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The Comparability Graph and the Graph of Linear Extensions of a Poset

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The Comparability Graph and the Graph of Linear Extensions of a Poset

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Abstract

For an ordered set P let $\mathrm{G}(P)$ be the graph of linear extensions of P with two of these being connected if they differ by an adjacent transposition and let $\mathrm{Comp}(P)$ denote the comparability graph of P. We prove that $\mathrm{G}(P) \cong \mathrm{G}(Q)$ implies $\mathrm{Comp}(P) \cong \mathrm{Comp}(Q)$ for finite posets P and Q with $|P| = |Q|$ and show that the reverse implication does not hold. Since the lattice $\mathrm{Ext}(P)$ of extensions of P is dually isomorphic to the lattice of convex sets of $\mathrm{G}(P)$, these results carry over to $\mathrm{Ext}(P)$, i.e., $\mathrm{Ext}(P) \cong \mathrm{Ext}(Q)$ implies $\mathrm{Comp}(P) \cong \mathrm{Comp}(Q)$, but not the other way around.

1 Introduction and Results

For a (partially) ordered set $(X, \leq)$ the graph $\mathrm{Comp}(P) = (X, \mathrm{comp}(P))$ with $\mathrm{comp}(P) = \{\{x, y\} | x \leq y \text{ or } y \leq x\}$ is the so-called comparability graph of P. Many parameters of an ordered set P (e.g. number of linear extensions, dimension, jump number) depend only on the comparability graph $\mathrm{Comp}(P)$ and many classes of ordered sets are closed (invariant) with respect to the comparability graphs, which means that $P \in \mathfrak{C}$ and $\mathrm{Comp}(P) \cong \mathrm{Comp}(Q)$ implies $Q \in \mathfrak{C}$. For a survey we refer to [3], [6], [5]. We restrict ourselves in the following to the finite case. That is, we make the general assumption of finiteness throughout this paper.

There is another fundamental graph related to a poset P, which we call the graph of linear extensions. Informally it consists of the linear extensions of P with two of these being connected if they differ by an adjacent pair. This is then a labeled graph $\mathrm{G}^{\mathrm{l}}(P)$ and we write $\mathrm{G}(P) = (\mathrm{V}(P), \mathrm{E}(P))$ if we assume that the labeling is not known. The formal definitions are given at the beginning of Section 2. We refer to [8] for detailed information and references on $\mathrm{G}(P)$. Here we only recall a theorem which is fundamental for our purpose. To do so, some definitions are needed. For $U \subseteq \mathrm{V}(P)$ let $\mathrm{conv}(U)$ be the (geodesic) convex hull of U, i.e., all vertices on shortest paths must be included. Let $\mathrm{Conv}(P) := (\{\mathrm{conv}(U) | U \subseteq \mathrm{V}(P)\}, \subseteq)$ denote the lattice of convex sets of $\mathrm{G}(P)$. By $\mathrm{Ext}(P)$ we denote the lattice of poset extensions of P (a poset

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$Q = (X, R)$ extends a poset $P = (X, S)$ if $R \supseteq S$), where an artificial top element $\top$ is added. Here is the theorem which is proven in [8] :

Theorem 1. *The lattice* $\mathrm{Ext}(P)$ *is dually isomorphic to the lattice* $\mathrm{Conv}(P)$.

An example which illustrates the definitions and the theorem is given in Figure 1. We are now in a position to state the main results of this paper:

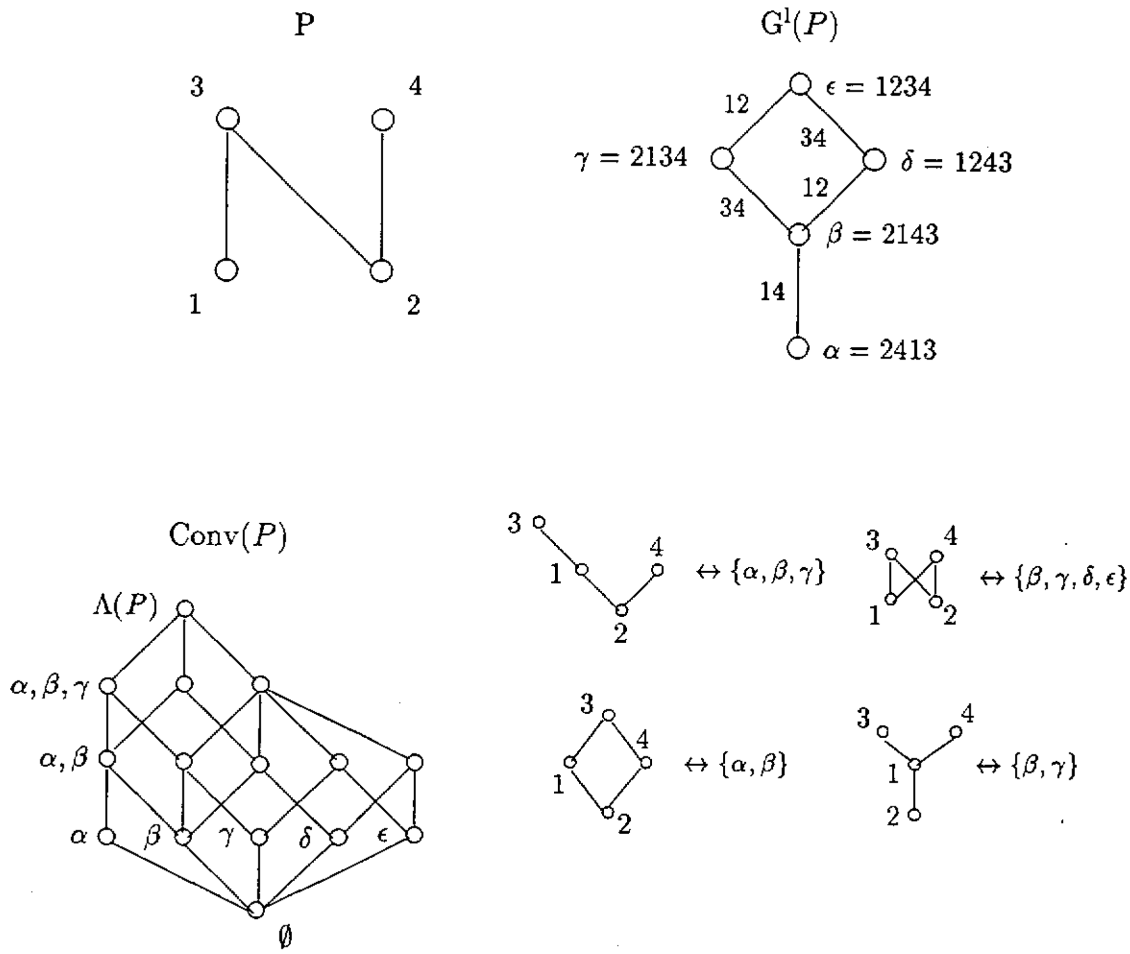


Figure 1

Theorem 2. *For posets* P *and* Q *with* $|P| = |Q|$ *it holds:*

$$\mathrm{G}(P) \cong \mathrm{G}(Q) \quad (\textit{resp.}\ \mathrm{Ext}(P) \cong \mathrm{Ext}(Q)) \ \textit{implies} \ \mathrm{Comp}(P) \cong \mathrm{Comp}(Q)$$

We shall state a bit more general result, in which the cardinalities of P and Q might differ, in the following Theorem 3. For an ordered set $P = (X, \leq)$ let $P^* =$

$(X^*, \leq_{|X^* \times X^*})$ denote the order in which all elements which are comparable to all others are removed.

Theorem 3. *For posets P and Q it holds:*

$$\mathrm{G}(P) \cong \mathrm{G}(Q) \quad (\textit{resp.}\ \mathrm{Ext}(P) \cong \mathrm{Ext}(Q)) \ \textit{implies}\ \mathrm{Comp}(P^*) \cong \mathrm{Comp}(Q^*)$$

An example which illustrates Theorem 3 is depicted in Figure 2.

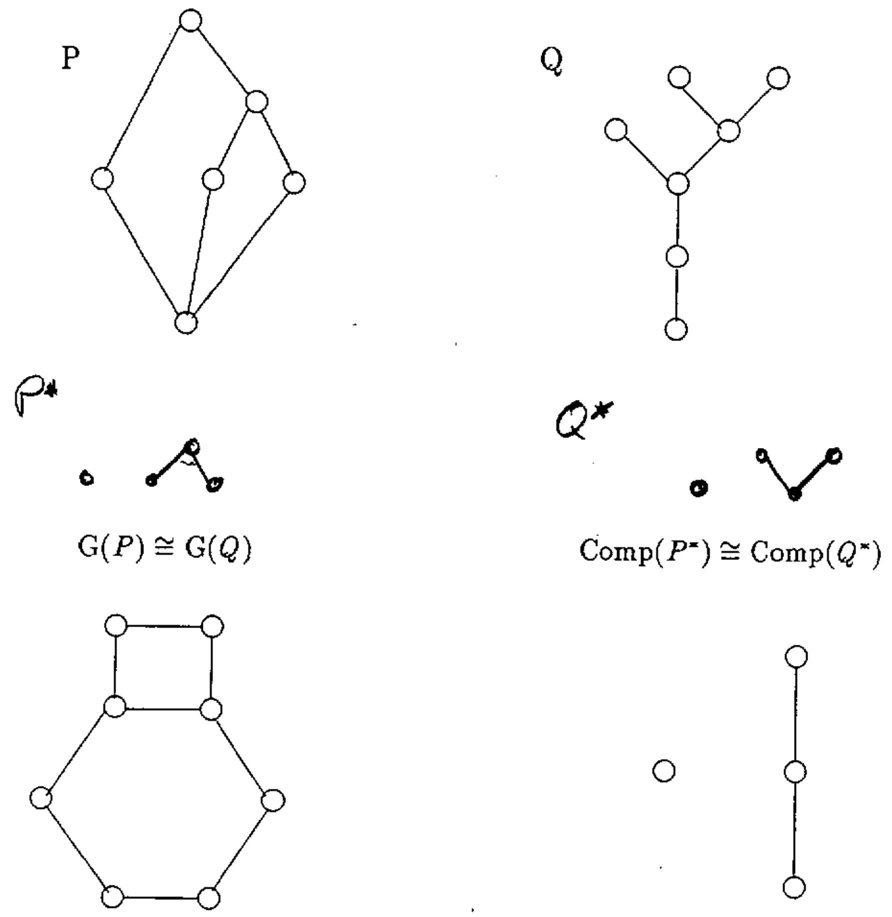


Figure 2

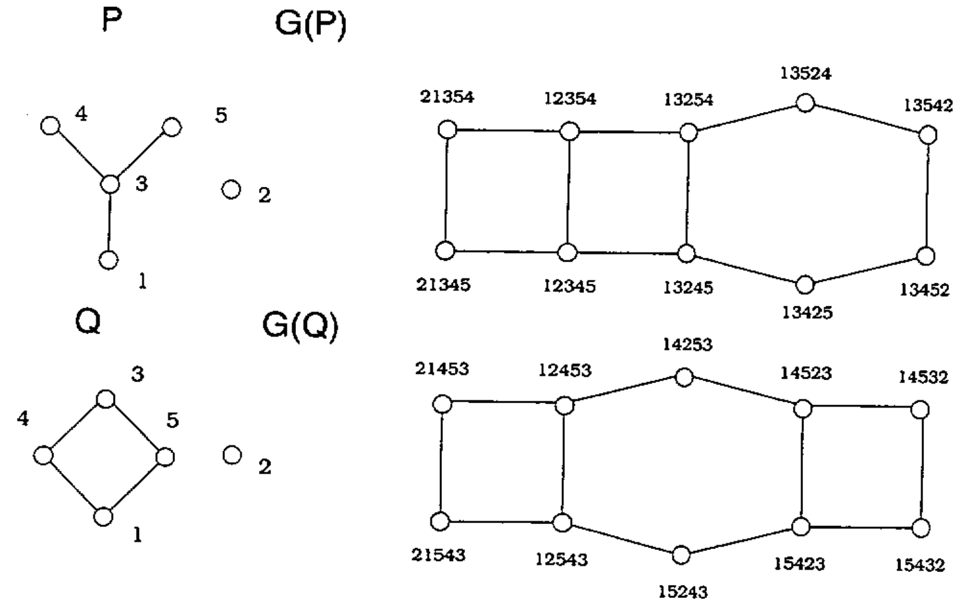
The reverse of Theorem 2 does not hold:

Theorem 4. *For the posets P and Q as depicted in Figure 3 it holds:*

$$\mathrm{Comp}(P) \cong \mathrm{Comp}(Q) \ \textit{and}\ \mathrm{G}(P) \ncong \mathrm{G}(Q)$$

2 Some convexity concepts

We describe a linear extension of an ordered set $P = (X, \leq)$ as a word (permutation of the elements of X) and use Greek letters to denote them. That



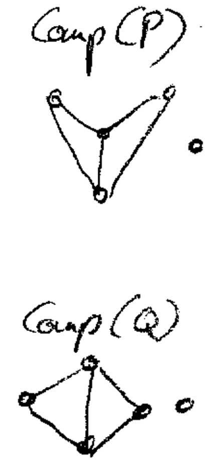


Figure 3:

is, α is a linear extension of $P = (X, \leq)$ if for all $x < y$ in P the position of x is left to that of y in the word α . The set of all linear extensions of P will be denoted by $\Lambda(P)$. If $\beta \in \Lambda(P)$ results from $\alpha \in \Lambda(P)$ by the switch of two adjacent elements x and y, then x and y must be incomparable. $\mathrm{inc}(P) := \{\{x, y\} | x, y \in X, x \not\leq y \text{ and } y \not\leq x\}$ denotes the set of all incomparable pairs of P.

Definition 1. *Let $P = (X, \leq)$ be an ordered set. The labeled graph of linear extensions of P is defined as*

$$\mathrm{G}^{\mathrm{l}}(P) = (\mathrm{V}(P), \phi, \mathrm{E}(P), \psi).$$

Here $\mathrm{V}(P)$ is a set (of vertices) which is bijective to $\Lambda(P)$. The bijection is given by $\phi : \mathrm{V}(P) \longrightarrow \Lambda(P)$. Furthermore,

$\mathrm{E}(P) := \{\{u, v\} | u, v \in \mathrm{V}(P), \phi(u)$ *and* $\phi(v)$ *differ by the switch of an adjacent pair* $\}$.

If xy is such a pair, then we set $\psi(\{u, v\}) := \{x, y\}$. Thus ψ maps $\mathrm{E}(P)$ to $\mathrm{inc}(P)$. The graph of linear extensions of P is now defined as $\mathrm{G}(P) = ((\mathrm{V}(P), \mathrm{E}(P))$.

At some places it will be of importance to express whether xy is switched to yx or vice versa. Then we assume that an edge $\{u, v\}$ actually consists of the two directed edges (u, v) and (v, u), and write $\psi((u, v)) = (x, y)$ if $\phi(u) = \ldots xy \ldots$ and $\phi(v) = \ldots yx \ldots$.

For posets $P = (X, \leq)$ and $Q = (Y, \leq)$ we define the ordinal sum $P \oplus Q$ to be the order on $X \uplus Y$, for which in addition to the comparabilities on X and Y all elements of X are less than all elements of Y.

Proposition 1. $\mathrm{G}(P \oplus Q) \cong \mathrm{G}(P) \times \mathrm{G}(Q)$ *for posets* P *and* Q.

Proof. The linear extensions of $P \oplus Q$ arise as concatenations of all linear extensions of P with all linear extensions of Q. Thus $\mathrm{V}(P \oplus Q) \cong \mathrm{V}(P) \times \mathrm{V}(Q)$. Since two linear extensions of $P \oplus Q$ differ by an adjacent pair if and only if this pair occurs on either the P-part or the Q-part of these linear extensions, the graph $\mathrm{G}(P \oplus Q)$ is indeed isomorphic to the cartesian product of the graphs $G(P)$ and $G(Q)$. □

As mentioned in the introduction $\mathrm{conv}(U)$ denotes the geodesic convex hull of U. We write $[u,v]$ for the interval $\mathrm{conv}(\{u,v\})$.

Lemma 1. *Let* P *be a poset and* $\mathrm{G}^{\mathrm{l}}(P)$ *its labeled graph of linear extensions. If* $(u,v),(w,z) \in \mathrm{E}(P)$ *with* $\psi((u,v)) = \psi((w,z))$*, then the intervals* $[u,w]$ *and* $[v,z]$ *are isomorphic as graphs.*

Proof. Let $(x,y) := \psi((u,v)) = \psi((w,z))$, then the isomorphism is given by interchanging x and y in the words corresponding to $[u,w]$ and $[v,z]$. □

We now introduce the concept of halfspaces in two ways, one using the labeling and the other not:

$$\mathrm{H}(x < y) := \{v \in V | x \text{ is left to } y \text{ in } \phi(v)\} \quad \text{for } \{x,y\} \in \mathrm{inc}(P)$$

and

$$\mathrm{H}(u,v) := \{w \in V | \operatorname{dist}(u,w) < \operatorname{dist}(v,w)\} \quad \text{for } \{u,v\} \in \mathrm{E}(P),$$

where $\operatorname{dist}(u,w)$ denotes the number of edges of a shortest path between u and w. We remark that the second definition is used in [1] in order to characterize isometric subgraphs of hypercubes. Since the halfspaces so defined are convex (cf. [8]) and $\mathrm{G}(P)$ is bipartite, it follows by the result of [1] that $\mathrm{G}(P)$ is an isometric subgraph of a hypercube.

Proposition 2. *i) Let* $\{x,y\},\{r,s\} \in \mathrm{inc}(P)$*, then* $\mathrm{H}(x < y) = \mathrm{H}(r < s)$ *implies* $x = r$ *and* $y = s$.

ii) Let $\{x,y\} \in \mathrm{inc}(P)$ *and* (u,v) *a directed edge of* $G(P)$*, then* $\psi((u,v)) = (x,y)$ *implies* $\mathrm{H}(u,v) = \mathrm{H}(x < y)$.

Proof. i) Let $\mathrm{H}(x < y) = \mathrm{H}(r < s)$. We claim that $r \leq x$ and $y \leq s$, since otherwise there exists a linear extension α with x left to y and s left to r, which means that $\alpha \in \mathrm{H}(x < y) - \mathrm{H}(r < s)$. In order to show this we assume first that $y \not\leq s$. Then we take as α a linear extension of $\{z \in X | z \leq s\} \oplus \{z \in X | z \not\leq s, z \not\geq y\} \oplus \{z \in X | z \geq y\}$. If $r \not\leq x$ then we choose as α a linear extension of $\{z \in X | z \leq x\} \oplus \{z \in X | z \not\leq x, z \not\geq r\} \oplus \{z \in X | z \geq r\}$. Similar $x \leq r$ and $s \leq y$ holds, since otherwise there exists a linear extension α with y left to x and r left to s which means that $\alpha \in \mathrm{H}(r < s) - \mathrm{H}(x < y)$.

ii) Let $w \in \mathrm{H}(u,v)$, i.e., $\operatorname{dist}(u,w) < \operatorname{dist}(v,w)$ and assume that $w \in \mathrm{H}(y < x)$. Then on a shortest path from u to w there must be u' and v' with $\psi(u',v') = (x,y)$. Since $[u,u']$ and $[v,v']$ are isomorphic by Lemma 1, we can construct a path from v to w via v' which is shorter, a contradiction. Now let $w \in \mathrm{H}(x < y)$. Then x is left to y in $\phi(w)$ as well as in $\phi(v)$. Then w is closer to u than to v, since otherwise x and y would be switched somewhere on a shortest path and we could shorten the path by an argument similar to the one given before. $\square$

$U \subseteq \mathrm{V}(P)$ is called biconvex if U and $\mathrm{V}(P) - U$ are convex. It can be shown that the biconvex subsets of $V(P)$ are just the halfspaces of $G(P)$. Since this is not really needed for this paper, we do not prove it here. See [8] for more details.

Now, we define an equivalence relation on the edges of $\mathrm{G}(P)$ for the directed as well as the undirected case.

Definition 2. *Two directed edges $(u,v),(w,z)$ of $E(P)$ are called parallel, written $(u,v)\|(w,z)$, if $H(u,v) = H(w,z)$. Two undirected edges $\{u,v\}$ and $\{w,z\}$ are called parallel, written $\{u,v\}\|\{w,z\}$ if the partitions $\mathrm{H}(u,v) \uplus \mathrm{H}(v,u)$ and $\mathrm{H}(w,z) \uplus \mathrm{H}(z,w)$ of $\mathrm{V}(P)$ are the same, i.e., if $\mathrm{H}(u,v) = \mathrm{H}(w,z)$ or $\mathrm{H}(u,v) = \mathrm{H}(z,w)$.*

Edges with same color are parallel.

It can be shown that the graph $G(P)$ has a drawing in which edges are parallel if and only if they are so in the drawing. One represents $\mathrm{G}^1(P)$ first in $\mathbb{R}^n$ with $n = |P|$ and then project it in general direction to $\mathbb{R}^2$. A word α (considered as a permutation) is mapped to its inverse α^{-1} and this is interpreted as a vector in $\mathbb{R}^n$. For example, $\alpha = 2413 \doteq \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$, $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} \doteq 3142$; thus α is mapped to $(3,1,4,2) \in \mathbb{R}^4$.

Definition 3. *Let E_1 and E_2 be two classes of the equivalence relation $\|$. We say that E_1 and E_2 are incident if there exist $u,v,w \in V$ with $\{u,v\} \in E_1, \{v,w\} \in E_2$ and $[u,w] = \{u,v,w\}$.*

Proposition 3. *Let $P = (X, \leq)$ be a poset and $\mathrm{G}^1(P) = (V, \phi, E, \psi)$ its labeled graph of linear extensions, then it holds*

(i) $\psi(E) = \mathrm{inc}(P)$,

(ii) $\{u,v\}\|\{w,z\}$ if and only if $\psi(\{u,v\}) = \psi(\{w,z\})$,

(iii) the mapping $\psi^{-1} : \mathrm{inc}(P) \longrightarrow E/\|$ is a bijection,

(iv) two edges $\{x,y\}$ and $\{r,s\}$ of $\mathrm{inc}(P)$ have an element in common, i.e., $|\{x,y\} \cap \{r,s\}| = 1$, if and only if $\psi^{-1}(\{x,y\})$ is incident to $\psi^{-1}(\{r,s\})$.

Proof. (i) it must be shown that ψ is surjectiv. For $x, y \in X$ with x incomparable to y there exist linear extensions of P of the form $\alpha = \ldots xy \ldots$ and $\beta = \ldots yx \ldots$ with the same beginning and end segments. Take e.g. for the beginning segment a linear extension of $\{z \in X | z < x \text{ and } z < y\}$ and for the end segment a linear extension of $\{z \in X | z \not\leq x \text{ and } z \not\leq y\}$. Now $\psi(\{\phi^{-1}(\alpha), \phi^{-1}(\beta)\}) = \{x,y\}$.

(ii) this follows from Proposition 1.

(iii) this is a consequence of (i) and (ii).

(iv) Let $\{x, y\}, \{r, s\} \in \text{inc}(P)$ and assume w.l.o.g. that $x = r$ and $s \not\leq y$. Then there exist three linear extensions of P:

Stimmt nicht!
Falls $y < s$, aber $\exists\, w$ mit $y < w < s$!

$$\phi(u) := \ldots xys \ldots$$

$$\phi(v) := \ldots yxs \ldots$$

$$\phi(w) := \ldots ysx \ldots$$

where the dotted segments are the same for $\phi(u), \phi(v)$ and $\phi(w)$. Just take suited linear extension of $\{z \in X | z < x \text{ or } z < y \text{ or } z < s\} \oplus \{x, y, s\} \oplus \{z \in X | z \not\leq x \text{ and } z \not\leq y \text{ and } z \not\leq s\}$. Thus there exist $u, v, w \in \mathrm{V}(P)$ with $\{u, v\} \in \psi^{-1}(\{x, y\}), \{v, w\} \in \psi^{-1}(\{r, s\})$ and $[u, w] = \{u, v, w\}$.

Now assume that there exist $u, v, w \in \mathrm{V}(P)$ with $\{u, v\} \in \psi^{-1}(\{x, y\}), \{v, w\} \in \psi^{-1}(\{r, s\})$ and $[u, w] = \{u, v, w\}$. If $\{x, y\} \cap \{r, s\} = \emptyset$,then we can assume w.l.o.g. that

$$\phi(u) = \ldots xy \ldots rs \ldots$$

$$\phi(v) = \ldots yx \ldots rs \ldots$$

$$\phi(w) = \ldots yx \ldots sr \ldots,$$

where the dotted segments are again the same for $\phi(u), \phi(v)$ and $\phi(w)$. But now there exists in addition the linear extension $\phi(v') = \ldots yx \ldots rs \ldots$ and it holds $[u, w] = \{u, v, v', w\}$. $\square$

We are now in a position to give the proves of Theorem 2 and Theorem 3.

Proves of Theorem 2 and Theorem 3. First we prove that $\mathrm{G}(P) \cong \mathrm{G}(Q)$ implies $\mathrm{Comp}(P^*) \cong \mathrm{Comp}(Q^*)$. Let $\mathrm{Coco}(P) = (X, \text{inc}(P))$ denote the cocomparability graph of P (this is the complementary graph of $\mathrm{Comp}(P)$) and $\mathrm{L}(\mathrm{Coco}(P))$ its line graph (see e.g. [4] for a survey on line graphs). Because of (iii) and (iv) in Proposition 3 $\mathrm{G}(P) \cong \mathrm{G}(Q)$ implies $\mathrm{L}(\mathrm{Coco}(P)) \cong \mathrm{L}(\mathrm{Coco}(Q))$. Up to isolated points and a small exception, a graph is reconstructable from its line graph. If the graph is connected, then the only exception is $\mathrm{L}(K_3) \cong \mathrm{L}(S_3) \cong K_3$ where K_3 is a triangle and S_3 is a 3-star. Up to the discussion of this exception, we can conclude that $\mathrm{Coco}(P)$ minus its isolated points is isomorphic to $\mathrm{Coco}(Q)$ minus its isolated points. Thus $\mathrm{Coco}(P^*) \cong \mathrm{Coco}(Q^*)$ and $\mathrm{Comp}(P^*) \cong \mathrm{Comp}(Q^*)$. The two posets which correspond to K_3 and S_3 are the three element antichain A_3 and the poset $C_{1,3}$ which consists of a three element chain and another element incomparable to all others. In fact, if $\mathrm{L}(\mathrm{Coco}(P)) \cong K_3$ then $P^* \cong A_3$ or $P^* \cong C_{1,3}$ holds. But this causes no problem, since $\mathrm{G}(A_3)$ is a 6-circle and $\mathrm{G}(C_{1,3})$ is a 4-path. If the line graph contains several K_3's, then the exceptional posets consist of ordinal sums of A_3's and $C_{1,3}$'s. If their cardinalities are different in P^* and Q^* then by Proposition 1 $\mathrm{G}(P)$ is not isomorphic to $\mathrm{G}(Q)$.

By Theorem 1 it holds:

$$\mathrm{G}(P) \cong \mathrm{G}(Q) \iff \mathrm{Conv}(P) \cong \mathrm{Conv}(Q) \iff \mathrm{Ext}(P) \cong \mathrm{Ext}(Q).$$

The statement of Theorem 2 follows from Theorem 3, since $\mathrm{Comp}(P^*) \cong \mathrm{Comp}(Q^*)$ and $|P| = |Q|$ imply $\mathrm{Comp}(P) \cong \mathrm{Comp}(Q)$. Observe that $\mathrm{Coco}(P)$ arises from $\mathrm{Coco}(P^*)$ by adding $|P| - |P^*|$ many isolated vertices.

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- Steckt hier irgendwo auch das Ergebnis drin, das $\psi(P)$ eindeutig ist und aus $G(P)$ konstruiert werden kann?
  M.E. nicht, zumindest nicht explizit.