SMOOTH POLYHEDRAL SURFACES

Felix Günther

Université de Genève and Technische Universität Berlin

Oberseminar Geometrie
Department of Islamic Art within the Musée du Louvre
Shape of a magic carpet
Original triangular mesh from architects
Planar quadrilateral mesh
Hybrid mesh from planar triangles and quads
Broken images in the reflections
Prelude

Złote Tarasy shopping mall in Warsaw
Złote Tarasy shopping mall in Warsaw
PRELUDE

Smooth and non-smooth realization of a cone and their Gauss images
Faces adapt to the sign of curvature
Transformation of faces during change of curvature
Episode I – The Phantom Lemma
Episode II – Attack of the Index
Episode III – Revenge of the Lemma
Storyboard

- Episode I – The Phantom Lemma
- Episode II – Attack of the Index
- Episode III – Revenge of the Lemma
- Episode IV – A New Normal
- Episode V – The Gauss Image Strikes Back
- Episode VI – Return of Projective Transformations
STORYBOARD

- Episode I – The Phantom Lemma
- Episode II – Attack of the Index
- Episode III – Revenge of the Lemma
- Episode IV – A New Normal
- Episode V – The Gauss Image Strikes Back
- Episode VI – Return of Projective Transformations
- Episode VII – The Applications Awaken
SMOOTH POLYHEDRAL SURFACES
Episode 1
THE PHANTOM LEMMA
Gaussian curvature
GAUSSIAN CURVATURE
POSITIVELY AND NEGATIVELY CURVED SURFACES

$K > 0$

$K < 0$
Motivation of discrete Gauss curvature

Peaks and passes
**Discrete Gauss Curvature**

\[ K(P) := 2\pi - \sum_i \alpha_i \]
**Theorem of Gauss-Bonnet**

Assumption: Polyhedron $M$ is triangulated.

$$K(M) = \sum_P K(P) = \sum_P 2\pi - \sum_P \sum_i \alpha_i$$
**Theorem of Gauss-Bonnet**

Assumption: Polyhedron $M$ is triangulated.

\[
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\]

\[
= 2\pi|V| - \pi|F|
\]
**Theorem of Gauss-Bonnet**

Assumption: Polyhedron $M$ is triangulated.

\[
K(M) = \sum_K K(P) = \sum_P 2\pi - \sum_P \sum_i \alpha_i \\
= 2\pi|V| - \pi|F| = 2\pi(|V| - |E| + |F|) + \pi(2|E| - 3|F|)
\]
THEOREM OF GAUSS-BONNET

Assumption: Polyhedron $M$ is triangulated.

\[ K(M) = \sum_P K(P) = \sum_P 2\pi - \sum_P \sum_i \alpha_i \]

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\[
= 2\pi |V| - \pi |F| = 2\pi (|V| - |E| + |F|) + \pi (2|E| - 3|F|)
\]

\[
= 2\pi (|V| - |E| + |F|) = 2\pi \chi(M) = 2\pi (2 - 2g)
\]
**Discrete Gauss curvature**

Assumption: Polyhedron $M$ is triangulated.

\[ K(P) := 2\pi - \sum \alpha_i \]

\[ K(M) = \sum K(P) = \sum 2\pi - \sum \sum \alpha_i = 2\pi |V| - \pi |E| + |F| = 2\pi (|V| - |E| + |F|) = 2\pi \chi(M) = 2\pi (2 - 2g) \]

\[ K(P) \]

\[ K(N_i) \]

\[ K(N_{i+1}) \]

\[ K(M) \]

\[ N_i \]

\[ N_{i+1} \]

\[ P \]

\[ \alpha_i \]

\[ \alpha' \]

\[ f_1, f_2, f_3 \]

\[ n_1, n_2, n_3 \]
MORE COMPLICATED GAUSS IMAGE

Self-intersecting Gauss image
Episode II

ATTACK OF THE INDEX
Critical point index

Indices sum up to Euler characteristic

\[
i(p_1, \xi) = 1
\]

\[
i(p_2, \xi) = -1
\]

\[
i(p_3, \xi) = -1
\]

\[
i(p_4, \xi) = 1
\]

\[
\therefore \chi(T^2) = 0
\]
INDEX OF A VERTEX OF A POLYHEDRAL SURFACE

\[ i(v, \xi) = 1 - \frac{1}{2} \text{ (number of } \triangle \text{ with } v \text{ middle for } \xi) \]

v is the middle vertex of \( \triangle \) for \( \xi \).
Monkey saddle

Index of a monkey saddle

\( i(\varphi, \xi) = -2 \)
Monkey saddle

Index of a monkey saddle

\[ i(\nu, \xi) = -2 \]
\[ \int_{S^2} m(v, \triangle, \xi) d\omega = 4 \] (interior angle of \( \triangle \) at \( v \)).
INTEGRATED INDEX

\[ \int_{S^2} m(v, \Delta, \xi) d\omega = 4 \text{ (interior angle of } \Delta \text{ at } v) \]

\[ \frac{1}{2} \int_{S^2} i(v, \xi) d\xi = \frac{1}{2} \int_{S^2} \left( 1 - \frac{1}{2} \sum_{\Delta \in M} m(v, \Delta, \xi) \right) \]

\[ = 2\pi - \sum_{\Delta \in M} \alpha_\Delta(v) = K(v) \]
RELATION TO ALGEBRAIC AREA OF GAUSS IMAGE

Positive discrete Gaussian curvature
RELATION TO ALGEBRAIC AREA OF GAUSS IMAGE

Negative discrete Gaussian curvature
Relation to algebraic area of Gauss image

Gauss image contains pair of antipodal points
Relation to algebraic area of Gauss image

Index of $\nu$ with respect to $\mathbf{n}$ equals -2
Index of $v$ with respect to $n$ equals -2

$$i(v, n) = w(g(v), n) + w(g(v), -n)$$
**Relation to algebraic area of Gauss image**

Definition of algebraic index
Relation to algebraic area of Gauss image

Behavior of the critical point index
Foldings lead to discontinuities
Foldings lead to discontinuities

\[ c + i(v, n) = w(g(v), n) + w(g(v), -n) \]

\[ d(g(v), n) = w(g(v), n) - w(g(v), -n) \]
Algebraic area versus intuitive area

\[ w(W', \xi) = 2 \] and \[ w(W', -\xi) = -1; \] intuitive winding numbers 3 and 0
Algebraic area versus intuitive area

\[ w(W', \xi) = 1 \text{ and } w_{\text{intuitive}}(W', \xi) = 2 \]
Algebraic area versus intuitive area

\[ w(W', \xi) = 1 \text{ and } w_{\text{intuitive}}(W', \xi) = 2(?) \]
ALGEBRAIC AREA VERSUS INTUITIVE AREA

\[ w(W', \xi) = 1 \text{ and } w_{\text{intuitive}}(W', \xi) = 1(!) \]
ALGEBRAIC AREA VERSUS INTUITIVE AREA
ALGEBRAIC AREA VERSUS INTUITIVE AREA
ALGEBRAIC AREA VERSUS INTUITIVE AREA
Episode III
REVENGE OF THE LEMMA
Spherical angle

$f = f_2$ is not an inflection face: $\hat{\alpha}_f = \alpha' = \pi - \alpha_f$
**Spherical angle**

\[ f = f_2 \text{ is an inflection face: } \hat{\alpha}_f = \alpha' = 2\pi - \alpha_f \]
Gauss image without self-intersections: $K > 0$

\[ \sum_{f \sim v} \hat{\alpha}_f - (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f \]
Gauss image without self-intersections: $K>0$

\[
\sum_{f \sim v} \hat{\alpha}_f - (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f
\]

\[
\implies \sum_{f \sim v} \hat{\alpha}_f = \sum_{f \sim v} (\pi - \alpha_f)
\]
GAUSS IMAGE WITHOUT SELF-INTERSECTIONS: $K>0$
\[ - \sum_{f \sim v} \hat{\alpha}_f + (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f \]
Gauss image without self-intersections: $K < 0$

$$- \sum_{f \sim V} \hat{\alpha}_f + (n - 2)\pi = K(v) = 2\pi - \sum_{f \sim V} \alpha_f$$

$$\implies \sum_{f \sim V} \hat{\alpha}_f = \sum_{f \sim V} (\pi + \alpha_f) - 4\pi$$
GAUSS IMAGE WITHOUT SELF-INTERSECTIONS: $K<0$

\[-\sum_{f \sim v} \hat{\alpha}_f + (n-2)\pi = K(v) = 2\pi - \sum_{f \sim v} \alpha_f\]

\[\implies \sum_{f \sim v} \hat{\alpha}_f = \sum_{f \sim v} (\pi + \alpha_f) - 4\pi\]

\[\hat{\alpha}_f = 2\pi - (\pi - \alpha_f) = \pi + \alpha_f \text{ if } f \text{ is not an inflection face}\]

\[\hat{\alpha}_f = 2\pi - (2\pi - \alpha_f) = \alpha_f \text{ if } f \text{ is an inflection face}\]
Gauss image without self-intersections: $K < 0$

Spherical pseudo-quadrilateral
Gauss image without self-intersections: $K<0$

Spherical pseudo-triangle (four inflection faces)
Gauss image without self-intersections: $K<0$

Spherical pseudo-triangle (two inflection faces)
Gauss image without self-intersections: $K<0$

Spherical pseudo-digon
Formula for general Gauss images

\[ I + 2C_+ - 2C_- = 2 + 2c \]
**Formula for General Gauss Images**

\[ I + 2C_+ - 2C_- = 2 + 2c \]

\[ I + 2C_+ - 2C_- = 4 + 2 \cdot 1 - 2 \cdot 2 = 2 = 2 + 2c \]
Episode IV
A NEW NORMAL
NORMAL VECTOR AND TRANSVERSAL PLANE

One-to-one projection of vertex star onto transversal plane
Dupin indicatrix

Intersection with a plane parallel and close to the tangent plane
Discrete Dupin indicatrix – $K>0$

Discrete ellipse in convex case
**Discrete Dupin indicatrix – $K<0$**

Inflection edge in the discreteDupin indicatrix

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Inflection edge in the discrete Dupin indicatrix
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola if Gauss image is star-shaped
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola (four inflection faces)
Discrete Dupin indicatrix – $K<0$

Discrete hyperbola (two inflection faces)
**Discrete Dupin indicatrix – K<0**

“Wrong” hyperbola
Star-shapedness and normals

Exterior star-shapedness (left) and different concepts of normals (right)
Episode V
THE GAUSS IMAGE STRIKES BACK
Gauss images do not intersect
Gauss images for positive curvature

Gauss images do not intersect

\[ \sum_{v \sim f_1} (\pi - \alpha_v) = n\pi - \sum_{v \sim f_1} \alpha_v \]

\[ = n\pi - (n - 2)\pi = 2\pi \]
Gauss images do not intersect
GAUSS IMAGES FOR NEGATIVE CURVATURE

Gauss images intersect transversally
GAUSS IMAGES FOR NEGATIVE CURVATURE

Branch point
Gauss images for negative curvature

Monkey saddle
GAUSS IMAGES FOR NEGATIVE CURVATURE

Polyhedral monkey saddle
Gauss images for negative curvature

Monkey friendly monkey saddle
GAUSS IMAGES FOR NEGATIVE CURVATURE

Deformation of a monkey saddle
Shapes of faces in negatively curved areas

\[ \hat{\alpha}_v = \alpha_v + \pi \text{ if } \alpha_v < \pi \text{ and } f_1 \text{ is not an inflection face}, \]
\[ \hat{\alpha}_v = \alpha_v \text{ if } \alpha_v < \pi \text{ and } f_1 \text{ is an inflection face}, \]
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\[ 2\pi = \sum_{v \sim f_1} \hat{\alpha}_v = \sum_{v \sim f_1} \alpha_v + c_1\pi - c_3\pi = (n - 2)\pi + c_1\pi - c_3\pi \]
Shapes of faces in negatively curved areas

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Thus, \( c_1 - c_3 = 4 - n \). Using \( c_1 + c_2 + c_3 + c_4 = n \),

\[ 2c_1 + c_2 + c_4 = 4. \]
Shapes of faces in negatively curved areas

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Thus, \( c_1 - c_3 = 4 - n \). Using \( c_1 + c_2 + c_3 + c_4 = n \),
\[ 2c_1 + c_2 + c_4 = 4. \]

\( f_1 \) planar: \( c_1 + c_2 \geq 3 \).
Shapes of faces in negatively curved areas

$c_1 = 0$, $c_2 = 4$, $c_3 = n - 4$, $c_4 = 0$
SHAPES OF FACES IN NEGATIVELY CURVED AREAS

\[ c_1 = 0, \ c_2 = 3, \ c_3 = n - 4, \ c_4 = 1 \]
Shapes of faces in negatively curved areas

$c_1 = 1, c_2 = 2, c_3 = n - 3, c_4 = 0$
Overlap of Gauss images
Gauss images for mixed curvature

Basic shapes of faces in areas of mixed curvature
GAUSS IMAGES FOR MIXED CURVATURE

Discrete parabolic curve
Smooth polyhedral surface

**Definition**

An orientable polyhedral surface $P$ immersed into $\mathbb{R}^3$ is *smooth* if:

1. $K(v) \neq 0$ for all $v$.
2. For all faces $f$, the sign of discrete Gaussian curvature changes at either zero or two edges.
3. $\forall f, \sum_{v \sim f} \hat{\alpha}_v$ is $\pm 2\pi$ or $0$ depending on whether $K(v)$ has the same sign for all $v \sim f$ or not. In the first case, we demand in addition that $f$ is star-shaped; in the latter case, we require that the convex hull of vertices of positive discrete Gaussian curvature does not contain any vertex of negative curvature.
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2. $\forall v \exists n, n' \in S^2$ such that $\langle n', n_f \rangle > 0 \ \forall n_f, f \sim v$, and such that the spherical polygon defined by $n_f$ is star-shaped with respect to $n$. 


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Episode VI
RETURN OF PROJECTIVE TRANSFORMATIONS
Theorem

Let $P$ be a smooth polyhedral surface immersed into $\mathbb{R}^3 \subset \mathbb{RP}^3$. Furthermore, let $\pi$ be a projective transformation that does not map any point of $P$ to infinity. Then, $\pi(P)$ is a smooth polyhedral surface.
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**Proof.**

- Infinitesimally, $\pi$ given by affine transformation
NOTION OF SMOOTHNESS IS PROJECTIVELY INVARIANT

THEOREM

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PROOF.

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- Gauss images remain free of self-intersections
**THEOREM**

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- $\pi$ maps normal direction (transversal plane) to normal direction
*Notion of smoothness is projectively invariant*

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- Gauss images remain free of self-intersections
- $\pi$ maps normal direction (transversal plane) to normal direction
- $\pi$ maps tangent plane to tangent plane
- star-shapedness of Gauss images encoded in intersection with parallel planes: $\pi$ maps tangent plane to tangent plane
**Theorem**

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**Notion of Smoothness is Projectively Invariant**

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- inflection faces are mapped to inflection faces
- “right” condition for mixed faces
Projective dual surfaces of smooth polyhedral surfaces are smooth.
Episode VII
THE APPLICATIONS AWAKEN
RESULTS OF OUR ALGORITHM

Smoothness not visible in all renderings
RESULTS OF OUR ALGORITHM

Optimization started from (a) non-smooth; (b) weakly smooth surface
RESULTS OF OUR ALGORITHM

Geometric constraints may lead to bad reflections and zigzag polylines
RESULTS OF OUR ALGORITHM

Optimization of Louvre surface
Geometric constraints for smoothness

Relevance of the alignment of the mesh along asymptotic directions
Episode VIII
FALL OF THE INITIALIZATION
THANK YOU VERY MUCH FOR YOUR ATTENTION!