SMOOTH POLYHEDRAL SURFACES

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Prelude

Złote Tarasy shopping mall in Warsaw
Złote Tarasy shopping mall in Warsaw
PRELUDE

Smooth and non-smooth realization of a cone and their Gauss images
STORYBOARD

- Episode I – The Phantom Lemma
- Episode II – Attack of the Index
- Episode III – Revenge of the Lemma
Storyboard

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- Episode II – Attack of the Index
- Episode III – Revenge of the Lemma
- Episode IV – A New Normal
- Episode V – The Gauss Image Strikes Back
- Episode VI – Return of Projective Transformations
Storyboard

- Episode I – The Phantom Lemma
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- Episode III – Revenge of the Lemma
- Episode IV – A New Normal
- Episode V – The Gauss Image Strikes Back
- Episode VI – Return of Projective Transformations
- Episode VII – The Applications Awaken
SMOOTH POLYHEDRAL SURFACES
Episode 1
THE PHANTOM LEMMA
Gaussian curvature
**DISCRETE GAUSSIAN CURVATURE**

\[ K(P) := 2\pi - \sum_i \alpha_i \]
Discrete Gaussian curvature

\[ K(P) := 2\pi - \sum \alpha_i \]
Self-intersecting Gauss image
Episode II
ATTACK OF THE INDEX
INDEX OF A VERTEX OF A POLYHEDRAL SURFACE

\[ i(v, \xi) = 1 - \frac{1}{2} \text{ (number of } \Delta \text{ with } v \text{ middle for } \xi) \]
\[ \int_{S^2} m(v, \Delta, \xi) d\omega = 4 \text{ (interior angle of } \Delta \text{ at } v). \]
INTEGRATED INDEX

$$\int_{S^2} m(v, \triangle, \xi) d\omega = 4 \text{ (interior angle of } \triangle \text{ at } v)$$

$$\frac{1}{2} \int_{S^2} i(v, \xi) d\xi = \frac{1}{2} \int_{S^2} \left( 1 - \frac{1}{2} \sum_{\triangle \in M} m(v, \triangle, \xi) \right)$$

$$= 2\pi - \sum_{\triangle \in M} \alpha_{\triangle}(v) = K(v)$$
Relation to algebraic area of Gauss image

Positive discrete Gaussian curvature
Relation to algebraic area of Gauss image

Negative discrete Gaussian curvature
Relation to algebraic area of Gauss image

Gauss image contains pair of antipodal points
Index of $v$ with respect to $n$ equals -2
Relation to algebraic area of Gauss image

Index of $v$ with respect to $n$ equals -2

$$i(v, n) = w(g(v), n) + w(g(v), -n)$$
Episode III
REVENGE OF THE LEMMA
Spherical angle

\[ f = f_2 \text{ is not an inflection face: } \hat{\alpha}_f = \alpha' = \pi - \alpha_f \]
**Spherical angle**

$f = f_2$ is an inflection face: $\hat{\alpha}_f = \alpha' = 2\pi - \alpha_f$
Gauss image without self-intersections: $K>0$

\[
\sum_{f \sim v} \hat{\alpha}_f - (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f
\]
Gauss image without self-intersections: \( K > 0 \)

\[
\sum_{f \sim v} \hat{\alpha}_f - (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f
\]

\[
\implies \sum_{f \sim v} \hat{\alpha}_f = \sum_{f \sim v} (\pi - \alpha_f)
\]
Gauss image without self-intersections: $K > 0$

Convex spherical polygon
Gauss image without self-intersections: $K < 0$

$$- \sum_{f \sim v} \hat{\alpha}_f + (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f$$
GAUSS IMAGE WITHOUT SELF-INTERSECTIONS: $K<0$

\[- \sum_{f \sim V} \hat{\alpha}_f + (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim V} \alpha_f\]

\[\implies \sum_{f \sim V} \hat{\alpha}_f = \sum_{f \sim V} (\pi + \alpha_f) - 4\pi\]
GAUSS IMAGE WITHOUT SELF-INTERSECTIONS: $K<0$

\[-\sum_{f \sim v} \hat{\alpha}_f + (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f\]

\[\implies \sum_{f \sim v} \hat{\alpha}_f = \sum_{f \sim v} (\pi + \alpha_f) - 4\pi\]

\[
\hat{\alpha}_f = 2\pi - (\pi - \alpha_f) = \pi + \alpha_f \text{ if } f \text{ is not an inflection face}
\]

\[
\hat{\alpha}_f = 2\pi - (2\pi - \alpha_f) = \alpha_f \text{ if } f \text{ is an inflection face}
\]
Gauss image without self-intersections: $K < 0$

Spherical pseudo-quadrilateral
Gauss image without self-intersections: $K<0$

Spherical pseudo-triangle (four inflection faces)
Gauss image without self-intersections: $K < 0$

Spherical pseudo-triangle (two inflection faces)
Gauss image without self-intersections: $K<0$

Spherical pseudo-digon
NORMAL VECTOR AND TRANSVERSAL PLANE

One-to-one projection of vertex star onto transversal plane
Dupin indicatrix

Intersection with a plane parallel and close to the tangent plane
Discrete Dupin indicatrix – $K>0$

Discrete ellipse in convex case
Discrete Dupin indicatrix – $K<0$

Inflection edge in the discrete Dupin indicatrix
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola if Gauss image is star-shaped
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola (four inflection faces)
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola (two inflection faces)
Episode V

THE GAUSS IMAGE STRIKES BACK
Gauss images for positive curvature

Gauss images do not intersect
Gauss images for negative curvature

Gauss images do not intersect
Gauss images for negative curvature

Polyhedral monkey saddle
Gauss images for negative curvature

Deformation of a monkey saddle
Shapes of faces in negatively curved areas

\[ \hat{\alpha}_V = \alpha_V + \pi \text{ if } \alpha_V < \pi \text{ and } f_1 \text{ is not an inflection face,} \]
\[ \hat{\alpha}_V = \alpha_V \text{ if } \alpha_V < \pi \text{ and } f_1 \text{ is an inflection face,} \]
\[ \hat{\alpha}_V = \alpha_V - \pi \text{ if } \alpha_V > \pi \text{ and } f_1 \text{ is not an inflection face,} \]
\[ \hat{\alpha}_V = \alpha_V \text{ if } \alpha_V > \pi \text{ and } f_1 \text{ is an inflection face.} \]
SHAPES OF FACES IN NEGATIVELY CURVED AREAS

\[ \hat{\alpha}_v = \alpha_v + \pi \] if \( \alpha_v < \pi \) and \( f_1 \) is not an inflection face,
\[ \hat{\alpha}_v = \alpha_v \] if \( \alpha_v < \pi \) and \( f_1 \) is an inflection face,
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\[ \hat{\alpha}_v = \alpha_v \] if \( \alpha_v > \pi \) and \( f_1 \) is an inflection face.

\[ 2\pi = \sum_{v \sim f_1} \hat{\alpha}_v = \sum_{v \sim f_1} \alpha_v + c_1 \pi - c_3 \pi = (n - 2)\pi + c_1 \pi - c_3 \pi \]
SHAPES OF FACES IN NEGATIVELY CURVED AREAS

\[ \hat{\alpha}_v = \alpha_v + \pi \text{ if } \alpha_v < \pi \text{ and } f_1 \text{ is not an inflection face,} \]
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\[ 2\pi = \sum_{v \sim f_1} \hat{\alpha}_v = \sum_{v \sim f_1} \alpha_v + c_1 \pi - c_3 \pi = (n - 2)\pi + c_1 \pi - c_3 \pi \]

Thus, \( c_1 - c_3 = 4 - n \). Using \( c_1 + c_2 + c_3 + c_4 = n \),

\[ 2c_1 + c_2 + c_4 = 4. \]
Shapes of faces in negatively curved areas

\( \hat{\alpha}_v = \alpha_v + \pi \) if \( \alpha_v < \pi \) and \( f_1 \) is not an inflection face,

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\[
2\pi = \sum_{v \sim f_1} \hat{\alpha}_v = \sum_{v \sim f_1} \alpha_v + c_1\pi - c_3\pi = (n - 2)\pi + c_1\pi - c_3\pi
\]

Thus, \( c_1 - c_3 = 4 - n \). Using \( c_1 + c_2 + c_3 + c_4 = n \),

\[
2c_1 + c_2 + c_4 = 4.
\]

\( f_1 \) planar: \( c_1 + c_2 \geq 3 \).
Shapes of faces in negatively curved areas

$c_1 = 0, c_2 = 4, c_3 = n - 4, c_4 = 0$
SHAPES OF FACES IN NEGATIVELY CURVED AREAS

\[ c_1 = 0, \quad c_2 = 3, \quad c_3 = n - 4, \quad c_4 = 1 \]
Shapes of faces in negatively curved areas

\[ c_1 = 1, \ c_2 = 2, \ c_3 = n - 3, \ c_4 = 0 \]
GAUSS IMAGES FOR MIXED CURVATURE

Overlap of Gauss images
Basic shapes of faces in areas of mixed curvature
GAUSS IMAGES FOR MIXED CURVATURE

Discrete parabolic curve
Smooth polyhedral surface

**Definition**

An orientable polyhedral surface $P$ immersed into $\mathbb{R}^3$ is *smooth* if:

1. $K(v) \neq 0$ for all $v$.
2. For all faces $f$, the sign of discrete Gaussian curvature changes at either zero or two edges.
3. $\forall f, \sum_{v \sim f} \hat{\alpha}_v$ is $\pm 2\pi$ or $0$ depending on whether $K(v)$ has the same sign for all $v \sim f$ or not. In the first case, we demand in addition that $f$ is star-shaped; in the latter case, we require that the convex hull of vertices of positive discrete Gaussian curvature does not contain any vertex of negative curvature.
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SMOOTH POLYHEDRAL SURFACE

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2. $\forall v \exists n, n' \in S^2$ such that $\langle n', n_f \rangle > 0 \forall n_f, f \sim v$, and such that the spherical polygon defined by $n_f$ is star-shaped with respect to $n$. 
**Smooth polyhedral surface**

**Definition**

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Episode VI
RETURN OF PROJECTIVE TRANSFORMATIONS
Theorem

Let $P$ be a smooth polyhedral surface immersed into $\mathbb{R}^3 \subset \mathbb{RP}^3$. Furthermore, let $\pi$ be a projective transformation that does not map any point of $P$ to infinity. Then, $\pi(P)$ is a smooth polyhedral surface.

Furthermore, discrete tangent planes, discrete asymptotic directions, discrete parabolic curves, and points of contact are mapped to the corresponding objects of $\pi(P)$. 
Projective dual surfaces of smooth polyhedral surfaces are smooth.
Episode VII

THE APPLICATIONS AWAKEN
RESULTS OF OUR ALGORITHM

Optimization started from (a) non-smooth; (b) weakly smooth surface
RESULTS OF OUR ALGORITHM

Department of Islamic Art within the Musée du Louvre
RESULTS OF OUR ALGORITHM

Broken images in the reflections
RESULTS OF OUR ALGORITHM

Optimization of Louvre surface
Episode VIII

THE LAST QUESTIONS