On the number of Tverberg partitions in the prime power case

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Abstract

We give an extension of the lower bound of [9] for the number of Tverberg partitions from the prime to the prime power case. Our proof is inspired by the $\mathbb{Z}_p$-index version of the proof in [3] and uses Volovikov's Lemma. Analogously, one obtains an extension of the lower bound for the number of different splittings of a generic necklace to the prime power case.

1 Introduction

In 1966, Helge Tverberg showed that any set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ admits a partition into q subsets such that the intersection of their convex hulls is non-empty. Such partitions are called Tverberg partitions; the result is best possible: For less than $(d+1)(q-1)+1$ points in $\mathbb{R}^d$ the implication of the statement does not hold. Moreover, it can be formulated in the following way.

Theorem 1 ([7]). *Let $q \geq 2$, $d \geq 1$, and put $N := (d+1)(q-1)$. For every affine map $f : \|\sigma^N\| \to \mathbb{R}^d$ there are q disjoint faces $F_1, F_2, \ldots, F_q$ of the standard N-simplex σ^N whose images under f intersect: $\bigcap_{i=1}^{q} f(\|F_i\|) \neq \emptyset$.*

Relaxing affine maps to continuous maps one gets a more general problem which is known as the Topological Tverberg Theorem. For q a prime this topological version was first proved by Bárány et al. [1]. The proof uses a Borsuk-Ulam type argument and can be found in Matoušek's book [3] on topological methods in combinatorics and geometry. In 1987, Özaydin proved the case q being a prime power in an unpublished manuscript [4], later Volovikov gave another proof in [8], see also [5] and [2]. All proofs make use of deep results from algebraic topology. For arbitrary q the problem is still open.

Theorem 1 establishes the existence of Tverberg partitions. Another natural question is to ask for a lower bound: How many Tverberg partitions into q subsets are there for a chosen affine or continuous map f? Sierksma conjectured that there are at least $((q-1)!)^d$ for any set of $(d+1)(q-1)+1$ points in $\mathbb{R}^d$. The conjecture is still not proved. The case $d = 1$ and arbitrary q can be proved for continuous maps using the intermediate value theorem. The only non-trivial lower bound is established for q being prime using a Borsuk-Ulam type argument (see [9]).

The following extends the result of [9] to the prime power case using Volovikov's lemma from [8].

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Theorem 2. *Let $q = p^r$ be a prime power and $d \geq 1$. For any continuous map $f : \|\sigma^N\| \to \mathbb{R}^d$, where $N = (d+1)(q-1)$, the number of unordered q-tuples $\{F_1, F_2, \ldots, F_q\}$ of disjoint faces of the N-simplex with $\bigcap_{i=1}^{q} f(\|F_i\|) \neq \emptyset$ is at least*

$$\frac{1}{(q-1)!} \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{N}{2} \rceil}$$

A simplified proof for the lower bound of [9] can be found in Section 6.6 of [3]. In the prime power case $q = p^r$, we cannot use the $\mathbb{Z}_q$-action by cyclic shifting of the q coordinates of the q-fold join as the space $(\mathbb{R}^d)^{*q}_{\Delta}$ is a non-free $\mathbb{Z}_q$-space so that $\mathrm{ind}_{\mathbb{Z}_q}((\mathbb{R}^d)^{*q}_{\Delta}) = +\infty$.

Lower bound \ q	prime	prime power	arbitrary
1	[1]	[4],[8]	open
[9]-type	[9]	✓	open
Sierksma	open	open	open

Table 1: Current state around the Topological Tverberg Theorem

Progress towards the general case has been slow. But recently T. Schöneborn [6] was able to connect the Topological Tverberg Theorem to geometric graph theory type questions. In particular, he showed that the $d = 2$ case is equivalent to the following conjecture. Moreover, any lower bound for the number of Tverberg partitions carries over to winding partitions.

Conjecture 3 (Winding number conjecture [6]). *For every drawing of the complete graph K_{3q-2} in general position there are*
- *either $q-1$ disjoint triangles (that is drawings of K_3 subgraphs) wind around one vertex,*
- *or $q-2$ disjoint triangles wind around the intersection of two edges,*

with the triangles, the edges and the vertex being pairwise disjoint in K_{3q-2}.

We give a proof of Theorem 2 in Section 3. In Section 4 we sketch how to extend the lower bound for splitting generic necklaces of [9] to the prime power case.

2 Preliminaries

Before proving our lower bound we repeat some definitions and results from [3], mainly for fixing our notation. We write $[n]$ for the set $\{1, 2, \ldots, n\}$. Let G be a finite group. A topological space X equipped with a (left) G-action via a group homomorphism $\Phi : G \to \mathrm{Homeo}(X)$ is called a *G-space*; we write $g\,x$ for $\Phi(g)(x)$. Continuous maps between G-spaces X and Y that respect the G-actions of X and Y are called *G-maps* or *equivariant maps*. For $x \in X$ the set $O_x = \{g\,x \mid g \in G\}$ is called the *orbit* of x. A G-space (X, Φ) where every O_x has at least two elements is called *fixed point free*, i. e. no point of X is fixed by all group elements. Let X be a fixed point free G-space and $Y \subset X$ closed under the G-action, then Y with the induced action of X is again a fixed point free G-space.

The *join* $X * Y$ of spaces X and Y is a standard construction in topology. One way of looking at it is to identify it with the set of formal convex combinations $tx \oplus (1-t)y$, where $t \in [0,1]$, $x \in X$, $y \in Y$. We use the symbol $\oplus$ to underline that the sum is formal and does not commute for $X = Y$. With this identification the n-fold join X^{*n} becomes the set of all formal convex combinations $t_1x_1 \oplus t_2x_2 \oplus \cdots \oplus t_nx_n$, where $t_1, t_2, \ldots, t_n$ are non-negative reals summing up to

1 and $x_1, x_2, \ldots, x_n \in X$. The join of simplicial complexes is again a simplicial complex. For abstract simplicial complexes K and L the join is defined as the set of simplices $\{F \uplus G \mid F \in \mathsf{K}, G \in \mathsf{L}\}$, where $F \uplus G = (F \times \{1\}) \cup (G \times \{2\})$ is the disjoint union of F and G. For subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ of Euclidean spaces the join can be represented *geometrically* in the following way: Embed $A \subset \mathbb{R}^n \subset \mathbb{R}^{n+m+1}$ in the standard way, and embed $B \subset \mathbb{R}^m \subset \mathbb{R}^{n+m+1}$ such that the first n coordinates are equal to 0 and the last one is equal to 1. The subspace $C \subset \mathbb{R}^{n+m+1}$ defined as the union of all segments joining a point of A with a point of B is homeomorphic to $A * B$. Finally, there is an inequality for the connectivity of the join $X * Y$ for topological spaces X and Y:

$$\operatorname{conn}(X * Y) \geq \operatorname{conn}(X) + \operatorname{conn}(Y) + 2, \tag{1}$$

where a disconnected space has connectivity -1, see [3, Section 4.4] for more details.

Let $n \geq k \geq 2$. We call an n-tuple $(x_1, x_2, \ldots, x_n)$ *k-wise distinct* if no k among the x_i are equal. The *n-fold k-wise deleted join* of a space X is

$$X^{*n}_{\Delta(k)} := X^{*n} \setminus \{\tfrac{1}{n}x_1 \oplus \tfrac{1}{n}x_2 \oplus \cdots \oplus \tfrac{1}{n}x_n \mid (x_1, x_2, \ldots, x_n) \text{ not } k\text{-wise distinct}\}.$$

In the case $k = n$ we delete the diagonal of X^{*n}, and for $k_1 < k_2$ we have $X^{*n}_{\Delta(k_1)} \subset X^{*n}_{\Delta(k_2)}$; we write X^{*n}_{Δ} for $X^{*n}_{\Delta(n)}$. For a simplicial complex K we define its *n-fold k-wise deleted join* as the following set of simplices:

$$\mathsf{K}^{*n}_{\Delta(k)} := \{F_1 \uplus F_2 \uplus \cdots \uplus F_n \in \mathsf{K}^{*n} \mid (F_1, F_2, \ldots, F_n) \ k\text{-wise disjoint}\},$$

where an n-tuple $(F_1, F_2, \ldots, F_n)$ is called k-wise disjoint if no k among them have a non-empty intersection. For simplicial complexes K we have $\|\mathsf{K}^{*n}_{\Delta(k)}\| \subset \|\mathsf{K}\|^{*n}_{\Delta(k)}$. In the proof, we are interested in the special cases $k = 2$ and $k = n$.

The group action. Here we use an elementary abelian p-group G because equivariant topology for such groups works best, as in Volovikov's Lemma below. The symmetric group S_q acts from the left on a (deleted) q-fold join by permuting the q coordinates: $\pi \mapsto x_{\pi^{-1}(1)} \oplus \cdots \oplus x_{\pi^{-1}(q)}$ for $\pi \in S_q$. The following result is the key lemma in [8] for the prime power case $q = p^r$, and it is proved for actions of the subgroup $G := (\mathbb{Z}_p)^r$ of S_q. G is a subgroup of S_q in the following way: every element $g \in G$ defines a permutation of the q elements of G by translation $h \mapsto g + h$ for $h \in G$. For this we identify the q elements of G with the set $[q]$ by ordering them in the lexicographic order. E. g. the element $(1, 1) \in (\mathbb{Z}_3)^2$ acts on X^{*9}:

$$t_1x_1 \oplus \cdots \oplus t_9x_9 \mapsto t_9x_9 \oplus t_7x_7 \oplus t_8x_8 \oplus t_3x_3 \oplus t_1x_1 \oplus t_2x_2 \oplus t_6x_6 \oplus t_4x_4 \oplus t_5x_5.$$

A cohomology n-sphere over $\mathbb{Z}_p$ is a CW-complex having the same cohomology groups with $\mathbb{Z}_p$-coefficients as the n-dimensional sphere S^n.

Proposition 4 (Volovikov's Lemma [8]). *Set $G = (\mathbb{Z}_p)^r$, and let X and Y be fixed point free G-spaces such that Y is a finite-dimensional cohomology n-sphere over $\mathbb{Z}_p$ and $\tilde{H}^i(X, \mathbb{Z}_p) = 0$ for all $i \leq n$. Then there is no G-map from X to Y.*

Volovikov [8] derives from this lemma a proof of the Topological Tverberg Theorem in the prime power case. The proof of Proposition 4 uses deeper results from bundle cohomology. Note that Proposition 4 is also Corollary 3.4 of [4].

3 The extension of the lower bound

The next two lemmas enable us to replace the index argument used in [3, Section 6.6] by Volovikov's Lemma. From now on let $q = p^r$ be a prime power and $G := (\mathbb{Z}_p)^r \subset S_q$ be as above.

Lemma 5. *Let X_Δ^{*q} be the q-fold q-wise deleted join for some space X equipped with the G-action defined as above. Then X_Δ^{*q} is a fixed point free G-space.*

Note that the G-action on $(\mathbb{R}^d)_\Delta^{*q}$ is not free.

Proof. Let $x = t_1x_1 \oplus t_2x_2 \oplus \cdots \oplus t_qx_q \in X_\Delta^{*q}$, then by definition there are indices i and j such that $t_i \neq t_j$ or $x_i \neq x_j$. Using our identification of G and $[q]$ the indices i and j correspond to elements a resp. b of $(\mathbb{Z}_p)^r$. Setting $g = b - a$, we get $x \neq g\,x$ hence $|O_x| > 1$. □

Lemma 6. *Let $q \geq 2$ and d be integers. Then we have $(\mathbb{R}^d)_\Delta^{*q} \simeq S^{(d+1)(q-1)-1}$.*

Proof. Using the geometric version of the join we get an embedding $(\mathbb{R}^d)_\Delta^{*q} \subset \mathbb{R}^{q(d+1)-1}$. More precisely, we can identify it with the subset $\{(t_1x_1, t_2x_2, t_2, \ldots, t_qx_q, t_q) \,|\, x_i \in \mathbb{R}^d, t_i \geq 0, \sum_1^q t_i = 1\}$. The diagonal of $(\mathbb{R}^d)^{*q}$ is embedded as $A = \{x, x, \frac{1}{q}, \ldots, x, \frac{1}{q} \,|\, x \in \mathbb{R}^d\}$, a d-dimensional affine subspace of $\mathbb{R}^{q(d+1)-1}$. Its orthogonal complement $A^\perp$ has dimension $(d+1)(q-1)$. The restriction of the orthogonal projection $p_{A^\perp}$ onto the complement maps $(\mathbb{R}^d)_\Delta^{*q}$ on $\mathbb{R}^{(d+1)(q-1)} \setminus \{\text{pt}\}$. This map is a homotopy equivalence. □

In the prime case, the following proof reduces to the Vućić-Živaljević proof, in the version of Matoušek [3, Section 6.6].

Proof. (of **Theorem 2**) Let K be the simplicial complex $(\sigma^N)_{\Delta(2)}^{*q}$. The vertex set of K is $[N+1]\times[q]$. A maximal simplex of K is of the form $F_1 \uplus F_2 \uplus \cdots \uplus F_q$, where the F_i are pairwise disjoint subsets of the vertex set $[N+1]$ of σ^N and $\bigcup_1^q F_i = [N+1]$. In other words, there is a one-to-one correspondence between the maximal simplices K and the ordered partitions $(F_1, F_2, \ldots, F_q)$ of the vertex set $[N+1]$. Another way of looking at K: The set of all maximal simplices can be identified with the complete $(N+1)$-partite hypergraph on the vertex set $[N+1]\times[q]$. For example, a maximal simplex in the case $d = 2$ and $q = 4$ encoding a Tverberg partition for $N + 1 = 10$ points in $\mathbb{R}^2$ is shown in Figure 1. A Tverberg partition is represented by a hyperedge consisting of 10 vertices.

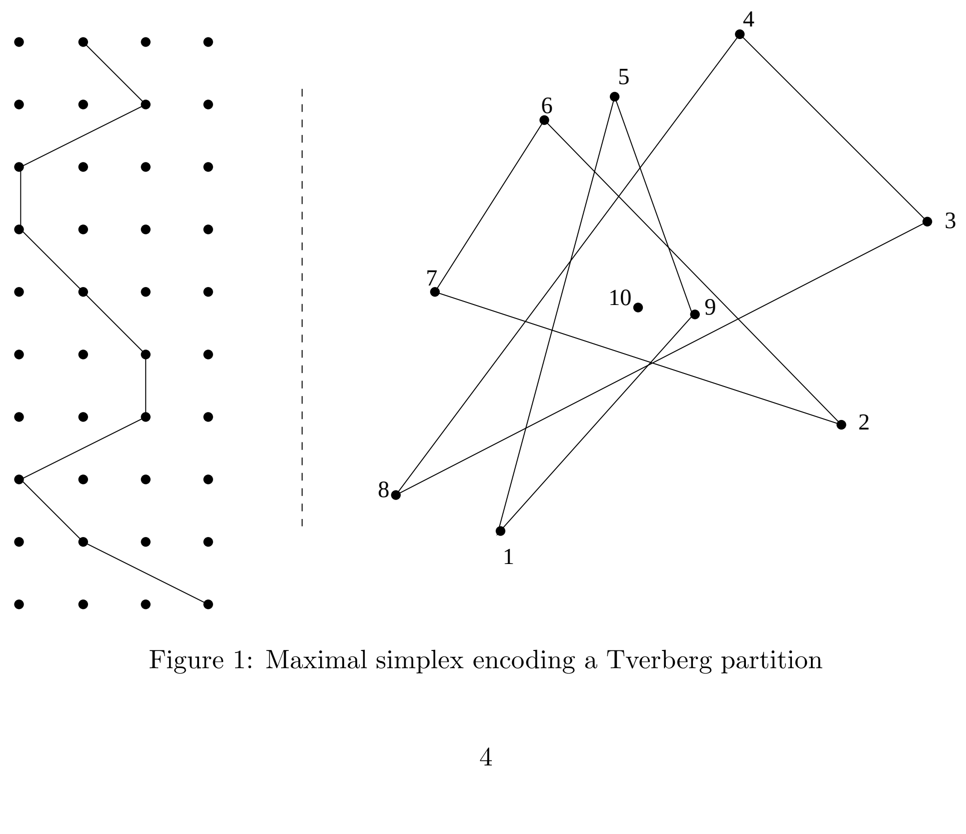

Figure 1: Maximal simplex encoding a Tverberg partition

The induced G-action permutes the q columns of vertices. We call a maximal face *good* if it encodes a Tverberg partition of the map f. Let $f^{*q} : \|\mathsf{K}\| \to (\mathbb{R}^d)^{*q}$ be the q-fold join of f restricted to $\|\mathsf{K}\|$, then it is a G-map. A maximal simplex S of K is good if its image $f^{*q}(\|S\|)$ intersects the diagonal of $(\mathbb{R}^d)^{*q}$. Proving a lower bound for the number of good simplices in K gives then a lower bound for the number of Tverberg partitions of f. If there are at least M good simplices we have at least $M/q!$ unordered Tverberg partitions.

In the next paragraph, we define a family $\mathcal{L}$ of subcomplexes $\mathsf{L} \subset \mathsf{K}$ having the properties: (i) L is closed under the G-action, and (ii) $\mathrm{conn}(\mathsf{L}) \geq N-1$. Then L is again a fixed point free G-space by (i) and Lemma 5. The reduced cohomology groups of L vanish in dimensions 0 to $N-1$ due to (ii). Now with Lemma 6 we get as a direct corollary of Volovikov's Lemma that L contains one good maximal simplex S; in fact, the entire orbit of S is good and we get q good simplices in L. Suppose Q is the number of $\mathsf{L} \in \mathcal{L}$ containing any given maximal simplex of K, then we obtain the lower bound

$$M \geq q \cdot |\mathcal{L}|/Q. \tag{2}$$

We define the family $\mathcal{L}$ and distinguish two cases: (i) N even, that is, p or d is odd, and (ii) N odd, that is, $p=2$ and even d. First we divide the $N+1$ rows into pairs such that we get $\frac{N}{2}$ pairs and one remaining row in the first case, and $\frac{N+1}{2}$ pairs in the second. Now we focus on the two rows of one pair; the simplices of K living on these two rows form bipartite graphs $K_{q,q}$. Suppose that we have chosen a connected G-invariant subgraph C_i of $K_{q,q}$, $i \in [\frac{N}{2}]$ resp. $i \in [\frac{N+1}{2}]$, for every pair. The maximal simplices of L to a given choice of row pairing and of the C_i, $i \in [\frac{N}{2}]$ resp. $i \in [\frac{N+1}{2}]$, are the maximal simplices of K that contain an edge of each C_i. L is G-invariant by construction. Topologically, we get in the first case

$$\mathsf{L} = C_1 * C_2 * \cdots * C_{N/2} * D_q,$$

and in the second

$$\mathsf{L} = C_1 * C_2 * \cdots * C_{(N+1)/2}.$$

Here D_q is the discrete space on q elements; in both cases one has $\mathrm{conn}(\mathsf{L}) \geq N-1$ using inequality (1).

In the next paragraph we will construct distinct G-invariant connect subgraphs C of the graph $G * G$ formed by two rows. Our aim is to get as much as possible subgraphs C so that $|\mathcal{L}|/Q$ - and at the same time our lower bound - gets as large as possible. The G-invariance implies that our subgraphs are regular, the connectivity implies that every vertex has at least degree $r+1$ (r is the smallest number of generators of the group G). We will construct

$$q(q-p^0)(q-p^1)(q-p^2)\cdots(q-p^{r-1})/(r+1)!$$

distinct G-invariant, connected subgraphs C having the smallest possible number of $q(r+1)$ edges.

To obtain a G-invariant subgraph choose edges and take their orbits, see Figure 2 for orbits in the case $q=3^2$. The vertices are elements of $(\mathbb{Z}_p)^r$ having order p as group elements. To make sure that we count an orbit without multiplicities choose its representative edge as the edge that is incident to the upper left vertex $O := (0, 0, \ldots, 0)$.

To prove the connectivity of the graph C we show that the component K_O of the vertex O is the whole graph C. Choosing $r+1$ representative edges consecutively such that in each step a new component is connected to the component K_O leads to a connected subgraph.

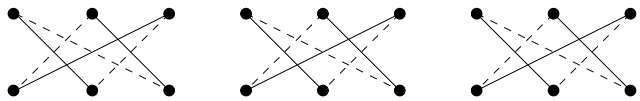

Figure 2: G-orbits of the edges $((0,0),(0,1))$ and $((0,0),(0,2))$.

More precisely, we will show inductively that after $1 \leq k \leq r+1$ steps: (i) there are $2p^{k-1}$ vertices in each component, p^{k-1} in each shore, and (ii) in total there are $p^{r-(k-1)}$ components. For $k=1$, the orbit of an edge consists of p^r vertex-disjoint edges, see Figure 2. For $k=2$, the graph of two orbits is equal to the disjoint union of p^{r-1} cycles of length $2p$, see Figure 2. Assume that for $1 \leq k \leq r$ edges the statement is true. Let the $(k+1)$-st edge be an edge connecting K_O with one of the other remaining $p^{r-(k-1)}-1$ components, there are $q-p^{k-1}$ many representative edges to do so. The graph of the $(k+1)$-st orbit and any of the k first orbits is again a union of cycles of length $2p$, hence each p components of the graph of the first k orbits get connected. Therefore the number of components decreases by a factor p, and the number of vertices increases by the factor p in each shore.

As the order in the $r+1$ steps of our construction does not play any role this process leads to the desired number of graphs C. Every given edge determines an orbit, hence there are

$$(q-p^0)(q-p^1)(q-p^2)\cdots(q-p^{r-1})/r!$$

connected, G-invariant graphs C containing this edge.

Finally, let π be the number of possibilities to do the row pairing in case (i) or (ii) (π cancels out in the end). Then in case (i) we get:

$$|\mathcal{L}| = \pi \cdot \left(q \cdot \prod_{i=0}^{r-1}(q-p^i)/(r+1)!\right)^{N/2},$$
$$Q = \pi \cdot \left(\prod_{i=0}^{r-1}(q-p^i)/r!\right)^{N/2},$$

and in case (ii):

$$|\mathcal{L}| = \pi \cdot \left(q \cdot \prod_{i=0}^{r-1}(q-p^i)/(r+1)!\right)^{(N+1)/2},$$
$$Q = \pi \cdot \left(\prod_{i=0}^{r-1}(q-p^i)/r!\right)^{(N+1)/2}.$$

Plugging these numbers into inequality (2) completes the proof. □

4 On the number of splitting necklaces

It is known that the methods introduced for the Topological Tverberg Theorem can also be applied to the splitting problem for necklaces for many thieves, see [3, Section 6.4]. We will extend the lower bound of [9] to the prime power case. A *necklace* is modeled in the following way: Given d continuous probability measures on $[0,1]$ and $q \geq 2$ thieves. A *fair splitting* of the necklace consists of a partition of $[0,1]$ into a number n of subintervals $I_1, I_2, \ldots, I_n$ and a partition of $[n]$ into q subsets $T_1, T_2, \ldots, T_q$ such that every thief has an equal amount of all d materials:

$$\sum_{j \in T_k} \mu_i(I_j) = \frac{1}{q}\ , \text{ for all } 1 \leq i \leq d \text{ and } 1 \leq k \leq q.$$

Noga Alon proved in 1987 that in general $d(q-1)$ is the smallest number of cuts for q thieves. A necklace is called *generic* if there is no fair splitting with less than

$d(q-1)$ cuts. The following result extends the lower bound of [9] for the number of fair splittings to the prime power case.

Theorem 7. *Let $q = p^r$ be a prime power. For generic necklaces made out of d continuously distributed materials the number of fair splittings with $d(q-1)$ cuts for q thieves is at least:*

$$q \cdot \left(\frac{q}{r+1}\right)^{\lceil \frac{d(q-1)}{2} \rceil}.$$

In the proof we will again face deleted joins, but also the deleted product $(\mathbb{R}^d)^q_\Delta$ that is the q-fold cartesian product of $\mathbb{R}^d$ without its diagonal. It is well-known that $(\mathbb{R}^d)^q_\Delta \simeq S^{d(q-1)-1}$, see e. g. [3, Section 6.3].

Proof. (sketch) In the proof of Theorem 6.4.1 of [3] there is a one-to-one correspondence between the set of splittings of a generic necklace for q thieves and the simplicial complex $\mathsf{K} = (\sigma^{d(q-1)+1})^{*q}_{\Delta(2)}$. The map $f : \|\mathsf{K}\| \to (\mathbb{R}^d)^q$, $z \mapsto f(z)_{i,k} := \sum_{j \in T_k} \mu_i(I_j)$ expressing the gains of the thieves is a G-map. If there is no fair splitting, f would miss the diagonal of $(\mathbb{R}^d)^q$. Now let $\mathcal{L}$ be a family of subcomplexes L satisfying: (i) L is closed under the G-action, and (ii) $\operatorname{conn}(\mathsf{L}) \geq d(q-1)-1$. Again with Volovikov's Lemma every L contains at least one fair splitting, but as above the whole orbit of size q is good. In conclusion, the whole construction for $\mathcal{L}$ and the counting as in the proof of Theorem 2 can be carried over. ☐

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