Visualizing Non-Euclidean Geometry
A Bolyai Bicentennial Survey

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1 Introduction

The setting Like many fundamental mathematical discoveries, non-euclidean geometry was first received as a bizarre oddity, but with time it has entered the mainstream of scientific thought. Indeed, there is growing acceptance of the idea that the universe we live in, is locally or globally non-euclidean. This makes the development of techniques for visualizing non-euclidean geometry of more than purely academic interest. This article will attempt to sketch what sorts of techniques have been developed, and what challenges still remain. The focus in the article is on hyperbolic geometry, although elliptic geometry is given attention also. After quickly reviewing the various mathematical models that have been developed for these geometries, we consider some simple examples of how graphics can aid the understanding of plane hyperbolic geometry. Then we will turn our attention to the more challenging case of three dimensions, and investigate in more depth two case studies of visualization projects in this area, one hyperbolic and one spherical. Here we indicate connections with the growing disciplines of 3D image synthesis and photorealistic rendering. Our exposition makes no attempt to be complete or rigorous, but aims to be a trail guide with references to sources for readers requiring more rigorous details.

Non-euclidean geometry challenges some of the deepest human preconceptions about space. Much

Figure 1: Orange, pyrite, kale: Natural models of spherical, euclidean, and hyperbolic geometry
ingenuity has been devoted to helping people understand it. Just as the reasons for wanting to form pictures of non-euclidean geometry vary widely, from pure research to popular education, so the means at hand to form those pictures run the gamut of technical sophistication. Fortunately Nature provides a collection of forms which can serve as an introduction to the theme. Perhaps the most accessible route to comprehending the discovery is provided by the surface of a sphere, for example, the surface of the Earth. One can speak of marching straight ahead, and accept that such a path, though it circles around and comes back to its starting point, is analogous to a straight line because it provides the shortest of all available paths between two points. With a little thought, one sees that any two such paths must cross, so parallel lines do not exist here. But much of the rest of geometry can be done: distances and angles measured, and triangles solved.

In fact, considering the importance of spherical trigonometry to astronomy, it might appear curious that no one thought of inventing “spherical geometry” as an independent alternative to Euclid until 180 years ago. But strictly speaking, spherical geometry is not a “non-euclidean geometry” since two intersecting lines intersect in two points, not one. It took the (re-)invention of projective geometry in the early 19th century to give a mathematical basis for a true non-euclidean geometry based on the sphere: elliptic geometry. This geometry is identical to spherical except that opposite points on the sphere are identified, and a single intersection point is thereby established. The resulting geometry has its own imaginative challenges, since it is non-orientable. The terms spherical and elliptic are often used interchangeably, but the reader should be aware of the distinction.

Nature provides many spheres for our edification in this regard; she is much more parsimonious with respect to hyperbolic surfaces. The closest one comes to finding a hyperbolic surface in nature is perhaps a kale leaf. (See the photographs in Figure 1.) Here, instead of rounding itself off modestly, the surface ripples and folds on itself, as if overflowing the space provided. In contrast to the sphere’s compact, convex surface, here the surface is saddle-shaped, curving here upwards, and there downwards. Following with the mind’s eye, the boundary of a kale leaf gives a convincing experience of how dramatically the circumference of a circle in hyperbolic space grows as the radius increases. Finally, the surface of a crystal such as pyrite is as close to a perfect euclidean plane that Nature comes: the surface of water, though it may appear flat, always takes the form of a sphere, though it can be as large as the earth itself!

Nineteenth century math, twenty-first century technology These simple objects provide primitive examples of how one can visualize non-euclidean geometry in two dimensions. But when one attempts to understand the situation in three dimensions, nature provides no simple analogies and one must create tools to assist the human imagination. Since Gauss’s measurement of a huge geodesic triangle to determine if the sum of the angles diverges from 180°, all efforts to detect a global non-euclidean reality have remained inconclusive. For all practical purposes, the space we live in is euclidean. However, in the last 20 years computer graphics techniques have been developed that open windows onto non-euclidean geometry in both two and three dimensions, so that now everyone can experience “virtual” hyperbolic and elliptic space. The essential ideas, however, spring from 19th century mathematics. In this paper I would like to describe the mathematics and the technical implementations in more detail. My focus here will be on the hyperbolic case, since that tends to present the most challenge, and the most fascination, for most people.
Figure 2: Tessellations of spherical, euclidean, and hyperbolic geometry by a $23n$ triangle ($n=5,6,7$)

**About tessellations** A common thread in the pioneering work of Klein, of Poincare, and also of the artist Escher, has been that the investigation of non-euclidean geometry has gone hand-in-hand with the investigation of tessellations of these spaces. There are practical reasons for this, since tessellations provide a simple and direct way to “fill-up” these new spaces with familiar motifs from our euclidean life, to give visual content where none is naturally present, and provide scenery which reveals the qualities and features of the underlying geometry by how it presents itself to the observer.

There are, however, more profound reasons to be interested in tessellations. Current theories about the origin of the universe in the Big Bang, posit a universe that has finite volume but is unbounded. Such a space cannot be the traditional infinite expanse of euclidean space. The alternatives are the object of study in three-dimensional topology, and are called manifolds. An inhabitant of a manifold will, in general, experience the manifold as a tessellation, in which everything appears multiple times; finitely many in an elliptic manifold, and infinitely many in a euclidean or hyperbolic one. An excellent introduction to this theme is [Wee85], or the associated instructional video [Wee00].

There are many tessellations available for use. Figure 2 shows three simple very similar tessellations, all generated by reflections in the sides of a triangle two of whose angles are $\pi/2$ and $\pi/3$. The other angles are $\pi/7$, $\pi/6$, and $\pi/5$. These are prototypes for the three “classical” geometries: elliptic, euclidean, hyperbolic. Most of the planar tessellations used in this article involve regular pentagons of various types; most of the three dimensional tessellations involve regular pentagonal dodecahedra. The reader interested in more details about how the tessellations themselves are calculated is referred to [Gun93] and [Lev92].

2 Models for hyperbolic geometry

One of the first anomalies of hyperbolic geometry was the realization that there was no isometric embedding in euclidean space, unlike the case of elliptic geometry. The 19th century witnessed the development of a family of non-isometric embeddings, or *models*, for hyperbolic geometry in any dimension. There are models embedded in a higher dimensional euclidean space, as well as ones in the same dimension. The latter can be divided into conformal and projective models. In the conformal camp belong the Poincare and upper-half space model; the projective model is also sometimes called the Klein or Cayley-Klein model (although to be accurate, Beltrami was the first
to propose it). As the name suggests, the conformal models represent angles faithfully, but the cost is that geodesics are represented by arcs of circles rather than straight lines. The situation is reversed in the projective case; here geodesics are shown as (euclidean) straight lines but angles are not faithfully shown. We look at four of these models in more detail now. Readers desiring more detail are referred to [Thu97].

2.1 The Klein, or Projective, Model

The Cayley-Klein derivation of the hyperbolic plane begins with the projective plane. We have homogeneous coordinates \((x, y, w)\) for the plane. We then choose a quadratic form, in this case \(Q^- = x^2 + y^2 - w^2\) (Other geometries arise with other choices). The condition \(Q^- = 0\) is called the Absolute Conic. In this case, it is, after dehomogenizing, the unit circle \(x^2 + y^2 = 1\). And on the interior of this circle, the unit disk \(D^2\), it is possible to establish a distance function based on the quadratic form \(Q^-\) and the projective invariant known as the cross ratio. This distance function can also be expressed using the indefinite inner product \(\bullet\) associated to \(Q^-\), defined for \(P = (x_1, y_1, w_1)\) and \(Q = (x_2, y_2, w_2)\) as \(P \bullet Q = x_1x_2 + y_1y_2 - w_1w_2\). This distance function can then be shown to give rise to a model of hyperbolic geometry, the so-called Klein or projective model. In this metric, the points of the Absolute Conic, the unit circle, are not accessible; they lie at an infinite distance from any point within the disk. The circle is sometimes referred to as the “circle at infinity” and can be useful in describing hyperbolic geometry even though it is not part of the model itself. (The same applies to the Poincare model described below).

2.2 Conformal Models

The most famous of the conformal models, the Poincare model, can be derived from the projective model by a sequence of simple projections: first project orthographically down onto the southern hemisphere of the unit sphere; then project back onto the unit disk by central projection from the North Pole of the unit sphere \((0,0,1)\). The result is to map the straight lines of the projective model onto circles orthogonal to the unit circle. Figure 4 illustrates this. A typical point \(P\) in the
projective model is first transformed to a point $H$ on the lower hemisphere, and then projected back up to $C$ in the Poincare model.

The points of intersection with the unit circle are left unchanged by the transformation. These circular arcs, then, are the straight lines of the Poincare model. It can be shown that the (euclidean) angle of intersection of these arcs agrees with the hyperbolic measure of these angles (derived from $\cdot$), hence it is a conformal model. It is also possible to develop hyperbolic geometry independently in this model, as was done historically, using complex numbers. The intermediate projection onto the northern hemisphere is also sometimes used as a model for hyperbolic geometry; it is also conformal and the isometries, considered as acting on the Riemann sphere, are simply Möbius transformations with real coefficients. Finally, the upper half plane model can be derived from the Poincare disk model by a conformal map (a Möbius transformation) that sends the unit circle to the real line, and the center of the circle to $i$. Now, the straight lines are represented by circles orthogonal to the real axis.

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2.3 The hyperboloid model

Beginning with $\mathbb{R}^3$ as a topological space, adjoin the quadratic form $Q^-$, mentioned above in the projective model. The condition $Q^- = -1$ defines a hyperboloid of two sheets. If we restrict attention to the sheet $H^-$ where $w > 0$ we get a simply connected model of the hyperbolic plane, the so-called hyperboloid model. While it is not itself used to visualize hyperbolic geometry, it is very important as a computational aid for the other models, so we devote some space to describing it.

Riemannian manifold Given a point $P \in H^-$, it is possible to show that the tangent space at $P$ is the orthogonal complement of $P$ with respect to $\cdot \cdot$. In fact, the inner product $\cdot \cdot$ is positive definite on the tangent space, and this gives rise, using standard techniques of differential geometry, to a Riemannian manifold. The geodesic between two points $p$ and $v$ is simply the intersection of $H$ with the plane through the origin, containing $p$ and $v$. And, this set of geodesics satisfies the axioms of hyperbolic geometry, that there are infinitely many lines through a point, which do not
intersect a given line.

Central projection from the origin \((0, 0, 0)\) establishes a 1-1 correspondence between the points of the projective model, and the points of \(H^-\). The cone \(H^0\) (where \(Q^- = 0\)) maps onto the unit circle. The metrics and the geodesics correspond under this projection. Using this hyperboloid in this way is analogous to using the unit sphere to model elliptic geometry; we might say the hyperboloid is the sphere of radius \(i\) when distances are determined by \(\bullet\) instead of the standard inner product.

**Minkowski coordinates** The hyperboloid model generalizes to any dimension. It can be thought of as a computational aid for the projective model. There is great freedom in choosing homogeneous coordinates; \(H^-\) is one way to select coordinates for the points in the projective model of hyperbolic geometry, in which the points have “unit length” = \(i\). These coordinates are referred to as Minkowski coordinates, and many formulae are simplified when points are represented by Minkowski coordinates. For example, the distance function between two points \(P\) and \(Q\) on \(H^-\) is given by \(\cosh^{-1}(P \bullet Q)\).

There are also Minkowski coordinates on tangent vectors. Consider for example the point \(O = (0, 0, 1)eH^-\). Its tangent plane consists of all vectors of the form \((a, b, 0)\). These vectors can be normalized to have unit length = 1, not \(i\); they lie on the surface \(H^+\) defined by \(Q^- = 1\); this is a hyperboloid of revolution (see Figure 5). Since \(O\) can be moved to any other point in \(H^-\) by a hyperbolic isometry which preserves \(Q^-\), the same observations hold for any point in \(H^-\). The region \(Q^- < 0\) is sometimes referred to as “time-like”; that where \(Q^- > 0\) is sometimes referred to as “space-like”. So we can describe the points of the hyperboloid model as time-like, while the tangent vectors are space-like. Three dimensional shading calculations (see Section 4.1) in hyperbolic space make heavy use of Minkowski coordinates for both points and vectors.

### 2.4 Comparisons

Three of the models mentioned above are shown in Figure 3, tessellated with the same pattern of regular right-angled pentagons. To be precise, the fundamental domain is such a pentagon; the pentagon that has been drawn is a slightly smaller pentagon, so that there is space from one copy to its neighbors. The drawn pentagon is not just reduced in size; its angles are actually slightly larger – only in euclidean space do similar figures exist.

Each of these models has advantages and disadvantages that make the one or the other more appropriate for particular applications. For example, the Poincare disk model, besides displaying accurate angles, has the advantage that it takes less euclidean area to display the same geometry, as the projective model, so that more is visible at the same time. This effect is especially pronounced as one approaches the circle at infinity. The upper half plane model excels at showing the boundary behavior. The projective model represents hyperbolic lines as euclidean lines – a feature that will become more important in three dimensions (see Section 4).

The disk has the advantage of being in itself a two-dimensional subspace of euclidean space. Another
One attractive feature of the disk model is that the exterior of the disk also provides a model for hyperbolic geometry.

One property of the projective model that will be important in the sequel is that it is conformal at one point, the center of the disk. For many computational purposes, this is almost as good as being conformal everywhere, since calculations can be done using euclidean angles at the origin, and then moved to another point in the disk using hyperbolic isometries. In this paper we will focus on the projective model since there are projective models for all three classical geometries, and there are simple and uniform procedures for calculating in the projective models. If pictures using the other models are desired, they may be derived as part of the display process, as outlined above. For example, a straight line segment in the projective model is stored as a euclidean line segment; to draw it in a conformal model, one simply needs to convert it to the arc of the appropriate circle.

3 Calculating in non-euclidean plane geometry

Once a non-euclidean metric has been defined, it is possible to develop a wide range of techniques for investigating geometric situations. For example, hyperbolic trigonometry can be developed with many results mirroring spherical trigonometry, except using hyperbolic trigonometric functions such as \( \cosh, \sinh, \) and \( \tanh. \) Areas and volumes can be calculated. Tessellations can be classified and described, as can the unique aspects of parallel and “ultraparallel” lines. Function theory can be carried out, and harmonic analysis of functions developed. Much of this program was already done, as noted above, in the 19th century.

An example of hyperbolic trigonometry

To get a feel for what is involved in calculating in hyperbolic geometry, we include an example. How does one calculate the regular right-angled pentagon shown in Figure 3? The first step is to notice that it can be broken into 10 congruent right-angled triangles with angles \( A = \pi/2, \) \( B = \pi/4, \) and \( C = \pi/5. \) In analogy to spherical trigonometry, there are formulae to solve hyperbolic triangles. In this case, we need to solve for \( a, \) the longest side of the triangle. This will be the hyperbolic distance of the vertex of the pentagon from the center of the pentagon. If we place the center at the origin, and the first vertex on the
y-axis, then the other vertices can be positioned by rotating the first by multiples of $2\pi/5$.

$$\cosh(a) = \frac{(\cos(B)\cos(C) + \cos(A))/\sin(B)\sin(C)}{(\cos(\pi/4)\cos(\pi/5) + \cos(\pi/2))/\sin(\pi/4)\sin(\pi/5)}$$

(1)  
(2)

If one carries this out, one arrives at the result that $a = 0.842482...$. By using the distance function described above, we conclude that the Minkowski coordinates of the pentagon’s vertex should be $(0, \sinh(a), \cosh(a)) = (0, 0.945742..., 1.37638...)$. Dehomogenizing yields $(0, 0.687123...) as the euclidean coordinates for the vertex. This is how the pentagon’s position and size were determined for the tessellations shown.

**Isometries**  
We let $H^n$ represent $n$-dimensional hyperbolic space. The isometries of the projective model are projective transformations which preserve the Absolute Conic $Q^- = 0$. The projective transformations can be represented by $(n + 1, n + 1)$ matrices with non-zero determinant, forming the group $PGL(R, n + 1)$, the projective general linear group. The isometries of n-dimensional hyperbolic geometry are then given by the subgroup $O(n, 1)$, the Lorenz group in dimension $n$. These isometries act transitively and isotropically on hyperbolic space. For example, any point $p$ in $H^2$ may be moved to the point $O = (0, 0, 1)$, and any element $v_p$ of the tangent space $T_p$ can be then brought into coincidence with a prescribed element of the tangent space at $O$ via a euclidean rotation, an element of $SO(n)$, which is a subgroup of $SO(n, 1)$. (Similar reasoning shows that the isometry group of elliptic geometry in dimension $n$ can be represented by the matrix group $SO(n + 1)$).

**An example isometry in $H^2$**  
Like the euclidean plane, planar hyperbolic isometries include rotations, translation, reflections and glide reflections. We give an example here to show what sorts of mathematics are required to construct and use such isometries. Figure 8 shows an example of a hyperbolic isometry acting on the pentagonal fundamental region seen in the previous tessellations. This isometry is like a euclidean translation in that it has no fixed points. But, unlike a euclidean translation, not all points move the same distance; there is a unique axis, in this case the vertical diameter of the disk, along which the distance moved is a minimum. The equation below shows the
matrix form $M$ of such a hyperbolic translation, and how it acts upon an arbitrary points $(x, y, w)$ in homogeneous coordinates:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cosh(d) & \sinh(d) \\
0 & \sinh(d) & \cosh(d)
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
w
\end{pmatrix}
=
\begin{pmatrix}
x \\
\cosh(d)y + \sinh(d)w \\
\sinh(d)y + \cosh(d)w
\end{pmatrix}
$$

The reader is encouraged to confirm that $M \overrightarrow{x} = \overrightarrow{x}'$ preserves the inner product $\cdot$ and hence is a hyperbolic isometry. In the case of the translation shown in Figure 8, $d = -0.842482...$ as explained above (the minus sign indicates the motion is downward), and the reader can confirm also that the uppermost vertex of the pentagon (with euclidean coordinates $(0, 0.687123...)$) indeed moves to the origin $(0,0)$ under this isometry.

Figure 11: Equidistant row of points

Figure 12: Vertical translation of the row of points, projective model

Figure 13: Same as Figure 12, in the Poincare model

The next set of figures shows how a vertical isometry acts on an evenly spaced horizontal row of points. These points are the result of translating the origin in the horizontal rather than vertical direction, by increments of .1, so that 21 points are collected, as seen in Figure 11. Then a vertical isometry that moves a distance .1 along the vertical axis is applied repeatedly to this row of points. The result (see Figure 12) is a grid of points representing the forward and backward orbits of these 21 points under the vertical translation. Notice that only the middle point follows a line; the other points follow equidistant curves, that maintain a constant distance to the vertical diameter. Compare Figure 13, which shows the same grid in the Poincare model. Although it may not be obvious from this figure, the equidistant curves in the conformal model are geometric circles, while in the projective model they are not.

For more information on isometries in hyperbolic geometry, see [Cox65] for a full account of them from a projective point of view. [Thu97] has a more intrinsic treatment of the three dimensional case, and [PG92] contains implementation details for the projective model.
4 Visualizing non-euclidean geometry in three dimensions

The outsider’s view  I would like now to turn from plane hyperbolic geometry, to three dimensional hyperbolic geometry. What new challenges, what new opportunities appear? Which model is the appropriate one for the task? Restricting our attention to two primary contenders, the projective ball model and the conformal ball model, what advantages and disadvantages are there? To begin with, notice that we have the freedom to position our viewing position either inside the ball or outside it. If we stand outside, then it is not so different than standing above the disk models of plane hyperbolic geometry, and we can learn to interpret what we see in either the conformal or projective model. We preserve a “spectator” consciousness in this way.

The insider’s view  What happens, however, if we position ourselves within the ball, within hyperbolic space itself? (After all, what is more natural than to want to be inside rather than outside the space we are studying?) We are no longer interested in making an image which can be read and decoded at “arm’s length”, but in making an image which simulates what a hypothetical inhabitant would actually see. This involves simulating an optical system like an eye or a camera. The creation of realistic images under these conditions is the goal of the newly developed field of realistic image synthesis. At the core of image synthesis is a central perspective projection, in which a bundle of lines joining the focal point with the three dimension scene geometry, is cut by a viewing plane. Each line contributes one point to the image on the viewing plane. Standard rendering software expects the paths of light to be straight lines, and is therefore compatible with the projective models of non-euclidean geometry. In a conformal representation, however, each path of light is represented by a circular arc, and so image synthesis using such a model would require an expensive ray-tracer customized to following circular arcs. This would in effect “straighten” out the curves; the result would be to yield the same image as the projective model, only at much higher cost – one point in favor of the projective model.

One possible objection to the use of the projective model to render images, is that since it is not conformal, the angles of the light rays arriving at the camera will have to be corrected, in order to create undistorted images. This is a reasonable objection, but nullified by the happy circumstance, mentioned above, that at the origin, in the center of the projective ball model, the model is in fact conformal (and only there!). Then the correct strategy for rendering in hyperbolic space is to leave the camera fixed at the origin, and move the world past the camera. This is actually the default behavior of current (euclidean) rendering systems anyway, so it does not require any adjustment in standard practice.

Conformality reconsidered  Otherwise, the major strength of the conformal models, that they represent angles correctly, carries little weight in the situation of an "inside" observer. Indeed, human beings rarely perceive visual reality in a conformal way in three dimensions. For example, if I look at a rectangular table top, I almost never see a right angle, unless I happen to be positioned directly above one of the corners. I learn to recognize right angles due to a variety of movement and depth cues. Similarly, when presented with virtual hyperbolic space in the projective model, the human visual system can learn to reconstruct accurate angles based on these sorts of cues.
An important coincidence  The fact that the isometries of 3-dimensional non-euclidean geometry in the projective ball model are represented as 4x4 projective matrices is also a vote in its favor, since the transformation pipeline of computer graphics technology, for independent reasons, is based on the same matrices. Historically, this “coincidence” has its roots in the birth of projective geometry out of the discovery of Renaissance perspective. In the case of computer graphics, the need to create perspectively accurate images requires that projective and not merely affine transformations be supported (see [FvDFH90]); it is the good fortune of non-euclidean geometric research that this allows isometric manipulations in virtual non-euclidean spaces at almost no extra cost. One caveat is that while projective matrices are fully supported, homogeneous coordinates are not; although non-euclidean geometry may be kept internally in homogeneous coordinates, when it is passed to standard rendering systems, it needs to be dehomogenized. For example, a point with Minkowski coordinates \((x, y, z, w)\) would have to be converted to \((\frac{x}{w}, \frac{y}{w}, \frac{z}{w})\) before calling rendering routines. The case \(w = 0\) does not arise, at least for hyperbolic geometry, since time-like Minkowski coordinates have \(w > 1\). For more details, see [Gun93].

4.1 Realistic shading in non-euclidean spaces

Hopefully the above arguments establish that the projective model is the correct one for the insider’s view of non-euclidean space. The extensive aspects of the scene will be rendered correctly, but there still remains the intensive aspects, such as color. Assigning colors to geometric objects based on a three dimensional model of light propagation is known as shading. The question then presents itself, how should shading be done in hyperbolic space?

Figure 14: Standard shading model involves calculations in the tangent space \(T_P\) at the point \(P\) where a line of sight from the camera intersects the surface

Review of shading  Standard shading procedures for three dimensional image synthesis are based on models of the interaction of the surface with light. If the image is imagined to consist of a rectangular array of pixels, then this task must be performed for each pixel. Each pixel determines a direction from the focal point of the camera into the three-dimensional scene. In a simple rendering system, this direction is followed out into the scene until a piece of geometry is encountered. Assume for the moment there is one triangle and one light source in the scene; then if the direction from the camera meets the triangle, a shading operation has to be performed to
determine how much light, and of what color, will be visible at the intersection point on the triangle. See Figure 14. Several vectors in the tangent plane $T_P$ of the intersection point $P$ play special roles. The surface normal $\vec{N}$, the vector pointing to the light source $\vec{L}$, the vector pointing to the camera $\vec{I}$, and the angle bisector $\vec{H}$ of $\vec{L}$ and $\vec{I}$. All are normalized to unit length. Then a simple shading model, assuming white light and white triangle, is given by intensity $= k_d (\vec{L} \cdot \vec{N}) + k_s (\vec{N} \cdot \vec{H})^e$. The first term calculates the diffuse, and the second, the specular, contribution. ($e^s$ is a “specular” exponent which is responsible for the bright highlights associated to shiny surfaces.) The interested reader is referred to [FvDFH90] for details. The same computations may also be carried out in hyperbolic space. Care must simply be taken that the inner product $\cdot$ is used to normalize vectors and calculate angles.

**Programmable hyperbolic shader** Such shading algorithms will not behave correctly when confronted with hyperbolic geometry, since the algorithms make heavy use of implicitly euclidean distance and angle calculations, as described above. The only solution is to replace the euclidean calculations with hyperbolic ones. This may sound drastic, but it is rather painless to do given existing tools. The solution adopted in making the images shown from “Not Knot” (see below) involved the use of Renderman, a commercial rendering package with programmable shaders (see [Ups89]). A hyperbolic shader was written in which the euclidean information provided from the rendering system was converted back into Minkowski coordinates, as described above. Then, using the inner product $\cdot$, accurate hyperbolic distance and angle calculations could be made, and shading values calculated. Recent advances in graphics technology indicate that programmable shaders will soon be available as standard feature of mainstream consumer products([PMTH01]).

### 4.2 Sample hyperbolic visualization: “Not Knot”

As indicated above, current cosmological myths (the Big Bang) have stimulated interest in three-dimensional manifolds, as candidates for the large-scale structure of our universe. In two dimensions, a sphere and a doughnut represent different manifolds, since they both are locally (looked at with a microscope) euclidean, but globally one surface cannot be deformed smoothly into the other. One of the unsolved classification problems of mathematics is to make a list of all three-dimensional manifolds. It turns out that, in some sense, “most” 3-manifolds are hyperbolic, that is, their natural metric is based on hyperbolic geometry.

“Not Knot” Naturally, this research has stimulated interest in visualizing hyperbolic space. As part of a research project at the University of Minnesota (see the web page [Cen] for more information) an animated video, “Not Knot”, was produced, ([GM91]. It narrates the story of one particular hyperbolic manifold arising out of the 3 linked Borromean rings. This leads in the video into a sequence of tessellations of hyperbolic space. All these tessellations (except the final one) involve a family of dodecahedra with pentagonal faces. One, in particular, involves a regular pentagon, all of whose angles are right angles. These pentagons can be seen tessellating the hyperbolic plane in Figure 3. In this video, they have been assembled into dodecahedra, and these dodecahedra fit together without overlap to tile hyperbolic space, just as ordinary cubes fit together to tile hyperbolic space.
Calculating a fundamental domain  Just as the initial right-angled pentagon was calculated using hyperbolic trigonometry, so the regular dodecahedron can be calculated. The position of the vertices and edges can be found beginning with the pentagonal face already found, and solving for parameters which allow 12 copies to fit together to form a regular dodecahedron. Since this polyhedron will be used to tessellate the whole space, a representation must be found that does not obstruct the viewer too much. The solution that was adopted in this case, was to leave the faces empty, and provide a solid beam covering each of the edges. The beams were designed as equidistant surfaces with respect to the edge within. They can be constructed as surfaces of revolution of a single equidistant curve (see Figure 12 for examples of equidistant curves). Furthermore, since in the tessellation four dodecahedra meet at each edge, the original geometry for the beam only has to include one quarter of a full surface.

Figure 15: Regular dodecahedron seen from “outside” in projective ball model

Figure 16: Regular dodecahedron seen from “outside” in conformal ball model

We skip over the details involved in the calculation. The result can be seen in Figure 15, which also shows the “outsider’s” view of the projective ball model. The circle represents the boundary of the ball. Figure 16 shows the same geometry in the Poincare model. These figures are not drawn using the realistic shading procedures described above.

Figure 17 shows, from inside hyperbolic space, how four of these dodecahedra actually do fit around one common edge without overlap, establishing that their dihedral angles are 90°! They have translucent walls in this figure. This process can then be repeated around other edges, ad infinitum, until the whole of hyperbolic space is filled with these dodecahedra (see Figure 19). Figures 17 and 19 were rendered with the hyperbolic Renderman shader described above. In this figure, the walls are completely transparent and the beams have been thickened. Some of the walls were originally colored based on the connection to the Borromean rings, and hence not all beams have the same shading.

The view of the full tessellation can reveal many interesting features of hyperbolic space. For example, there are many hyperbolic planes to be seen, unlike in euclidean space. Define the visual diameter of an object to be the largest angle subtended by any two rays joining it to the eye position. Then the visual diameter of any euclidean plane is always π, but in hyperbolic space a plane not containing the viewing point will have a visual size that drops off with the distance to the eye. As a result, many complete hyperbolic planes are visible in the background, all tessellated with the right-angled pentagons. (The largest such plane in Figure 19 contains the large, central...
An inhabitant of hyperbolic space could ask the interesting question: “Are there embedded euclidean planes in hyperbolic space?” The answer is yes; they are called horospheres and one is seen in Figure 18. The angles that look like right angles really are right angles. The horosphere itself is a convex surface with one point contact with the bounding sphere of the ball model. (The corners in this figure are obstructed by beams, but the checkerboard tessellation really goes on forever.)

Other features of hyperbolic “reality” can be experienced best in the animated sequences offered in the video. For example, in one sequence the camera flies through the tessellation along a curve equidistant to one beam direction; this experience gives a visceral meaning to equidistant curve that transcends what is available merely looking at one statically from outside.

### 4.3 Visualization case study in spherical space: the 120-cell

It is interesting to compare this tessellation of hyperbolic space with one of spherical space. Again, regular pentagon dodecahedra are the tessellating forms. Because we have concentrated above on the hyperbolic case, we only sketch the technical details involved; they are very similar to the hyperbolic case. The differences can be traced back to the fact that the Absolute Conic is the positive definite quadratic form $x^2 + y^2 + z^2$, not $Q^{-}$. 

A regular pentagon dodecahedron, considered as a euclidean shape, consists of euclidean regular
pentagons. Each angle of each pentagon measures 108°. But it’s possible to think of these 12 pentagons as composing a tessellation of the surface the sphere. If we “inflate” them until they are lying on the surface of the sphere, we find that we have expanded the angles at each corner to 120°. This “inflation” is the opposite gesture from what happened in hyperbolic space when we created regular pentagons with 90° corners. There the angles decreased; here they increase. To actually calculate the coordinates, one could proceed analogous to the calculation of the hyperbolic pentagon described above, but using formulae of spherical trigonometry to create a spherical regular pentagon with interior angle 120°.

It’s interesting to explore the same principle in three dimensions. We can play the same game with a regular pentagon dodecahedron. In euclidean space, it has dihedral angles of 116.56°, less than 120°. In analogy with the pentagon considered above, it’s possible, by entering spherical space, to “inflate” these solid dihedral angles until they are exactly 120°. After the inflation, three of the figures can be positioned around one edge, and it turns out that a fourth fits exactly to enclose a vertex, in a tetrahedral arrangement. Continuing to fit these spherical polyhedra together, one finds that, it takes 120 of them to completely tessellate the 3-sphere. This arrangement goes by the name of the “120-cell” or dodecahedral honeycomb. It has many interesting properties. For example, the 120 dodecahedra can be separated into 12 connected rings of 10 each. Each ring is linked with every other ring. This decomposition into rings can be done in 6 different ways! It is the insider’s view of a historic three dimensional elliptic manifold known as the Poincare homology sphere. There are also connections to the famous Hopf fibration of the 3-sphere.

Figure 20 shows how 13 of these dodecahedra appear, one in the center and a layer of 12 covering each of its faces. Figure 21 shows the same arrangement but rendered differently to allow better visibility. Figure 22 shows the effect of moving away from the configuration. Paradoxically, the parts of the object farthest away, appear largest! This is an archetypal experience in spherical space. Why? In spherical space, all geodesics from a point converge at the antipodal point, just as great circles passing through the North Pole of the earth also pass through the South Pole. The optical effect is that, if the viewer is at the North Pole, an object appears smallest when it crosses the equator, and then begins to increase in apparent size. Finally, any object which occupies the antipodal position in the sphere will appear in every direction and highly magnified. This effect is more pronounced in Figure 23. Finally, Figure 24 shows an “insider’s” view of the 120-cell manifold. To be able to see through the tessellation, the display strategy of Figure 21 has been adopted here; the dodecahedra have been shrunk; the original dimensions have been rendered in
Figure 23: Further away ... and larger!

Figure 24: The full 120-cell; the antipodal dodecahedron fills the background of the view.

Figure 25: A tree structure embedded in hyperbolic space “wireframe” only. The viewing position in this image is actually located at the center of one of the dodecahedra, which has not been drawn. There is an “antipodal” dodecahedron located at the opposite side of the 3-sphere; it fills the background of the image.

For both the three dimensional case studies, we refer the interested reader to [Thu97]; and for the 120-cell, also to [Cox73].

4.4 Other research

One aspect of hyperbolic visualization with practical applications is the use of hyperbolic geometry to represent data structures such as trees. Such structures, as is well known, have exponential growth: a binary tree of depth \( n \) may have as many as \( 2^n \) nodes. Since trees are used to represent many aspects of modern knowledge bases, such as directory structures on a computer or the connections on the world wide web, finding convenient graphical representations for trees is a topic of active research. Lately, such trees have been embedded into both hyperbolic two- and three-dimensional space – see Figure 25. The exponential growth of area and volume with linear dimension in hyperbolic geometry matches the growth rate of trees nicely, and promising results have been obtained. For details see [Mun98].

5 Outlook

I hope this article has given the reader a feeling for the nature of visualization of non-euclidean geometry in two and three dimensions, and the conviction that there is still much interesting work to be done in this direction. In no particular order, here are some themes/questions/topics which could serve to drive further explorations:

- Can stereo perspective effects be utilized in non-euclidean space and if so, how?
• Can 3D modelers be constructed that produce consistent results for all three classical geometries?

• What aspects of physics need to be revised in order to simulate motion and dynamics in non-euclidean spaces?

• How can fractal textures be generated in non-euclidean spaces?

• Can hyperbolic space be used as the setting for a video game that could not be played anywhere else?

• Can 3-sphere visualizations help understand unit quaternions?

• Can sound be used with computer graphics to help "audialize" non-euclidean geometry?

• Does the absence of similarity in non-euclidean space imply hostile conditions for life and growth?

A Further resources

Readers interested in exploring the world of non-euclidean visualization have a variety of resources available. I have tried to indicate reference books and articles where the underlying mathematical topics are treated in detail. Additionally, there are software resources available. For two dimensional geometry, the dynamic geometry package Cinderella ([RGK99] ) provides an interactive environment for exploring all three classical geometries. The 3D viewer Geomview ([MLP+]) is freely available software for Linux, and supports interactive three dimensional viewing in the three classical geometries. Its interactivity is a great aid in developing geometric intuition in these unfamiliar spaces. It was the tool used to prepare the two 3D examples discussed in detail above. For those users with access to Mathematica, [Goo] is a Mathematica package for exploring n-dimensional hyperbolic geometry, with especial support for generating graphics in 2 and 3 dimensions, in a variety of models.

Mathematica is trademarked by Wolfram Research, Inc.; Renderman is a trademark of Pixar, Inc.

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References


