Remark on a Resolvent Inequality for the
Second Order Central Divided Difference Operator

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In the recent paper [1] H.W.J. Lenferink and M.N. Spijker proved in Corollary 3.2 the resolvent inequality

$$(*) \qquad \| (\zeta I - A_h)^{-1} \|_\infty \le C_\alpha |\zeta|^{-1}, \quad \zeta \in \mathbb{C}\backslash\{0\}, \quad |\arg(\zeta)| \le \alpha$$

where $\alpha \in [0,\pi)$ and A_h is the difference operator

$$A_h u_h(x_j) := (u_h(x_{j+1}) - 2u_h(x_j) + u_h(x_{j-1}))/\mathrm{h}^2, \quad j = 1, \dots, n-1,$$

$n \ge 2$, acting on grid-functions u_h defined on the uniform grid $\{x_j = x_0 + jh, j = 0, ..., n\}$ such that $u_h(x_0) = u_h(x_n) = 0$. $\| \cdot \|_\infty$ denotes the maximum norm for grid functions resp. the corresponding operator norm. The constant C_α is shown to be independent of n if $x_0 = 0, x_n = 1, nh = 1$. The proof relies on a result by V. Thomée [2]. Another proof using some clever direct calculations (see [1]) was suggested by M. Crouzeix. In this note we also give a direct proof which is quite elementary and at the same time shows that C_α also can be taken independent of x_0, x_n.

Proposition. *Let* $\alpha \in [o,\pi)$ *be given. Then* $(*)$ *holds with a constant* C_α *independent of* x_0, x_n, $h, n \in \mathbb{N}, n \ge 2$.

P r o o f . The discrete Green's function $g_h = g_h(x_j, x_k\, ; \zeta)$ is defined by the equations

$$(\zeta I - A_h)\, g_h(x_j, x_k\, ; \zeta) = h^{-1}\, \delta_{jk}, \qquad j,k = 1, \dots, n-1$$
$$g_h(x_o, x_k\, ; \zeta) = g_h(x_n, x_k\, ; \zeta) = 0, \qquad k = 1, \dots, n-1,$$

δ_{jk} denoting Kronecker's symbol. The solution of

$$(\zeta I - A_h) u_h = f_h, \qquad u_h(x_0) = u_h(x_n) = 0$$

is then given by

$$u_h(x_j) = h \sum_{k=1}^{n-1} g_h(x_j, x_k\, ; \zeta)\, f_h(x_k), \quad j = 0, \dots, n.$$

Clearly,

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$$\| (\zeta I - A_h)^{-1} \|_\infty = \max_{j=1,\dots,n-1} h \sum_{k=1}^{n-1} | g_h(x_j, x_k ; \zeta) |.$$

Let λ be the solution with $|\lambda| > 1$ of the quadratic equation $\lambda^2 - (2 + h^2\zeta)\lambda + 1 = 0$, i.e.

$$\lambda = 1 + \frac{h^2\zeta}{2} + \sqrt{\left(\frac{h^2\zeta}{2}\right)^2 + h^2\zeta}$$

(here we take the branch of the function $\sqrt{z(z+2)}$ which is positive for $z = 1$, where the complex plane is slit along (–2,0)).

It is easy to check that g_h is given by

$$g_h(x_j, x_k ; \zeta) = \frac{h}{\lambda - \lambda^{-1}} \frac{1}{\lambda^n - \lambda^{-n}} \begin{cases} (\lambda^{n-k} - \lambda^{-n+k})(\lambda^j - \lambda^{-j}), & j = 0, \dots, k \\ (\lambda^k - \lambda^{-k})(\lambda^{n-j} - \lambda^{-n+j}), & j = k, \dots, n. \end{cases}$$

Because of the symmetry in g_h it is sufficient to estimate

$$R := \max_{j=1,\dots,n-1} h \sum_{k=1}^{j} | g_h(x_j, x_k ; \zeta) |.$$

Since $|\lambda| > 1$ we have

$$|\lambda^k - \lambda^{-k}| \le 2|\lambda|^k, \quad |\lambda - \lambda^{-1}| \ge |\lambda| - 1, \quad |\lambda^n - \lambda^{-n}| \ge |\lambda|^n - 1$$

and, consequently, summing up a geometrical series,

$$R \le \max_{j=1,\dots,n-1} \frac{4h^2|\lambda|}{(|\lambda| - 1)^2} \frac{|\lambda|^n - |\lambda|^{n-j}}{|\lambda|^n - 1} \le \frac{4h^2|\zeta||\lambda|}{(|\lambda| - 1)^2} \frac{1}{|\zeta|} .$$

Since $\lambda = \lambda(h^2\zeta)$ we have to bound the right-hand side as a function of $z := h^2\zeta$ in the sector $|\arg(z)| \le \alpha$, $z \ne 0$. Evidently, $|\lambda(z)| \ne 1$ for all these z. Moreover,

$$\limsup_{\substack{z \to 0 \\ |\arg(z)| \le \alpha}} \frac{|z||\lambda(z)|}{(|\lambda(z)| - 1)^2} \le \frac{1}{\cos^2 \frac{\alpha}{2}} , \quad \lim_{z \to \infty} \frac{|z||\lambda(z)|}{(|\lambda(z)| - 1)^2} = 1 .$$

Hence, the assertion follows.

Remark 1. For $\alpha \in [0,\pi)$ also

$$\binom{*}{*} \qquad \| (\zeta I - A_h)^{-1} \|_p \le C_\alpha |\zeta|^{-1}, \quad \zeta \in \mathbb{C}\setminus\{0\}, \quad |\arg(\zeta)| \le \alpha$$

holds true for $p \in [1, \infty]$. Here

$$\| u_h \|_p := \Big(h \sum_{j=1}^{n-1} | u_h(x_j) |^p \Big)^{1/p} .$$

Remark 2. Let $\{x_j , j = 0, \dots , n\}$, $n \geq 2$, be a nonuniform grid with mesh-sizes $h_j := x_{j+1} - x_j$, $j = 0, \dots , n-1$. Let A_h be the second divided difference operator acting on grid-functions u_h with $u_h(x_0) = u_h(x_n) = 0$, i.e.

$$A_h u_h(x_j) = \frac{2}{h_{j-1} + h_j} \left[\frac{u_h(x_{j+1}) - u_h(x_j)}{h_j} - \frac{u_h(x_j) - u_h(x_{j-1})}{h_{j-1}} \right], \quad j = 1, \dots , n-1.$$

Defining

$$\| u_h \|_2 := \Big(\sum_{j=1}^{n-1} \frac{h_{j-1} + h_j}{2} | u_h(x_j) |^2 \Big)^{1/2}$$

the above estimate $\binom{*}{*}$ holds true for $p = 2$ with C_α independent of the grid and $x_0, x_n, n \in \mathbb{N}$, $n \geq 2$ (in fact $C_\alpha = 1 / \sin \alpha$ for $\alpha \in [\pi/2, \pi)$ and $C_\alpha = 1$ for $\alpha \in [0, \pi/2]$).

Remark 3. Let $\alpha \in [0,\pi)$ and $m, M \in \mathbb{R}$ be given with $m \leq x_0 < x_n \leq M$. Then, there exists a constant $C = C(\alpha, m, M)$ independent of $x_0, x_n, h, n \in \mathbb{N}$, $n \geq 2$, such that

$$(+) \qquad \| (\zeta I - A_h)^{-1} \|_p \leq \frac{C}{1 + | \zeta |}, \quad \zeta \in \mathbb{C} \setminus \{0\}, \quad | \arg(\zeta) | \leq \alpha$$

for $p \in [1, \infty]$. The estimate (+) holds true also for nonequidistant grids with $p = 2$ where C does not depend on the grid as well as not on $x_0, x_n, h, n \in \mathbb{N}$, $n \geq 2$.

References

[1] H.W.J. Lenferink, M.N. Spjker: A generalization of the numerical range of a matrix. Rapportnr. TW–88–09, Rijksuniversiteit te Leiden, Netherlands, Division of Applied Mathematics

[2] V. Thomée: Stability of difference schemes in the maximum-norm. J. Different. Equ. **1**, 273 – 292 (1965)