(2, \mathbb{C})$) is given by the set of unordered 4–tuples $(p_1, .., p_4)$ in $\mathbb{C}\mathrm{P}^1$ such that the points $p_1, .., p_4 \in \mathbb{C}\mathrm{P}^1$ have double ratio $e^{\pm i\frac{\pi}{3}}$. But $\mathrm{SL}(2, \mathbb{C})$ acts transitive on this set, i.e for 4–tuples $(p_1, .., p_4)$ and $(q_1, .., q_4)$ with double ratio $e^{i\frac{\pi}{3}}$ or $e^{-i\frac{\pi}{3}}$ there exists a Moebius transformation $G \in \mathrm{SL}(2, \mathbb{C})$ such that $G(q_i) = p_i$, $i = 1, .., 4$, and since $G\gamma_4 = \gamma_4 \circ G$ by definition of γ_4 and the action of $\mathrm{SL}(2, \mathbb{C})$ on Q^3, every meromorphic null immersion F with 4 simple poles is given by stereographic projection of $\gamma_4 \circ G$ for a suitable Moebius transformation $G \in \mathrm{SL}(2, \mathbb{C})$ at a fixed point $[\xi] \in Q^3$ which does not lie on the tangent surface of γ_4. One easily computes that if this $[\xi] \in Q^3$ is given by 4 distinct points $p_1, .., p_4 \in \mathbb{C}\mathrm{P}^1$ with double ratio $e^{\pm i\frac{\pi}{3}}$, $\gamma_4 \circ G$ meets the hyperplane $P\xi^\perp$ exactly at the points $q_i \in \mathbb{C}\mathrm{P}^1$ with $G(q_i) = p_i$. By 10.2 the stereographic projections at this point are only determined up to translation, multiplication by a complex number $a \neq 0$ and up to the action of $\mathrm{SO}(3, \mathbb{C})$ on $\mathbb{C}^3$. Thus, if we fix a stereographic projection $\pi_{[\xi]}\colon Q^3 \setminus P\xi^\perp \to \mathbb{C}^3$, for any meromorphic null immersion $F\colon\ \mathbb{C}\mathrm{P}^1 \to \mathbb{C}^3$ there exist a Moebius transformation $G \in \mathrm{SL}(2, \mathbb{C})$, a complex number $a \neq 0$, a transformation $A \in \mathrm{SO}(3, \mathbb{C})$ and a vector $v \in \mathbb{C}^3$ such that

$$F = aA \circ \pi_{[\xi]} \circ \gamma_4 \circ G + v.$$

□

11.9. Theorem. *The space of Willmore spheres with Willmore functional 12π is given by*

$$\mathcal{M}_4 = (\mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C}))/(\mathbb{R}^+ \times \mathrm{SO}(3,\mathbb{R})) \cong S^1 \times H^3,$$

where H^3 is the hyperbolic 3–space.

Proof. A proof of the second equality is given in [**KuSch95**]. By 9.7 there is a one–to–one correspondence between Willmore spheres in S^3 up to Moebius transformations L with $\mathcal{W}(L) = 12\pi$ and minimal surfaces up to rotation, translation and dilatation $f\colon \mathbb{CP}^1 \setminus \{p_1, .., p_4\} \to \mathbb{R}^3$ with flat ends at $p_1, .., p_4$. But every such minimal surface is the real part of a meromorphic null immersion $F\colon \mathbb{CP}^1 \to \mathbb{C}^3$ with simple poles at $p_1, .., p_4$. With $\mathrm{Re}(F + v) = \mathrm{Re}(F) + \mathrm{Re}(v)$ for any $v \in \mathbb{C}^3$ and with 11.8 this is exactly the space

$$(\mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C}))/(\mathbb{R}^+ \times \mathrm{SO}(3,\mathbb{R})).$$

□

12. Examples

Bryant showed in [**Br88**] that there exist immersed non–planar null curves of degree $2k$ and genus 0 for $k \geq 2$. For completeness we state the examples.

12.1. Lemma. *For $k \geq 2$ there exist immersed non–planar null curves $\gamma\colon \mathbb{CP}^1 \to Q^3$ of degree $2k$.*

Proof. By 11.1 an immersed non–planar null curve of degree $2k$ exists if and only if a non–degenerate contact curve $\lambda\colon \mathbb{CP}^1 \to \mathbb{CP}^3$ of degree $2k - 1$ with total ramification index $2k - 4$ exists. Let $e_0, .., e_3$ be a basis of C^4 such that $\Omega = e_0^* \wedge e_3^* + e_1^* \wedge e_2^*$ and $z\colon \mathbb{CP}^1 \setminus \{\infty\} \to \mathbb{C}$ a holomorphic coordinate. Then for $k \geq 2$ the curve λ_k given by

$$z \mapsto [-\frac{1}{2k-1} e_0 + e_1 z^{k-1} + e_2 z^k + e_3 z^{2k-1}]$$

is a holomorphic non–degenerate curve of degree $2k - 1$. Moreover it is a contact curve and its ramification points are $z = 0$ and ∞ with ramification index $\beta(0) = \beta(\infty) = k - 2$, so $\beta_{\lambda_k} = 2k - 4$. Thus the Lie transform γ_k of λ_k is a curve with the required properties. □

We want to construct immersed Willmore spheres L in S^3 with Willmore functional $\mathcal{W}(L) = 8\pi k$ for any $k \in \mathbb{N}$, $k \geq 4$. This are exactly these surfaces which correspond under affine coordinates to minimal surfaces $f\colon \mathbb{CP}^1 \setminus \{p_1, .., p_{2k+1}\} \to \mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ with flat ends at $p_1, .., p_{2k+1}$. We now construct spinors s_1, s_2 such that the corresponding minimal surface f, given by 8.12.1, has an end with normal vector $\mathbb{i}$. Thus it is not necessary that both spinors s_1, s_2 have poles at $p_1, .., p_{2k+1}$. We use an idea of Peng in [**Pe86**] to construct the spinors.

12.2. Lemma. *For $k \geq 4$ there exist points $p_1, .., p_{2k+1} \in \mathbb{CP}^1$ and meromorphic spinors s_1, s_2 with pole divisors $D(s_1) + p_1 = D(s_2) = \sum_{l=1}^{2k+1} p_l$, such that the Abelian differentials $s_i s_j$, $i, j = 1, 2$ have no residues at the points $p_1, .., p_l$ and have no common zeros.*

PROOF. Let $z\colon \mathbb{CP}^1 \setminus \{\infty\} \to \mathbb{C}$ be the (affine) holomorphic coordinate. It is well–known that on $\mathbb{CP}^1$ exists only one (up to holomorphic isomorphism) spinor bundle. A meromorphic section with simple pole at ∞ is given by $\sqrt{dz}$, where $\sqrt{dz}^2 = dz$. We try to find $a, b, c, \lambda \in \mathbb{C}$ such that the spinors

$$
\begin{aligned}
s_1(z) &:= \frac{z(z^k - c)}{(z^k - \lambda)(z^k - 1)} \sqrt{dz} \\
s_2(z) &:= \frac{(z^k - a)(z^k - b)}{z(z^k - \lambda)(z^k - 1)} \sqrt{dz},
\end{aligned}
\tag{12.2.1}
$$

satisfy all conditions. s_1 has a zero of order $k-2$ at ∞ and s_2 does not vanish at ∞. Thus the poles of s_1 and s_2 can only be 0 or the kth roots of 1 respectively λ. Denote these points by $p_1, .., p_{2k+1} \in \mathbb{C}$. First note that if s_1 and s_2 have no common zeros and poles at $p_2, .., p_{2k+1}$ resp. $p_1, .., p_{2k+1}$, the numbers a, b, c, λ must be pairwise distinct and cannot be 0 or 1. The condition that the Abelian differentials $s_i s_j$ have no residues at the points p_l is, for $k \geq 4$, equivalent to the condition that the constant terms of the Laurent expansions of s_1, s_2 at the points p_l vanish, compare with 8.19. The conditions that the constant term of the Laurent expansion at the kth roots of 1 for s_1 and s_2 vanishes are exactly the equations

$$
0 = \lambda(k + 3 - 3c + ck) + k - 3 + 3c - 3ck \tag{12.2.2}
$$

respectively

$$
\begin{aligned}
0 =& \lambda(3k - ka - kb - 1 + b + a - ab - abk) \\
&- k - ka - kb + 1 - b - a + ab + 3abk.
\end{aligned}
\tag{12.2.3}
$$

For the kth roots of λ we obtain for s_1 the equation

$$
0 = \lambda^2(3 - k) + \lambda(-k - 3 - 3c + 3ck) + 3c - ck \tag{12.2.4}
$$

and for s_2 the equation

$$
\begin{aligned}
0 =& \lambda^3(k - 1) + \lambda^2(-3k + ka + kb + 1 + b + a) \\
&+ \lambda(ka + kb - b - a - ab - 3abk) + ab + abk,
\end{aligned}
\tag{12.2.5}
$$

while the condition that the constant terms of the Laurent expansion of the spinors s_1 and s_2 at $z = 0$ vanish is automatically satisfied. But 12.2.2 and 12.2.4 are equivalent to

$$
\begin{aligned}
0 =& \lambda^2(k^2 - 6k + 9) + \lambda(2k^2 + 12k - 18) + (k^2 - 6k + 9) \\
0 =& c(4k - 6) + (3 - k)\lambda + 3 - k
\end{aligned}
$$

hence all possible solutions for λ and c are for $k \geq 4$

$$
\begin{aligned}
\lambda_{1,2} = \lambda_{1,2}(k) &= -\frac{k^2 + 6k - 9 \pm 2k\sqrt{3}\sqrt{2k - 3}}{(k - 3)^2} \\
c_{1,2} = c_{1,2}(k) &= \frac{-6k + 9 \mp k\sqrt{3}\sqrt{2k - 3}}{(2k - 3)(k - 3)}.
\end{aligned}
\tag{12.2.6}
$$

The equations 12.2.3 and 12.2.5 are equivalent to the system

$$
\begin{aligned}
0 &= b^2(\lambda^2(k+1)^2 + 2\lambda(k^2-2k-1) + (k+1)^2) \\
&\quad + b(\lambda^3(k^2-1) + \lambda^2(1-5k^2) + \lambda(1-5k^2) + k^2 - 1) \\
&\quad + \lambda^3(k-1)^2 + 2\lambda^2(k^2+2k-1) + \lambda(k-1)^2 \\
0 &= a(\lambda^2(k+1)^2 + 2\lambda(k^2-2k-1) + (k+1)^2) \\
&\quad + \lambda^3(k^2-1) + \lambda^2(b(k+1)^2 + 1 - 5k^2) \\
&\quad + \lambda(1 - 4kb - 2b - 5k^2 + 2k^2 b) + k^2 - 1 + b(k+1)^2.
\end{aligned} \tag{12.2.7}
$$

Using $\lambda = \lambda_{1,2}(k)$, see 12.2.6, and the first equation of 12.2.7 we can determine all possible solutions $b_{i,j}(k),\ i,j \in \{1,2\}$ for b in dependence of k and the choice of $\lambda_i(k)$. Then all possible solutions $a_{i,j}(k)$ for a can be computed with the second equation of 12.2.7 in dependence of k and the choice of solutions $\lambda = \lambda_i(k),\ b = b_{i,j}$. One finds that for any $k \geq 4$ there are always solutions $\lambda_i,\ c_i,\ b_{i,j}$ and $a_{i,j}$ for $\lambda,\ c,\ b$ respectively a of the system 12.2.2, 12.2.3, 12.2.4 and 12.2.5, which are pairwise distinct and not equal to 0 or 1. For these values the spinors s_1 and s_2 defined by 12.2.1 satisfy all required conditions. □

12.3. Theorem. *For any $n \in \mathbb{N} \setminus \{2,3,5,7\}$ there exists an immersed minimal surface f of genus 0 with n flat ends, consequently the spaces $\mathcal{M}_n$ of immersed Willmore spheres with Willmore functional $4\pi(n-1)$ are not empty.*

Proof. Because of 9.7 it is enough to show the first statement. For $n = 1$ the flat plane is an immersed minimal surface. For $n = 2k,\ k \geq 2$, 12.1, 10.4 and 9.7 show the existence of immersed minimal surfaces with n flat ends.

It remains to show the existence for $n = 2k+1$ for $k \geq 4$. Let s_1, s_2 be the meromorphic spinors of 12.2. Then the $\mathbb{C}^3$–valued meromorphic 1–form

$$\omega := e_1 \otimes (-2i s_1 s_2) + e_2 \otimes (s_1^2 + s_2^2) + e_3 \otimes (-i(s_2^2 - s_1^2))$$

is exact on $M \setminus \{p_1, .., p_{2k+1}\}$ because the differentials $s_i s_j$ have no residues, compare with 9.5. Let F be a solution of $dF = \omega$. The same argument as in 9.5 shows that F has null tangents. Since s_1 and s_2 have no common zeros, F must be an immersion. F has simple poles at the points $p_1, .., p_{2k+1}$ since ω has poles of order 2 at $p_1, .., p_{2k+1}$. Now 9.6 implies that $\mathrm{Re}(F)$ is a minimal immersion of genus zero with flat ends at $p_1, .., p_{2k+1} \in \mathbb{C}\mathrm{P}^1$. □

12.4. Example. Finally we construct a minimal surface with 9 flat ends explicitly. First we compute spinors of the form 12.2.1 for $k = 4$. Thus we have to determine possible values for $\lambda,\ a,\ b,\ c$. With 12.2.6 we get

$$\lambda = \lambda_1(4) = -31 - 8\sqrt{15} \quad \text{and} \quad c = c_1(k) = -3 - \frac{4}{5}\sqrt{15}.$$

Then we can solve 12.2.7 and obtain possible values for a and b :

$$b = b_{1,1}(4) = \frac{1890 + 488\sqrt{15} + 3\sqrt{833497 + 215208\sqrt{15}}}{3(31 + 8\sqrt{15})}$$

and

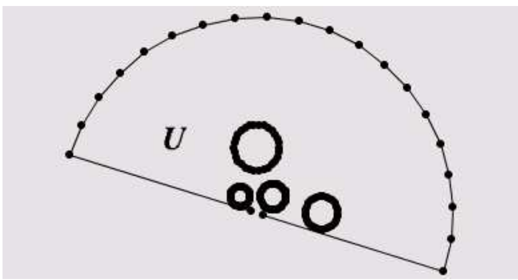
$$a = a_{1,1}(4) = \frac{-117150 - 30248\sqrt{15} + 3\sqrt{833497 + 215208\sqrt{15}}(31 + 8\sqrt{15})}{3(31 + 8\sqrt{15})^2}.$$

With these values we obtain the meromorphic spinors $s_1,\ s_2$ of 12.2.1 and we can compute the meromorphic null immersion in dependence of the holomorphic coordinate $z\colon\ \mathbb{C}\mathrm{P}^1 \setminus \{\infty\}$ for suitable $z_0 \in \mathbb{C}$:

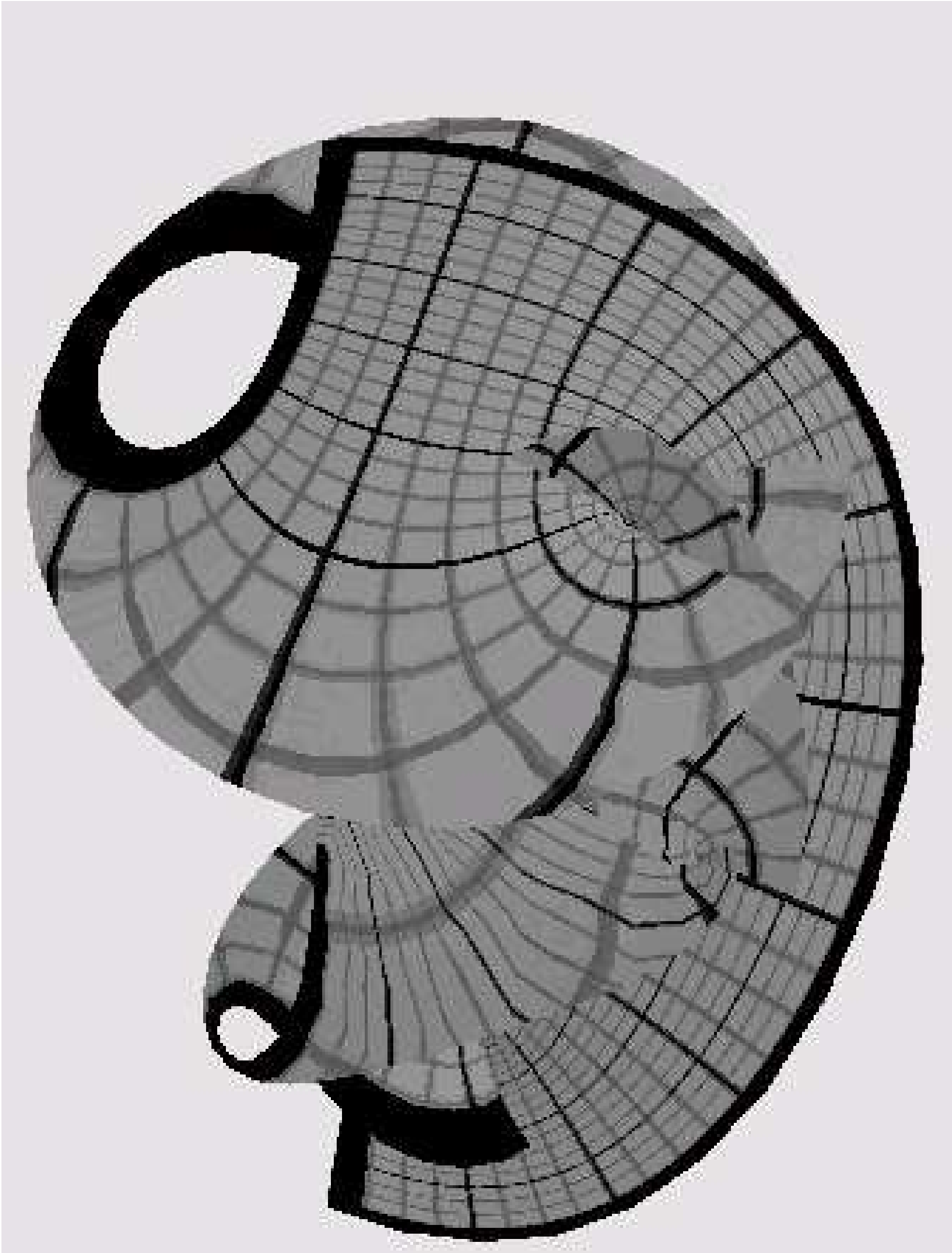
$$\begin{aligned}
F(z) = &\int_{z_0}^{z} -2i(s_1 s_2)e_1 + (s_1^2 + s_2^2)e_2 + i(s_2^2 - s_1^2)e_3 \\
=&v - \frac{2iz(4687220z^4 + 1210235z^4\sqrt{15} - 7380481\sqrt{15} - 28584480)}{15(31 + 8\sqrt{15})^3(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})(z^4 - 1)}e_1 \\
&+ \frac{z^8(223244789044 + 57641556673\sqrt{15})}{(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_2 \\
&+ \frac{z^4(206224160182596 + 53246849198187\sqrt{15})}{45(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_2 \\
&+ \frac{-69187876305620 - 17864232846075\sqrt{15}}{45(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_2 \\
&+ \frac{iz^8(223244789044 + 57641556673\sqrt{15})}{(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_3 \\
&+ \frac{iz^4(202205753979804 + 52209301178073\sqrt{15})}{45(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_3 \\
&+ \frac{i(-69187876305620 - 17864232846075\sqrt{15})}{45(31 + 8\sqrt{15})^6(4 + \sqrt{15})(z^4 + 31 + 8\sqrt{15})z(z^4 - 1)}e_3,
\end{aligned}$$

where e_1, e_2, e_3 is the standard basis of $\mathbb{C}^3$ and arbitrary $v \in \mathbb{C}^3$.

If $\omega = 1/z\colon\ \mathbb{C}\mathrm{P}^1 \setminus \{0\} \to \mathbb{C}$ takes values as indicated in the following graphic



then the minimal surface $f = \mathrm{Re}(F)\colon U \to \mathbb{R}^3$ looks like that:



Clearly, $f = \mathrm{Re}(F)\colon \mathbb{C}\mathrm{P}^1 \setminus \{0, \pm 1, \pm i, \pm \sqrt[4]{\lambda_1}, \pm i \sqrt[4]{\lambda_1},\} \to \mathbb{R}^3$ is an immersed minimal surface with 9 flat ends of genus 0, compare with 9.6.

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