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Discrete complex analysis on quad-graphs

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(joint work with Alexander I. Bobenko)

We deal with a linear discretization of complex analysis on quad-graphs. Discrete holomorphic functions on the square lattice were studied by Isaacs [7], where he proposed two different definitions for holomorphicity. One of them was reintroduced by Ferrand [6] and studied extensively by Duffin [4], who extended the notion of discrete holomorphicity to rhombic lattices [5]. The investigation of discrete complex analysis on rhombic lattices was resumed by Mercat [9], Kenyon [8], Chelkak and Smirnov [3]. For a survey on the theory of discrete complex analysis based on circle patterns and its relation to the linear theory, see the book of Bobenko and Suris [2].

Our setting is a bipartite quad-graph Λ corresponding to a strongly regular and locally finite cell decomposition of a Riemann surface consisting of quadrilaterals only. Mainly, we are interested in bipartite quad-graphs embedded in the complex plane. We denote by Γ respectively Γ^* the maximal independent sets of Λ. In addition to the graph Λ, its dual $\Diamond := \Lambda^*$ will come up in the definitions of discrete derivatives.

Being consistent with the previous definitions of discrete holomorphicity, a function $H : \Lambda \to \mathbb{C}$ is *discrete holomorphic* on the face $z \in \Diamond$, iff

$$\frac{H(u_+) - H(u_-)}{u_+ - u_-} = \frac{H(w_+) - H(w_-)}{w_+ - w_-}$$

for the two diagonals u_-u_+ and w_-w_+ of z. Based on this notion of holomorphicity, we generalize the *discrete derivatives* $\partial, \bar{\partial}$ of [3] to arbitrary quadrilaterals. These discrete derivatives map functions on Λ to functions on $\Diamond$ or vice versa. As in the rhombic setting, we can find *discrete primitives* of discrete holomorphic functions on simply-connected domains of $\Diamond$.

For functions on Λ, we show the factorization $4\partial\bar{\partial} = 4\bar{\partial}\partial = \triangle$ where $\triangle$ is the *discrete Laplacian* introduced by Mercat [10] and studied by Skopenkov [11]. As a corollary, ∂H is discrete holomorphic if $H : \Lambda \to \mathbb{C}$ is *discrete harmonic*, i.e. $\triangle H \equiv 0$. Also, the real and the imaginary part of a discrete holomorphic function $H : \Lambda \to \mathbb{C}$ are discrete harmonic. Moreover, if H is discrete holomorphic on a simply-connected domain, the imaginary part of H is determined uniquely by its real part up to two additive constants on Γ and Γ^*. In particular, a discrete holomorphic and purely real or purely imaginary function H on a simply-connected domain is constant on Γ and constant on Γ^*.

Skopenkov proved existence and uniqueness of solutions to the discrete Dirichlet boundary value problem [11]. Basing on this result, we prove surjectivity of the operators $\partial, \bar{\partial}, \Diamond$ on discrete domains homeomorphic to a disk or the plane. Especially, *discrete Green's functions* $G(\cdot; v_0) : \Lambda \to \mathbb{R}$ and *discrete Cauchy kernels* $K(\cdot, v_0) : \Diamond \to \mathbb{C}$ and $K(\cdot, z_0) : \Lambda \to \mathbb{C}$ exist for all $v_0 \in \Lambda$ and $z_0 \in \Diamond$ in the case that Λ discretizes the complex plane. For all $v \in \Lambda$ and $z \in \Diamond$, these functions fulfill

$$G(v_0; v_0) = 0 \text{ and } \triangle G(v; v_0) = \frac{1}{2\mu_\Lambda(v_0)}\delta_{vv_0},$$

$$\bar{\partial} K(\cdot; z_0) = \delta_{zz_0}\frac{\pi}{\mu_\Diamond(z_0)} \text{ and } \bar{\partial} K(\cdot; v_0) = \delta_{vv_0}\frac{\pi}{\mu_\Lambda(v_0)}.$$

Here δ is the Kronecker delta, and $\mu_\Lambda(v_0)$ are $\mu_\Diamond(z_0)$ are geometric weights associated to vertices of Λ and $\Diamond$ which already appear in the definitions of ∂ and $\bar{\partial}$. Note that we do not require any certain asymptotic behavior. However, we construct discrete Green's functions and Cauchy kernels with asymptotics analogous to the rhombic [3, 8] and close to the smooth case if all quadrilaterals are parallelograms with bounded interior angles and bounded ratio of side lengths. The construction of these functions is closely related to discrete complex analysis on quasicrystallic parallelogram-graphs [1] and uses the connection to discrete integrable systems [2].

To state discrete Cauchy formulas in a simpler way than in [3], we introduce the *medial graph* X of Λ which is defined as follows. The vertex set is given by the set of midpoints of all edges of Λ and two vertices are adjacent iff the corresponding edges belong to the same face and have a vertex in common. The set of faces of X is in bijective correspondence with the vertex set $\Lambda \cup \Diamond$: A face corresponding to a vertex v of Λ consists of the midpoints of all edges incident to v and a quadrilateral face corresponding to a face z of $\Diamond$ consists of the four midpoints of edges belonging to z. So given two functions $F : \Diamond \to \mathbb{C}$ and $H : \Lambda \to \mathbb{C}$, we can define a product $F \cdot H$ on the edges of X in a canonical way. Such functions can then be integrated along paths of edges of X.

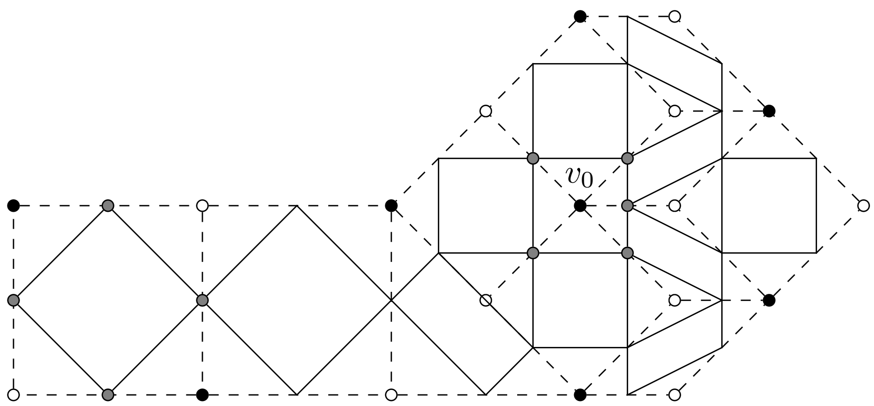

FIGURE 1. Bipartite quad-graph Λ (dashed) with medial graph X

Note that choosing $F \equiv 1$ or $H \equiv 1$, F respectively H is discrete holomorphic iff all closed discrete line integrals of $1 \cdot H$ respectively $F \cdot 1$ vanish, yielding a discrete Morera's theorem. Also, all closed discrete line integrals of $F \cdot H$ vanish if F and H are both discrete holomorphic. Though, $F \cdot H$ is not everywhere discrete holomorphic in the sense of the theory above. More precisely, $F \cdot H$ is defined on the vertices of the medial graph of X, which is the dual of a bipartite quad-graph having $\Lambda \cup \Diamond$ as one maximal independent vertex set and the midpoints of edges of Λ as the other. But in general, $F \cdot H$ is discrete holomorphic on the vertices of $\Lambda \cup \Diamond$ only.

Let $F : \Diamond \to \mathbb{C}$ and $H : \Lambda \to \mathbb{C}$ be discrete holomorphic, and $v_0 \in \Lambda$, $z_0 \in \Diamond$. If $K(\cdot, z_0) : \Lambda \to \mathbb{C}$ and $K(\cdot, v_0) : \Diamond \to \mathbb{C}$ are discrete Cauchy kernels, then for any discrete contour C_{z_0} or C_{v_0} on the medial graph X surrounding z_0 respectively v_0 once in counterclockwise order (e.g. the paths determined by the gray vertices in Figure 1), the *discrete Cauchy formula* holds true:

$$F(z_0) = \frac{1}{2\pi i} \oint\limits_{C_{z_0}} F \cdot K(\cdot; z_0),$$

$$H(v_0) = \frac{1}{2\pi i} \oint\limits_{C_{v_0}} K(\cdot; v_0) \cdot H.$$

If additionally C_{z_0} does not pass through any vertex incident to z_0, then

$$\partial H(z_0) = \frac{1}{2\pi i} \oint\limits_{C_{z_0}} -\partial K(\cdot; z_0) \cdot H.$$

In the special case of the $\mathbb{Z}^2$-lattice of a skew coordinate system in the complex plane, $\Diamond \cong \Lambda$ and all derivatives of a discrete holomorphic functions are discrete holomorphic themselves. We derive discrete Cauchy formulas for all discrete derivatives and show that the nth derivative of the discrete Cauchy kernel with asymptotics $O(x^{-1})$ has asymptotics $O(x^{-(n+1)})$.

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Consistent discretizations of the Laplace–Beltrami operator and the Willmore energy of surfaces

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(joint work with Konrad Polthier)

A fundamental aspect when translating classical concepts from smooth differential geometry, such as differential operators or geometric functionals, to corresponding discrete notions is consistency. A discretization is consistent if the discrete operator or functional converges to its smooth counterpart in the limit of refinement. Here, we are concerned with the construction of consistent counterparts to the Laplace–Beltrami operator and the Willmore energy on polyhedral surfaces in $\mathbb{R}^3$. Our starting point is the weak form of the Laplace–Beltrami operator (LBO) on a smooth surface M. This is the continuous linear operator that maps any $u \in H^1(M)$ to the distribution $\Delta u \in H^1(M)'$ that is given by

$$\langle \Delta u | \varphi \rangle = -\int_M g(\operatorname{grad} u, \operatorname{grad} \varphi)\, \mathrm{d}vol \tag{1}$$

for all $\varphi \in H^1(\mathcal{M})$. Here $H^1(M)$ denotes the Sobolev space of functions whose first derivative is square integrable, $H^1(M)'$ is the (topological) dual space, and $\langle\cdot|\cdot\rangle$ denotes the pairing of $H^1(M)'$ and $H^1(M)$. This operator can be rigorously defined on polyhedral surfaces, and, in [1], convergence of this operator to its smooth counterpart in an appropriate operator norm was shown. To discretize the operator, we restrict u and φ to be functions in the finite dimensional subspace S_h of H^1 consisting of continuous functions that are piecewise linear over the triangles. Then, the discrete weak LBO is a map from S_h to S_h' (the dual space of S_h). It can be used to discretize second order differential equation on surfaces, and convergence of solutions of discrete Dirichlet problems of Poisson's equation was shown in [2]. In contrast to the weak form, discretizations of the strong LBOs are endomorphisms of S_h. Based on the discrete weak LBO, different constructions of discrete strong LBOs are used in practice, but none of them was proven to be consistent and counterexamples to pointwise convergence have been reported.

Here, we introduce a consistent discretization of the strong LBO. As a tool for the construction, we use functions that we call r-local functions.

Definition 1. *Let $\mathcal{M}$ be a smooth or a polyhedral surface in $\mathbb{R}^3$, and let C_D be a positive constant. For any $x \in \mathcal{M}$ and $r \in \mathbb{R}^+$, we call a function $\varphi : \mathcal{M} \mapsto \mathbb{R}$* r-local *at x (with respect to C_D) if the criteria*