Elementary Proof of a Result by M.L. Treuden on Collectively Compact Operator Sequences

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In his thesis M.L. Treuden [3] proved the following

Theorem. *Assume one of the following two conditions to hold:*

(i) F is a normed space such that a bounded sequence $(P_i)_{\mathbb{N}}$ of finite dimensional projections P_i exists with

$$P_i f \longrightarrow f \quad (i \in \mathbb{N}), \qquad f \in F.$$

(ii) F is a Hilbert space.

Let E be a normed space. Then a sequence of linear mappings $K_n : E \to F$, $n \in \mathbb{N}$, is discretely compact if and only if there exists a sequence $T_n : E \to F$, $n \in \mathbb{N}$, such that $\|T_n\| \to 0$ $(n \in \mathbb{N})$ and $(K_n - T_n)_{\mathbb{N}}$ is collectively compact.

Treuden gave a proof of this theorem using the measure of noncompactness. The purpose of this note is to prove it in an elementary way.

Let us recall some definitions. By B we denote the unit ball in E. A sequence $K_n : E \to F$, $n \in \mathbb{N}$, is said to be collectively compact [1] if the set

$$\bigcup_{n \in \mathbb{N}} K_n B$$

is relatively compact. For a sequence of subsets $S_n \subset F$, $n \in \mathbb{N}$, let

$$\operatorname{Lim\,sup}_{\mathbb{N}} S_n := \{f \in F \mid \exists\, \mathbb{N}' \subset \mathbb{N}, \quad f_n \in S_n, \quad n \in \mathbb{N}' : f_n \to f \quad (n \in \mathbb{N}')\}.$$

A sequence $(S_n)_{\mathbb{N}}$ is said to be discretely compact if for each subsequence $\mathbb{N}' \subset \mathbb{N}$ the set $\operatorname{Lim\,sup}_{\mathbb{N}'} S_n$ is nonvoid. A sequence of mappings $K_n : E \to F$,

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$n \in \mathbb{N}$, is defined to be discretely compact [2] if each K_n, $n \in \mathbb{N}$, is bounded and $(K_n B)_{\mathbb{N}}$ is discretely compact. Clearly $(K_n)_{\mathbb{N}}$ is collectively compact if and only if it is discretely compact and K_n is compact for each $n \in \mathbb{N}$.

The following lemma is well-known [4], for completeness we give the short proof.

Lemma. *For $(S_n)_{\mathbb{N}}$ discretely compact the set $\operatorname{Lim\,sup}_{\mathbb{N}} S_n$ is compact.*

Proof. Let $f^{(j)} \in S$, $j \in \mathbb{N}$, be an arbitrary sequence. Then there exist subsequences $\mathbb{N}'_j \subset \mathbb{N}$ and elements $f_n^{(j)} \in S_n$, $n \in \mathbb{N}'_j$, with

$$f_n^{(j)} \longrightarrow f^{(j)} \quad (n \in \mathbb{N}'_j), \qquad j \in \mathbb{N}.$$

Let $\epsilon^{(j)} > 0$, $\epsilon^{(j)} \to 0$ $(j \in \mathbb{N})$. Choose indices $n_1 < n_2 < n_3 < \ldots$ such that

$$(1) \qquad ||f^{(j)} - f_{n_j}^{(j)}|| < \epsilon^{(j)}, \quad j \in \mathbb{N}.$$

Evidently, $f_{n_j}^{(j)} \in S_{n_j}$, $j \in \mathbb{N}$, and hence by the discrete compactness of $(S_n)_{\mathbb{N}}$ we find a subsequence $\mathbb{N}' \subset \mathbb{N}$ and a $f \in F$ with

$$(2) \qquad f_{n_j}^{(j)} \longrightarrow f \quad (j \in \mathbb{N}').$$

We have $f \in S$ and taking (1), (2) into account we obtain

$$||f - f^{(j)}|| \leq ||f - f_{n_j}^{(j)}|| + ||f_{n_j}^{(j)} - f^{(j)}|| \to 0 \quad (j \in \mathbb{N}').$$

□

As an easy consequence of the foregoing lemma we state the following corollary interesting in this context although not needed for the proof of the theorem.

Corollary. *Let $(K_n)_{\mathbb{N}}$ be discretely compact and pointwise convergent to a mapping $K : E \to F$. Then K is compact.*

Proof. The result follows from the fact that the convergence $K_n \to K$ $(n \in \mathbb{N})$ implies

$$KB \subset \operatorname{Lim\,sup}_{\mathbb{N}} K_n B,$$

the latter set being compact as is seen by applying the lemma to $S_n := K_n B$.
□

Proof of the theorem. Clearly the discrete compactness is stable with respect to additive perturbations of sequences $(T_n)_{\mathbb{N}}$ with $||T_n|| \to 0$ $(n \in \mathbb{N})$. Hence only the necessary part of the theorem is left to be proved.

Case (i). Let $T_n := (I - P_n)K_n$. We show

$$(3) \qquad ||T_n|| \longrightarrow 0 \qquad (n \in \mathbb{N}).$$

Assuming this not to hold we find $\delta > 0$, a subsequence $\mathbb{N}' \subset \mathbb{N}$ and elements $u_n \in B$, $n \in \mathbb{N}'$, such that

$$||T_n u_n|| = ||(I - P_n)K_n u_n|| \geq \delta, \quad n \in \mathbb{N}'.$$

For a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and some $f \in F$

$$K_n u_n \longrightarrow f \qquad (n \in \mathbb{N}'').$$

This leads to the contradiction

$$(4) \qquad (I - P_n)K_n u_n \longrightarrow f - f = 0 \quad (n \in \mathbb{N}'').$$

As a consequence of (3) the sequence $(K_n - T_n)_{\mathbb{N}}$ is also discretely compact and since each $K_n - T_n = P_n K_n$, $n \in \mathbb{N}$, is compact as a bounded mapping of finite rank we conclude its collective compactness.

Case (ii). As a consequence of the lemma, the set

$$\operatorname{Lim\,sup}_{\mathbb{N}} K_n B$$

is compact. Hence its linear hull, denoted by F_0, is separable. Let P be the orthogonal projection on the closure of F_0. Due to the separability of F_0 there exist orthogonal projections P_i of finite rank with $P_i \to P$ $(i \in \mathbb{N})$. The assertion now follows defining T_n as in case (i) taking this time the convergence

$$P_n K_n u_n \longrightarrow Pf = f \qquad (n \in \mathbb{N}'')$$

into account resulting from the property $f \in F_0$. □

References

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