On a topological fractional Helly theorem

Stephan Hell *

Institut für Mathematik, MA 6–2, TU Berlin,
D–10623 Berlin, Germany, hell@math.tu-berlin.de

Abstract

We prove a new fractional Helly theorem for families of sets obeying topological conditions. More precisely, we show that the nerve of a finite family of open sets (and of subcomplexes of cell complexes) in $\mathbb{R}^d$ is k-Leray where k depends on the dimension d and the homological intersection complexity of the family. This implies fractional Helly number $k+1$ for families $\mathcal{F}$: For every $\alpha > 0$ there is a $\beta(\alpha) > 0$ such that for sets $F_1, F_2, \dots, F_n \in \mathcal{F}$ with $\bigcap_{i \in I} F_i$ for at least $\lfloor \alpha \binom{n}{k+1} \rfloor$ sets $I \subseteq \{1, 2, \dots, n\}$ of size $k+1$, there exists a point which is common to at least $\lfloor \beta n \rfloor$ of the F_i. Moreover, we obtain a topological (p, q)-theorem. Our result contains the (p, q)-theorem for good covers of Alon, Kalai, Matoušek, and Meshulam [2] as a special case. The proof uses a spectral sequence argument. The same method is then used to reprove a homological version of a nerve theorem of Björner.

1 Introduction

Helly's theorem is a classical theorem in convex geometry: For every finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ in which every $d+1$ or fewer sets have a common point we have $\bigcap \mathcal{F} \neq \emptyset$. Numerous Helly-type results are known; see [10] for a survey.

In this paper we are mainly concerned with topological conditions for fractional Helly theorems. Recently new fractional Helly theorems have been derived using different approaches; see [2], [4], [15]. A finite or infinite family $\mathcal{F}$ of sets has *fractional Helly number* k if for each $\alpha \in (0, 1]$ there is a $\beta(\alpha) > 0$ such that the following implication holds: For all $F_1, F_2, \dots, F_n \in \mathcal{F}$ such that $\bigcap_{i \in I} F_i \neq \emptyset$ for at least $\lfloor \alpha \binom{n}{k} \rfloor$ index sets $I \in \binom{[n]}{k}$, there exists a point which is in at least $\lfloor \beta n \rfloor$ of the sets F_i. Here $[n]$ is short for the set $\{1, 2, \dots, n\}$, and $\binom{X}{k}$ for the set of k-element subsets of a set X. There is two main tasks concerning fractional Helly theorems:

- For a family $\mathcal{F}$ find $\beta(\alpha) > 0$ optimal, as large as possible.
- Determine new families of sets that admit a fractional Helly theorem. What is their fractional Helly number?

In this paper, we focus on the second problem motivated by a question of Kalai and Matoušek: *Is there a homological analog of VC-dimension?* We give a positive answer to this question in Theorem 3 where homological conditions imply a fractional Helly theorem.

The original fractional Helly theorem for convex sets in $\mathbb{R}^d$ by Katchalski and Liu can then be stated in the following way.

*This research was supported by the Deutsche Forschungsgemeinschaft within the European graduate program 'Combinatorics, Geometry, and Computation' (No. GRK 588/2).

Theorem 1 (Fractional Helly theorem for convex sets [12]). *For every $d \geq 1$, the family of convex sets in $\mathbb{R}^d$ has fractional Helly number $d+1$.*

This result was generalized by Alon et al. [2] to *good covers* in $\mathbb{R}^d$ where a finite family of sets in whose members are either all open or all closed, is called a *good cover* if $\bigcap \mathcal{G}$ is either empty or contractible for all subfamilies $\mathcal{G} \subseteq \mathcal{F}$. Moreover, it was shown by Kalai [11] that the optimal $\beta(\alpha)$ equals $1-(1-\alpha)^{1/(d+1)}$, so the case $\alpha = 1$ implies Helly number $d+1$. Their proof uses the nerve theorem and the following proposition for d^*-Leray families.

Proposition 2 (Fractional Helly theorem for Leray families [2]). *Let $\mathcal{F}$ be a finite d^*-Leray family, and let $\mathcal{F}^\cap$ the family of all intersections of the sets of $\mathcal{F}$. Then $\mathcal{F}^\cap$ has fractional Helly number $d+1$. Moreover, one can take $\beta(\alpha) = 1-(1-\alpha)^{1/(d+1)}$.*

Here a family $\mathcal{F}$ is called *d^*-Leray* if its nerve complex $N(\mathcal{F})$ is d-Leray: For all induced subcomplexes $L \subseteq N(\mathcal{F})$ the homology groups $H_n(N(\mathcal{F}))$ vanish for all $n \geq d$. As a special case the fractional Helly property holds for the family $\mathcal{F}$. Proposition 2 plays a key role in proving one of the main results of [2], a (p, q)-theorem for finite good covers.

In this paper we identify topological/homological conditions for families of sets which imply that their nerve complex is k-Leray, where k depends on the above conditions. Having Proposition 2 in mind we extend the fractional Helly theorem for good covers to families with higher topological intersection complexity; see Figure 3 for the relations between the results.

Theorem 3 (Topological fractional Helly theorem). *Let $\mathcal{F}$ be a finite family of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$, and $k \geq d$ such that for all subfamilies $\mathcal{G} \subseteq \mathcal{F}$ one of the following conditions holds:*

(i) $\bigcap \mathcal{G}$ is empty, or

(ii) the reduced homology groups of $\bigcap \mathcal{G}$ vanish in dimension at least $k - |G|$, that is

$$\tilde{H}_n(\bigcap \mathcal{G}) = 0 \text{ for all } n \geq k - |\mathcal{G}|.$$

Then $\mathcal{F}^\cap$ has fractional Helly number $k+1$. Moreover, we can choose $\beta(\alpha) = 1-(1-\alpha)^{1/(k+1)}$.

As in the case of good covers this implies fractional Helly number $k+1$ for the family $\mathcal{F}$. We call a family of sets as in Theorem 3 satisfying conditions (i) and (ii) a $(k - |\mathcal{G}|)$-*acyclic* family.

For $k = d$ we obtain fractional Helly number $k+1$ in a more general setting than good covers. Table 1 shows in the first column the conditions in the good cover case: All non-empty intersections are contractible, so their homology vanishes in all dimensions. In the second column the conditions in the $(k - |\mathcal{G}|)$-acyclic case are shown: Non-empty intersections of $i < k$ sets can have arbitrary homology groups in dimension less or equal than $k - i - 1$.

The case $k > d$ admits even more general families of sets in $\mathbb{R}^d$. The price one has to pay for increasing the intersection complexity of the F_i is a higher fractional Helly number. See Figure 2 for an example of the intersection pattern of $k = 3$ sets of a $(3 - |\mathcal{G}|)$-acyclic family in $\mathbb{R}^2$. There the F_{i_j} can have an arbitrary number of 0- and of 1-dimensional holes. The intersection of two elements $F_{i_j} \cap F_{i_k}$ of our family still can have an arbitrary number of 0-dimensional holes.

Matoušek showed a fractional Helly theorem for families with bounded VC-dimension; see [15]. Matoušek's results is not a special case of our result. Bounded

	good cover	$(k - \|\mathcal{G}\|)$-acyclic
$\tilde{H}_n(F_i)$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq k-1, \\ \text{arbit.} & \text{for } n = 0, 1, \ldots, k-2. \end{cases}$
$\tilde{H}_n(F_{i_1} \cap F_{i_2})$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq k-2, \\ \text{arbit.} & \text{for } n = 0, 1, \ldots, k-3. \end{cases}$
$\vdots$	$\vdots$	$\vdots$
$\tilde{H}_n(F_{i_1} \cap F_{i_{k-1}})$	0 for all $n \geq 0$	$\begin{cases} 0 & \text{for all } n \geq 1, \\ \text{arbit.} & \text{for } n = 0. \end{cases}$
$\tilde{H}_n(F_{i_1} \cap \cdots \cap F_{i_t})$	0 for all $n \geq 0$	0 for all $n \geq 0$ and $t \geq k$.

Figure 1: Homological conditions for good covers and for $(k - |\mathcal{G}|)$-acyclic families

VC-dimension does not guarantee any Helly property, e. g. the family $\{[n] \setminus \{i\} \mid i \in [n]\}$ has bounded VC-dimension, but no Helly property. Another important example of families with bounded VC-dimension is the family of all semialgebraic subsets in $\mathbb{R}^d$ of bounded description complexity. Let's look at a concrete example: Define a semialgebraic set $F_i = \{x \in \mathbb{R}^d \mid x_1^2 + x_2^2 - i \geq 0\}$, then the family $\{F_1, F_2, \ldots, F_n\}$ has fractional Helly number $d + 1$. However, we have $\tilde{H}_1(\bigcap_I F_i) = \mathbb{Z} \neq 0$ for all index sets $\emptyset \neq I \subseteq [n]$.

Bárány and Matoušek showed in [4] that the family of convex lattice sets has fractional Helly number $d + 1$, using a Ramsey-type argument. We can not hope to obtain the same fractional Helly number using our results as its Helly number is known to be 2^d.

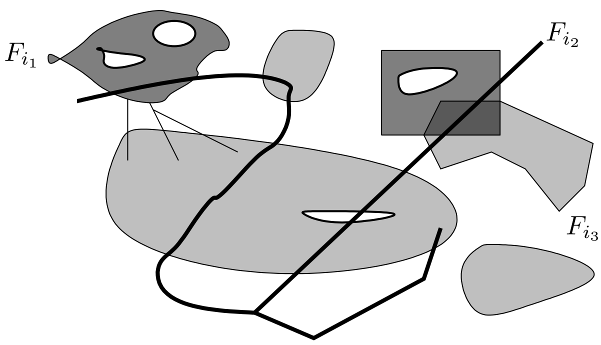

Figure 2: Example of $k = 3$ sets of a $(3 - |\mathcal{G}|)$-acyclic family in $\mathbb{R}^2$

The (p, q)-theorem for convex sets was conjectured by Hadwiger and Debrunner, and proved by Alon and Kleitman [3]. For this let p, q, d be integers with $p \geq q \geq d + 1 \geq 2$. Then there exists a number $\mathrm{HD}(p, q, d)$ such that the following holds: Let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$ satisfying the (p, q)-condition; that is, among any p sets of $\mathcal{F}$, there are q sets with a non-empty intersection. Then $\tau(\mathcal{F}) \leq \mathrm{HD}(p, q, d)$, where $\tau(\mathcal{F})$ denotes the *transversal number* of $\mathcal{F}$, i. e. the smallest cardinality of a set $X \subseteq \bigcup \mathcal{F}$ such that $F \cap X \neq \emptyset$ for all $F \in \mathcal{F}$. It was observed in [2] that the crucial ingredient in the proof is a fractional Helly theorem for $\mathcal{F}^{\cap}$. Therefore Theorem 3 implies immediately a new (p, q)-theorem using the general tools developed in [2].

Theorem 4 ((p,q)-theorem for $(k-|\mathcal{G}|)$-acyclic families). *The assertions of the (p,q)-theorem also hold for finite $(k-|\mathcal{G}|)$-acyclic families of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$ where $p \geq q \geq k \geq d+1 \geq 2$.*

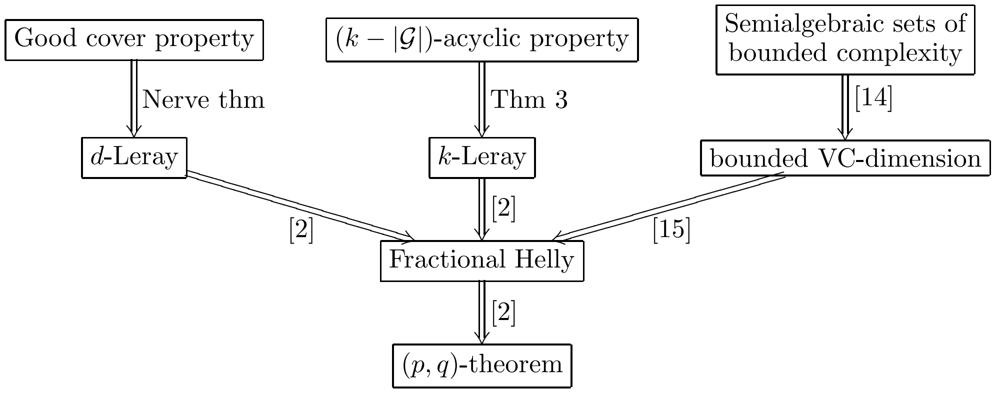

Figure 3: Diagram of the abstract machinery

The proof of Theorem 3 uses a spectral sequence argument. Section 2 comes with a crash course on spectral sequences. In Section 3 we prove Theorem 3. The same technique is used in Section 4 to reprove a homological version of a nerve theorem of Björner.

2 Preliminaries

As a start we give some background and fix our notation. Spanier's book [17] is a good reference for (algebraic) topology. A topological space X is *connected* if it is not the disjoint union of two non-empty open subsets. If X is a connected space then one has $H_0(X,G) = G$ for singular homology with coefficients in G. A topological space X is *contractible* if the identity $i : X \to X$ is homotopic to a constant map $c : X \to X$. The singular homology of a contractible space vanishes in all dimensions except for dimension 0 where it equals the coefficient group G.

Let Δ^p be the standard p-dimensional simplex with vertex set $[p+1]$. The *nerve* $N(\mathcal{F})$ of a family of sets $\mathcal{F}$ is the simplicial complex with vertex set $\mathcal{F}$ whose simplices are all $\sigma \subseteq \mathcal{F}$ such that $\bigcap_{F\in\sigma} F \neq \emptyset$. For a family $\mathcal{F} = \{F_i \,|\, i \in I\}$ of subspaces we define the group of singular n-chains:

$$S_n\{\mathcal{F}\} = \mathbb{Z}^{\{\sigma:\Delta^p \to \bigcup \mathcal{F} \,|\, \mathrm{im}(\sigma)\subseteq F_i \text{ for some } i\}}$$

For finite families of open sets (or of subcomplexes of a cell complex) the inclusion of the singular chain groups $S_*\{\mathcal{F}\} \hookrightarrow S_*(\bigcup \mathcal{F})$ induces an isomorphism in homology. In the following we write $H_*(X) := H_*(X,\mathbb{Z})$ for the singular homology with integer coefficients of a topological space X. For simplicity we use for non-empty spaces X also the reduced singular homology groups $\tilde{H}_n(X)$, thus saving extra considerations for the case $n = 0$. Most of our work also holds for arbitrary (co)homology theories and arbitrary coefficients. As this paper is mainly addressed to people who use algebraic topology as a tool box we abstain from a more general formulation.

Singular homology however shows anomalies first noticed in [5]. Subspaces $A \subseteq \mathbb{R}^d$ which are *not nice* can have non-vanishing homology in infinitely many dimensions. For this let A be the union of countable many spheres of fixed dimension $r > 1$ all having one point in common with their diameter going to zero, then A is such subspace which is not nice. In the literature this example with $r = 1$

is also known as the *Hawaiian earring*. To exclude such not-nice phenomena we consider only families $\mathcal{F}$ of open sets in $\mathbb{R}^d$, and of subcomplexes of CW-complexes (cell complexes) in $\mathbb{R}^d$. In both cases one has $H_n(\bigcup \mathcal{F}) = 0$ for all $n \geq d$.

Spectral sequence of a double complex. Spectral sequences are not standard tools in combinatorics so we repeat some definitions from [16] on *spectral sequences for homology*; see also [6] for a short introduction. Let $C_{*,*}$ a double complex with two differentials $\partial^I : C_{p,q} \to C_{p-1,q}$ and $\partial^{II} : C_{p,q} \to C_{p,q-1}$ such that $\partial^I\partial^{II} + \partial^{II}\partial^I = 0$. We associate to a double complex its total complex $\mathrm{Tot(C)_n} = \bigoplus_{\mathrm{p+q=n}} \mathrm{C_{p,q}}$ with differential $d := \partial^I + \partial^{II}$. The above relation on ∂^I and ∂^{II} implies $d \circ d = 0$. In this paper we are interested in first quadrant sequences so $C_{p,q} = 0$ for $p < 0$ or $q < 0$. Before going into more details we state the following tool for the homology $H_*(\mathrm{Tot(C), d})$.

Theorem 5 (Spectral sequence of a double complex; [16, Theorem 2.15]). *Given a double complex $(C_{*,*}, \partial, \tilde{\partial})$ there are two spectral sequences $(E^r_{*,*}, d^r)$ and $(\tilde{E}^r_{*,*}, \tilde{d^r})$ with*

$$E^2_{*,*} \cong H^{\partial}_{*,*} H^{\tilde{\partial}}(C) \quad \textit{and} \quad \tilde{E}^2_{*,*} \cong H^{\tilde{\partial}}_{*,*} H^{\partial}(C).$$

If $C_{p,q} = 0$ for $p < 0$ and $q < 0$ then both spectral sequences converge to $H_*(\mathrm{Tot(C), d})$.

Here $H^{\partial}_*(C)$ stands for the homology of $C_{*,*}$ with respect to the boundary ∂. The boundary $\tilde{\partial}$ induces a boundary on $H^{\partial}_*(C)$ so that $H^{\tilde{\partial}}_{*,*} H^{\partial}(C)$ is well defined. We now repeat in detail the construction of the spectral sequences to a given double complex.

Definition 6 (Spectral sequence of homological type). *A spectral sequence is a collection of differential bigraded modules, that is, for $r = 1, 2, 3, \ldots$, and $p, q \geq 0$ we have a module $E^r_{p,q}$, and furthermore differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$. Finally, $E^{r+1}_{*,*} \cong H(E^r_{*,*}, d_r)$ for all $r \geq 1$.*

Figure 4 shows the differentials d^r; for r big enough d^r hits the row 0. Let $(C_{*,*}, \partial, \tilde{\partial})$

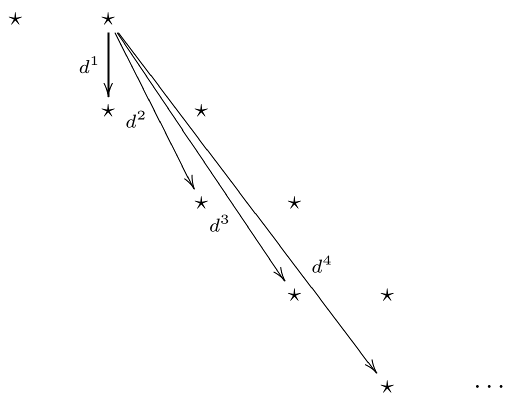

Figure 4: The differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

be a double complex with $\partial : C_{p,q} \to C_{p-1,q}$, $\tilde{\partial} : C_{p,q} \to C_{p,q-1}$, and total differential $d = \partial + \tilde{\partial}$. We define two filtrations of the total complex $(\mathrm{Tot(C)_n, d})$ as follows:

$$F_m(\mathrm{Tot(C)_n}) = \bigoplus_{\mathrm{p \leq m}} \mathrm{C_{p,n-p}} \quad \text{and} \quad \mathrm{\tilde{F}_m(Tot(C_n))} = \bigoplus_{\mathrm{q \leq m}} \mathrm{C_{n-q,q}}.$$

From now on we do construction for the first filtration. This goes over to the construction for the second filtration by reindexing the double complex as its transpose

$C^T_{p,q} = C_{q,p}$, $\partial^T = \tilde{\partial}$, $\tilde{\partial}^T = \partial$. The filtration is increasing and respects the total differential d:

$$0 \cdots \subseteq F_{m-1}(\mathrm{Tot(C)_n}) \subseteq \mathrm{F_m(Tot(C)_n)} \subseteq \cdots \mathrm{Tot(C)_n}$$

and,

$$d(F_m(\mathrm{Tot(C)_n})) \subseteq \mathrm{F_m(Tot(C)_n)}.$$

Since d respects the filtration the homology of (C, d) inherits a filtration

$$0 \subseteq \cdots \subseteq F_{m-1}(H(\mathrm{Tot(C), d})) \subseteq \mathrm{F_m(H(Tot(C), d))} \subseteq \cdots \subseteq \mathrm{H(Tot(C), d)},$$

where $F_m(H(\mathrm{Tot(C), d})) := \iota^{\mathrm{m}}_*(\mathrm{H(F_m(Tot(C)), d)}) \subseteq \mathrm{H(Tot(C), d)}$, and $\iota^m : F^m(\mathrm{Tot(C)}) \hookrightarrow \mathrm{Tot(C)}$ is the inclusion. We know from Theorem 5 that the associated spectral sequence converges to $H(\mathrm{Tot(C), d})$; more precisely, there is a $r > 0$ such that $d_r = 0$ is trivial. This is called *the sequence collapses at term r*. Therefore we have

$$E^{r+1}_{p,q} = E^{\infty}_{p,q} \cong F^p(H_{p+q}(\mathrm{Tot(C), d}))/\mathrm{F^{p-1}(H_{p+q}(Tot(C), d))}.$$

Hence, one gets $H_n(C, d) \cong \bigoplus_{p+q=n} E^{\infty}_{p,q}$ in the case of homology with field coefficients. In the case of integer coefficients this leads to an extension problem. In this paper the extension problem is trivial as only one of the groups $E^{\infty}_{p,q}$ with $n = p + q$ is different from zero, namely either $E^{\infty}_{n,0}$ or $E^{\infty}_{0,n}$. Therefore $H_n(\mathrm{Tot(C), d})$ equals either $E^{\infty}_{n,0}$ or $E^{\infty}_{0,n}$.

Consider the following definitions for $r \geq 0$

$$\begin{aligned} Z^r_{p,q} &= F_p(\mathrm{Tot(C)_{p+q}}) \cap \mathrm{d^{-1}(F_{p-r}(Tot(C)_{p+q-1}))} \\ B^r_{p,q} &= F_p(\mathrm{Tot(C)_{p+q}}) \cap \mathrm{d(F_{p+r}(Tot(C)_{p+q+1}))} \\ Z^{\infty}_{p,q} &= F_p(\mathrm{Tot(C)_{p+q}}) \cap \mathrm{ker(d)} \\ B^{\infty}_{p,q} &= F_p(\mathrm{Tot(C)_{p+q}}) \cap \mathrm{im(d)} \end{aligned}$$

Then this leads to a tower of submodules

$$B^0_{p,q} \subseteq B^1_{p,q} \subseteq \cdots \subseteq B^{\infty}_{p,q} \subseteq Z^{\infty}_{p,q} \subseteq \cdots \subseteq Z^1_{p,q} \subseteq Z^0_{p,q}.$$

For a first quadrant sequence we get that for $r > \max\{p, q\}$

$$Z^r_{p,q} = Z^{\infty}_{p,q} \text{ , and } B^r_{p,q} = B^{\infty}_{p,q}$$

hold. This insures the convergence of our spectral sequence. Define

$$E^r_{p,q} = Z^r_{p,q}/(Z^{r-1}_{p-1,q} + B^{r-1}_{p,q})$$

It can be checked that the differential $d : Z^r_{p,q} \to Z^r_{p-r,q+r-1}$ induces a unique homomorphism d^r such that the following diagram commutes:

$$\begin{array}{ccc} Z^r_{p,q} & \xrightarrow{d} & Z^r_{p-r,q+r-1} \\ \downarrow{\scriptstyle \eta^r_{p,q}} & & \downarrow{\scriptstyle \eta^r_{p-r,q+r-1}} \\ E^r_{p,q} & \xrightarrow{d^r} & E^r_{p-r,q+r-1} \end{array}$$

where the maps $\eta^r_{p,q} : Z^r_{p,q} \to E^r_{p,q}$ are the canonical projections. The same constructions leads to the spectral sequence $(\tilde{E}^r, \tilde{d}^r)$, $\tilde{d}^r : \tilde{E}^r_{p,q} \to \tilde{E}^r_{p+r-1,q-r}$, for the second filtration.

3 On $(k - |\mathcal{G}|)$-acyclic families

In this section we prove our main result Theorem 3. The following lemma is the key argument in extending the fractional Helly theorem from good covers to $(k - |\mathcal{G}|)$-acyclic families.

Lemma 7. *For $k \geq 0$ let $\mathcal{F}$ be a finite $(k - |\mathcal{G}|)$-acyclic family of open sets (or of subcomplexes of a cell complex) in $\mathbb{R}^d$. Then $H_n(\bigcup \mathcal{F}) \cong H_n(N(\mathcal{F}))$ for all $n \geq k$.*

To prove Lemma 7 we first define a suitable double complex $(C_{*,*}, \partial, \tilde{\partial})$. Then we compute its $E^2_{*,*}$- and $\tilde{E}^2_{*,*}$-term which are shown in Figures 5 and 6. Finally, we apply Theorem 5 to get the conclusion.

Proof. For $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ define a double complex

$$C_{p,q} := \bigoplus_{J \subseteq [n],\, |J| = q+1} S_p(\bigcap_{j \in J} F_j) \quad \text{for } p, q \geq 0.$$

Let $\partial := \oplus \partial : C_{p,q} \to C_{p-1,q}$ be the usual singular boundary operator. For index sets $J' \subseteq J \subseteq [n]$ let $i^{J,J'} : \bigcap_{j \in J} F_j \to \bigcap_{j \in J'} F_j$ be the inclusion. The group $S_p(\bigcap_{j \in J} F_j)$ is freely generated by the set of singular p-simplices $\sigma : \Delta^p \to \bigcap_{j \in J} F_j$. Define $\tilde{\partial}$ component-wise on the elements $c = \sum r_\sigma \sigma$ of $S_p(\bigcap_{j \in J} F_j)$:

$$c \mapsto \tilde{\partial}(\sigma) := (-1)^p \sum_{\sigma} \sum_{i=0}^{q} (-1)^i r_\sigma i_*^{J,J_i}(\sigma) \ \in \bigoplus_{J \subseteq [n], |J| = (q-1)+1} S_p(\bigcap_{j \in J} F_j),$$

where $J_i = \{j_0 < j_1 < \ldots < \hat{j}_i < \cdots < j_q\}$ is the set obtained from J by deleting the element j_i, and the factor $(-1)^p$ is added to guarantee $\partial\tilde{\partial} + \tilde{\partial}\partial = 0$.

We show that (i)

$$E^2_{p,q} = H^{\partial}_{p,q} H^{\tilde{\partial}}(C) = \begin{cases} H_p(\bigcup \mathcal{F}) & \text{for } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that (ii)

$$\tilde{E}^2_{p,q} = H^{\tilde{\partial}}_{p,q} H^{\partial}(C) = \begin{cases} H_q(N(\mathcal{F})) & \text{for } p = 0,\ q \geq k \\ 0 & \text{for } p = i,\ q = n - i, \text{ for all } 1 \leq i \leq n \text{ and } n \geq k-1. \end{cases}$$

The E^2-term is equal to zero except for the zeroth column where we get the homology of $\bigcup \mathcal{F}$, see Figure 5. The $\tilde{E}^2$-term contains in the zeroth row the homology of $N(\mathcal{F})$ for dimensions greater or equal than k, at the same time vanish all terms above the $(k-1)$-th anti-diagonal, see also Figure 6. Hence the $\tilde{E}^r$-sequence collapses at term 2 above this anti-diagonal. Using Theorem 5 we obtain

$$H_n(\bigcup \mathcal{F}) \cong H_n(\mathrm{Tot(C)}) \cong \mathrm{H_n(N(\mathcal{F}))} \text{ for all n} \geq \text{k}.$$

To obtain (i) we first compute $H^{\tilde{\partial}}(C_{p,*})$. Using the definition of singular homology one gets:

$$C_{p,q} = \bigoplus_{J \subseteq [n], |J| = q+1} \ \bigoplus_{\sigma : \Delta^p \to X, \mathrm{im}(\sigma) \subseteq \bigcap_{j \in J} F_j} \mathbb{Z}$$

For $\sigma : \Delta^p \to X$ let J_σ be the maximal subset $J \subseteq [n]$ with $\mathrm{im}(\sigma) \subseteq \bigcap_{j \in J} F_j$. Then this leads to:

$$C_{p,q} = \bigoplus_{\sigma : \Delta^p \to X} \ \bigoplus_{J \subseteq [n],\, J \subseteq J_\sigma,\, |J| = q+1} \mathbb{Z}$$

$H_n(\bigcup \mathcal{F})$	0	0	0	$\cdots$
$\vdots$	0	0	0	$\cdots$
$H_1(\bigcup \mathcal{F})$	0	0	0	$\cdots$
$H_0(\bigcup \mathcal{F})$	0	0	0	$\cdots$

Figure 5: E^2-term in the proof in the proof of Lemma 7

Notice that for $J_\sigma = \emptyset$ a (innocent) zero is added to the direct sum. The boundary $\tilde{\partial}$ has no effect on σ so that we can look at every component

$$\bigoplus_{J \subseteq [n],\, J \subseteq J_\sigma,\, |J|=q+1} \mathbb{Z}$$

separately. For $J_\sigma \neq \emptyset$ one can check that this equals the simplicial chain complex of a $(|J_\sigma|-1)$-dimensional simplex. The simplicial homology of the simplex vanishes in all dimensions except dimension 0 where it equals $\mathbb{Z}$. Using that

$$\bigoplus_{\sigma:\Delta^p \to X} \bigoplus_{J \subseteq [n],\, J \subseteq J_\sigma,\, |J|=1} \mathbb{Z} = S_p(\mathcal{F}),$$

we obtain:

$$H_q^{\tilde{\partial}}(C_{p,*}) = \begin{cases} S_p(\mathcal{F}) & \text{for } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The induced boundary ∂ can be identified with the usual boundary on $S_*(\mathcal{F})$. For families of open sets or of subcomplexes of a cell complex the inclusion $S_*(\mathcal{F}) \to S_*(X)$ induces an isomorphism $H_*(\mathcal{F}) \to H_*(X)$ in homology.

To prove (ii) we start with computing $H^\partial(C_{*,q})$, we make use of the $(k-|\mathcal{G}|)$-acyclicity of our family. By the definition of ∂ we know that

$$H_p^\partial(C_{*,q}) = \bigoplus_{J \subseteq [n],\, |J|=q+1} H_p(\bigcap_{j \in J} F_j).$$

If $\bigcap_{j\in J} F_j = \emptyset$ then the contribution to this sum equals 0. In the case $\bigcap_{j \in J} F_j \neq \emptyset$ the summand depends on our assumption $\tilde{H}_n(\bigcap_{j\in J} F_j) = 0$ for $n \geq k - |J|$. E. g. for $p = 0$ and $|J| \geq k$ the summand equals $\mathbb{Z}$, and in general this leads to

$$H_p^\partial(C_{*,q}) = \begin{cases} \bigoplus_{J \subseteq [n],\, |J|=q+1,\, \bigcap_{j\in J} F_j \neq \emptyset} \mathbb{Z} & \text{for } p = 0,\ q \geq k-1 \\ 0 & \text{for } p = i,\ q = n-i, \text{ for all } 1 \leq i \leq n \text{ and } n \geq k-1. \end{cases}$$

It is easy to check that the induced chain complex $(H_0^\partial(C_{*,*}), \tilde{\partial})$ and the simplicial chain complex $C_*(N(\mathcal{F}))$ are isomorphic in dimension $\geq k$. The chain groups are even isomorphic in dimensions $\geq k-1$, but the differential differ in general in dimension $k-1$. To see that differentials coincide up to a sign in dimensions $\geq k$ remember that $H_0(\bigcap_{j\in J} F_j)$ is freely generated by the class of any 0-simplex, and i_*^{J,J_i} maps the class of a 0-simplex on a class of a 0-simplex. Hence row 0 of the E^2-term contains the homology $H_n(N(\mathcal{G}))$ for $n \geq k$. □

0	0	0	0	$\cdots$
$\star$	0	0	0	$\cdots$
$\star$	$\star$	0	0	$\cdots$
$\star$	$\star$	$\star$	$H_k(N(\mathcal{G}))$	$\cdots$

Figure 6: The $\tilde{E}^2$-term in the proof of Lemma 7

After these preparations the proof of Theorem 3 is short. In the next section we make use of the same double complex $(C_{*,*}, \partial, \tilde{\partial})$. Similar computations lead to a nice proof of a homological version of a nerve theorem of Björner.

Proof. (of **Theorem 3**) We prove that our family $\mathcal{F}$ is $k^\star$-Leray, hence Theorem 3 follows immediately from Theorem 2. For this let $L \subseteq N(\mathcal{F})$ be an induced subcomplex. Then L is of the form $N(\mathcal{G})$ for some $\mathcal{G} \subseteq \mathcal{F}$. The family $\mathcal{G}$ is again $(k - |\mathcal{G}|)$-acyclic so that Lemma 7 implies

$$H_n(\bigcup \mathcal{G}) \cong H_n(N(\mathcal{G})) \text{ for all } n \geq k.$$

Finally we have $H_n(\bigcup \mathcal{G}) = 0$ for all $n \geq d$ as $\bigcup \mathcal{G}$ is a *nice* subset of $\mathbb{R}^d$. □

4 On nerve theorems

The nerve theorem is a standard tool in topological combinatorics. It was first obtained by Leray [13]; see Björner [7] for a survey on nerve theorems. Here we prove a homological version of a nerve theorem of Björner. For this recall that for $k \geq -1$ a topological space X is *k-connected* if for every $l = -1, 0, 1, \ldots, k$, each continuous map $f : S^l \to X$ can be extended to a continuous map $\bar{f} : B^{l+1} \to X$. Here the (-1)-dimensional sphere S^{-1} is interpreted as $\emptyset$, and the 0-dimensional ball B^0 as a single point. Hence (-1)-connected means non-empty, and 0-connected means pathwise connected.

Theorem 8 (Nerve Theorem II [8], homology version). *For $k \geq 0$ let $\mathcal{F}$ be a family of open sets (or a finite family of subcomplexes of a cell complex) such that every $\bigcap \mathcal{G}$ is empty or $(k - |\mathcal{G}| + 1)$-connected for all non-empty subfamilies $\mathcal{G} \subseteq \mathcal{F}$. Then $H_n(X) = H_n(N(\mathcal{F}))$ holds for $X = \bigcup \mathcal{F}$ and all $n \leq k$.*

Proof. As in the proof of Lemma 7 we have that $H_*(X) \cong H_*(\mathrm{Tot(C)}, \mathrm{d})$. For a k-connected space X we know from a famous theorem of Hurewicz that $H_n(X) = 0$ for all $n \leq k$. Using the conditions on the connectivity of $\mathcal{G}$ and analogous arguments as in the proof of Lemma 7 the $\tilde{E}^2$-term looks as in Figure 7. Hence we see that $H_n(N(\mathcal{F})) \cong H_n(C, d)$ for all $n \leq k$. □

Acknowledgement

I thank Carsten Schultz for suggesting the use of the spectral sequence argument, and my advisor Günter M. Ziegler for helpful discussions. Many thanks also to Jiří

0	$\star$			
0	0	$\star$		
0	0	0	$\star$	
0	0	0	0	$\star$
$H_0(N(\mathcal{F}))$	$H_1(N(\mathcal{F}))$	$\cdots$	$H_{k-1}(N(\mathcal{F}))$	$H_k(N(\mathcal{F}))$

Figure 7: The $\tilde{E}^2$-term in the proof of Theorem 8

Matoušek for his warm hospitality at KAM, Prague, and for introducing me to the problem.

References

[1] N. Alon and G. Kalai, *Bounding the piercing number*, Discr. Comp. Geom. **13** (1995), pp. 245–256.

[2] N. Alon, G. Kalai, J. Matoušek, and R. Meshulam, *Transversal numbers for hypergraphs arising in geometry*, Adv. Appl. Math. **29** (2003), pp. 79–101.

[3] N. Alon and D. J. Kleitman, *Piercing convex sets and the Hadwiger Debrunner (p, q)–problem*, Adv. Math. **96** (1992), pp. 103–112.

[4] I. Bárány and J. Matoušek, *A fractional Helly theorem for convex lattice sets*, Adv. Math. **174** (2003), pp. 227–235.

[5] M. G. Barratt and J. Milnor, *An example of anomalous singular homology*, Proc. Amer. Math. Soc. **13** (1962), pp. 293–297.

[6] S. Basu, *Different bounds on the different Betti numbers of semi–algebraic sets*, Discr. Comp. Geom. **30** (2003), pp. 65–85.

[7] A. Björner, *Topological methods*, in Handbook of Combinatorics, R. Graham, M. Grötschel, and L. Lovász, eds., North Holland, Amsterdam, 1995, pp. 1819–1872.

[8] A. Björner, *Nerves, fibers and homotopy groups*, J. Combinatorial Theory Ser. A **102** (2003), pp. 88–93.

[9] G. Chichilnisky, *Intersecting families of sets and the topology of cones in economics*, Bull. Am. Math. Soc., New Ser. **29** (1993), pp. 189–207.

[10] J. Eckhoff, *Helly, Radon and Carathéodory type theorems*, in Handbook of Convex Geometry, P. M. Gruber and J. M. Wills, eds., North–Holland, 1993, pp. 389–448.

[11] G. Kalai, *Intersection patterns of convex sets*, Isr. J. Math. **48** (1984), pp. 161–174.

[12] M. Katchalski and A. Liu, *A problem of geometry in* $\mathbb{R}^d$, Proc. Amer. Math. Soc. **75** (1979), pp. 284–288.

[13] J. Leray, *Sur la forme des espaces topologiques et sur les points fixes des représentations*, J. Math. Pures Appl. **24** (1945), pp. 95–167.

[14] J. Matoušek, *Lectures on Discrete Geometry*, Graduate Texts in Mathematics 212, Springer, 2002.

[15] J. Matoušek, *Bounded VC–dimension implies a fractional Helly theorem*, Discr. Comp. Geom. **31** (2004), pp. 251–255.

[16] J. McCleary, *User's Guide To Spectral Sequences*, Cambridge studies in advanced mathematics 58, Cambridge University Press, 2 ed., 2001.

[17] E. H. Spanier, *Algebraic Topology*, McGraw–Hill, 1966.