Elementary Proof of a Result by M.L. Treuden on Collectively Compact Operator Sequences

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In his thesis M.L. Treuden [3] proved the following

**Theorem.** Assume one of the following two conditions to hold:

(i) $F$ is a normed space such that a bounded sequence $(P_i)_{i \in \mathbb{N}}$ of finite dimensional projections $P_i$ exists with

$$P_i f \rightarrow f \quad (i \in \mathbb{N}), \quad f \in F.$$

(ii) $F$ is a Hilbert space.

Let $E$ be a normed space. Then a sequence of linear mappings $K_n : E \to F$, $n \in \mathbb{N}$, is discretely compact if and only if there exists a sequence $T_n : E \to F$, $n \in \mathbb{N}$, such that $\|T_n\| \to 0$ ($n \in \mathbb{N}$) and $(K_n - T_n)_{\mathbb{N}}$ is collectively compact.

Treuden gave a proof of this theorem using the measure of noncompactness. The purpose of this note is to prove it in an elementary way.

Let us recall some definitions. By $B$ we denote the unit ball in $E$. A sequence $K_n : E \to F$, $n \in \mathbb{N}$, is said to be collectively compact [1] if the set

$$\bigcup_{n \in \mathbb{N}} K_n B$$

is relatively compact. For a sequence of subsets $S_n \subseteq F$, $n \in \mathbb{N}$, let

$$\limsup_{\mathbb{N}} S_n := \{ f \in F \mid \exists \mathbb{N}' \subseteq \mathbb{N}, \quad f_n \in S_n, \quad n \in \mathbb{N}' : f_n \to f \quad (n \in \mathbb{N}') \}.$$

A sequence $(S_n)_{\mathbb{N}}$ is said to be discretely compact if for each subsequence $\mathbb{N}' \subseteq \mathbb{N}$ the set $\limsup_{\mathbb{N}'} S_n$ is nonvoid. A sequence of mappings $K_n : E \to F$,

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\[ n \in \mathbb{N}, \text{ is defined to be discretely compact } [2] \text{ if each } K_n, \ n \in \mathbb{N}, \text{ is bounded and } (K_n, B)_\mathbb{N} \text{ is discretely compact. Clearly } (K_n)_\mathbb{N} \text{ is collectively compact if and only if it is discretely compact and } K_n \text{ is compact for each } n \in \mathbb{N}. \]

The following lemma is well-known [4], for completeness we give the short proof.

**Lemma.** For \((S_n)_\mathbb{N}\) discretely compact the set \(\limsup_\mathbb{N} S_n\) is compact.

**Proof.** Let \(f^{(j)} \in S, \ j \in \mathbb{N}\), be an arbitrary sequence. Then there exist subsequences \(N_j' \subseteq \mathbb{N}\) and elements \(f_{n_{j}}^{(j)} \in S_n, \ n \in N_j',\) with

\[ f_{n_{j}}^{(j)} \to f^{(j)} \quad (n \in N_j'), \quad j \in \mathbb{N}. \]

Let \(\varepsilon^{(j)} > 0, \ \varepsilon^{(j)} \to 0 \ (j \in \mathbb{N}).\) Choose indices \(n_1 < n_2 < n_3 < \ldots\) such that

(1) \[ \| f^{(j)} - f_{n_{j}}^{(j)} \| < \varepsilon^{(j)}, \quad j \in \mathbb{N}. \]

Evidently, \(f_{n_{j}}^{(j)} \in S_{n_j}, \ j \in \mathbb{N},\) and hence by the discrete compactness of \((S_n)_\mathbb{N}\) we find a subsequence \(N' \subseteq \mathbb{N}\) and a \(f \in F\) with

(2) \[ f_{n_{j}}^{(j)} \to f \quad (j \in N'). \]

We have \(f \in S\) and taking (1), (2) into account we obtain

\[ \| f - f^{(j)} \| \leq \| f - f_{n_{j}}^{(j)} \| + \| f_{n_{j}}^{(j)} - f^{(j)} \| \to 0 \quad (j \in N'). \]

\[ \square \]

As an easy consequence of the foregoing lemma we state the following corollary interesting in this context although not needed for the proof of the theorem.

**Corollary.** Let \((K_n)_\mathbb{N}\) be discretely compact and pointwise convergent to a mapping \(K : E \to F.\) Then \(K\) is compact.

**Proof.** The result follows from the fact that the convergence \(K_n \to K \ (n \in \mathbb{N})\) implies

\[ KB \subseteq \limsup_\mathbb{N} K_n B, \]

the latter set being compact as is seen by applying the lemma to \(S_n := K_n B.\)

\[ \square \]

**Proof of the theorem.** Clearly the discrete compactness is stable with respect to additive perturbations of sequences \((T_n)_\mathbb{N}\) with \(\| T_n \| \to 0 \ (n \in \mathbb{N}).\) Hence only the necessary part of the theorem is left to be proved.
Case (i). Let $T_n := (I - P_n)K_n$. We show

(3) \[ ||T_n|| \to 0 \quad (n \in \mathbb{N}). \]

Assuming this not to hold we find $\delta > 0$, a subsequence $\mathbb{N}' \subset \mathbb{N}$ and elements $u_n \in B$, $n \in \mathbb{N}'$, such that

\[ ||T_n u_n|| = ||(I - P_n)K_n u_n|| \geq \delta, \quad n \in \mathbb{N}'. \]

For a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ and some $f \in F$

\[ K_n u_n \to f \quad (n \in \mathbb{N}''). \]

This leads to the contradiction

(4) \[ (I - P_n)K_n u_n \to f - f = 0 \quad (n \in \mathbb{N}''). \]

As a consequence of (3) the sequence $(K_n - T_n)_{\mathbb{N}}$ is also discretely compact and since each $K_n - T_n = P_n K_n$, $n \in \mathbb{N}$, is compact as a bounded mapping of finite rank we conclude its collective compactness.

Case (ii). As a consequence of the lemma, the set

\[ \text{Limsup}_{\mathbb{N}} K_n B \]

is compact. Hence its linear hull, denoted by $F_0$, is separable. Let $P$ be the orthogonal projection on the closure of $F_0$. Due to the separability of $F_0$ there exist orthogonal projections $P_i$ of finite rank with $P_i \to P$ ($i \in \mathbb{N}$). The assertion now follows defining $T_n$ as in case (i) taking this time the convergence

\[ P_i K_n u_n \to Pf = f \quad (n \in \mathbb{N}'') \]

into account resulting from the property $f \in F_0$. \qed

References


