Let $G$ be a simply connected region in $\mathbb{C}$ with a non-trivial sufficiently smooth boundary $\partial G$ parametrized by $c : [0, 1] \to \mathbb{C}$, and let $z_0 \in G$. We are looking for a continuous function $f : \overline{G} \to \mathbb{C}$, holomorphic on $G$, with $f(z_0) = 0$ and $|f(z)| = 1$ for all $z \in \partial G$. \hfill (1)

By the maximum principle then $f(G) \subset \mathbb{D}$. Moreover, the winding number

$$n(f \circ c, a) = \frac{1}{2\pi i} \int_0^1 \frac{f'(c(t))c'(t)}{f(c(t))} \, dt = \frac{1}{2\pi i} \int_c f(z) - a \, dz$$

is constant on $\mathbb{D}$, all values $a \in \mathbb{D}$ are attained the same number of times. With the ansatz

$$f(z) = (z - z_0)e^{g(z)} \hfill (2)$$

$f : G \to \mathbb{D}$ has exactly one zero, and hence is bijective.

How to find such an $f$?

We split $g$ into real and imaginary part:

$$g = u + iv.$$ 

Then \hfill (1) is equivalent with

$$\log |z - z_0| + u(z) = 0 \text{ on } \partial G.$$ 

We therefore want a continuous function $u$ on $\tilde{G}$, harmonic on $G$, with prescribed boundary values $u_0(z) := -\log |z - z_0|$ on $\partial G$. This is now commonly called a Dirichlet problem.

Put $\mathcal{F} := \{ \tilde{u} : \tilde{G} \to \mathbb{R} \mid \tilde{u}|_{\partial \tilde{G}} = u_0 \}$, where the regularity of the $\tilde{u}$ is assumed to suffice for the following arguments. On $\mathcal{F}$ we consider the functional

$$E(\tilde{u}) := \int_{\tilde{G}} \| \text{grad } \tilde{u} \|^2 \, dxdy$$

and take $u \in \mathcal{F}$ which minimizes this functional:

$$E(u) := \min_{\tilde{u} \in \mathcal{F}} E(\tilde{u}).$$

Then for any function $v : \tilde{G} \to \mathbb{C}$ with compact support in $G$

$$E(u + tv) = \int_{\tilde{G}} \| \text{grad } u + t \text{ grad } v \|^2 \, dxdy$$

$$= \int_{\tilde{G}} \| \text{grad } u \|^2 + 2t \int_{\tilde{G}} \langle \text{grad } u, \text{ grad } v \rangle \, dxdy + t^2 \int_{\tilde{G}} \| \text{grad } v \|^2 \, dxdy.$$
This is minimal for $t = 0$ if and only if $\int_{G} \langle \text{grad } u, \text{grad } v \rangle \, dx \, dy = 0$.

But

$$\int_{G} \langle \text{grad } u, \text{grad } v \rangle \, dx \, dy = \int_{G} \text{div}(v \, \text{grad } u) \, dx \, dy - \int_{G} v \Delta u \, dx \, dy$$

$$= \int_{\partial G} \left\langle v \, \text{grad } u, \text{unit normal } \nu \right\rangle \, dO - \int_{G} v \Delta u \, dx \, dy.$$

Besides the product rule for $\text{div } v \, \text{grad } u$ we used Stokes and the information $v|_{\partial G} = 0$. Now

$$\int_{G} v \Delta u \, dx \, dy = 0$$

for all compactly supported $v$ implies $\Delta u = 0$, i.e. $u$ is harmonic. This is referred to as the Dirichlet principle.

Locally, $u$ is the real part of a holomorphic function $g$ with derivative

$$g' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}.$$

But this is defined on all of $G$, and therefore $g$ can be analytically continued along any curve in $G$. Now $G$ is simply connected, and the monodromy theorem applies to give a holomorphic function $g$ with real part $u$ globally defined on $G$. Let us assume it to be continuous on $\bar{G}$. Then (2) defines $f$ with the required properties.

This “proof” has several unclear points: The regularity requirements for the boundary, the regularity of the functions involved, but overall the existence of the minimizer $u \in \mathcal{F}$ for the energy functional $E$. 