Supraconvergence of a finite difference scheme for elliptic third kind boundary value problems in fractional order Sobolev spaces

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Abstract
In this paper we study the convergence of a finite difference discretization of a second order elliptic equation with variable coefficients subject to general boundary conditions. We prove that the scheme exhibits the phenomenon of supraconvergence on non-uniform grids, i.e. although the truncation error is in general of first order only, one has second order convergence. More precisely, for \( s \in (1/2, 2] \) optimal order \( O(h^s) \)-convergence of the finite difference solution and its gradient is shown if the exact solution is in the Sobolev-Slobodetskij space \( H^{1+s}(\Omega) \). All error estimates are strictly local.

Another result of the paper is a close relationship of the finite difference scheme and a linear finite element method combined with a special kind of quadrature. As a consequence, the results of the paper can be viewed as introducing a fully discrete finite element method for which the gradient is superclose, i.e., the error of the approximate gradient with respect to a linear interpolation of the solution \( u \) is of second order if \( u \in H^3(\Omega) \). A numerical example is given.


Keywords: non-uniform grid, elliptic finite difference scheme, stability, supraconvergence, supercloseness of gradient, fully discrete linear FEM.

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