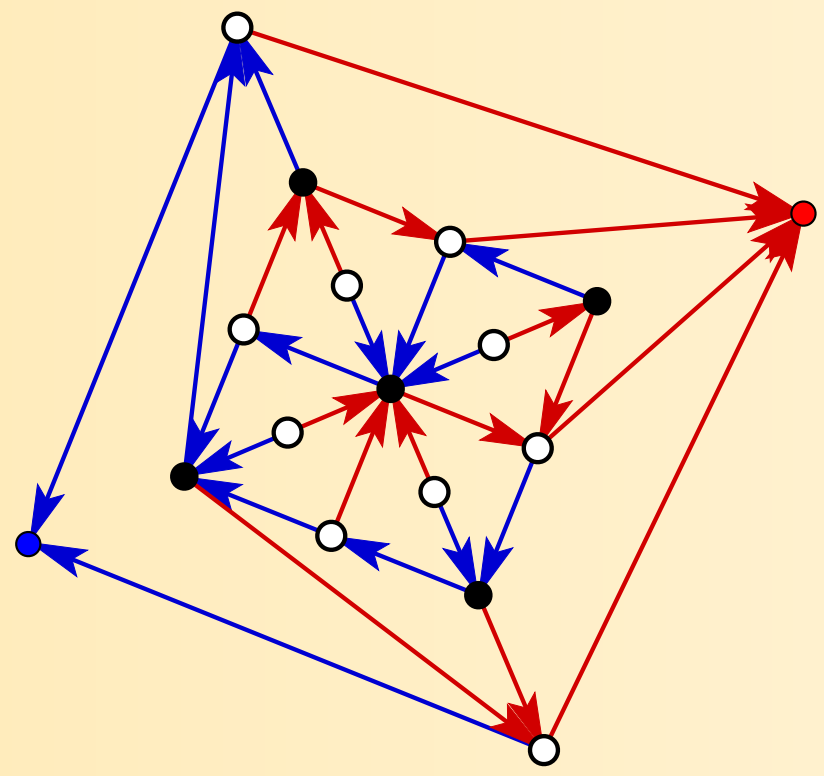


Orientations of Planar Graphs

Doc-Course
Bellaterra
March 11, 2009

Stefan Felsner

Technische Universität Berlin
felsner@math.tu-berlin.de



Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

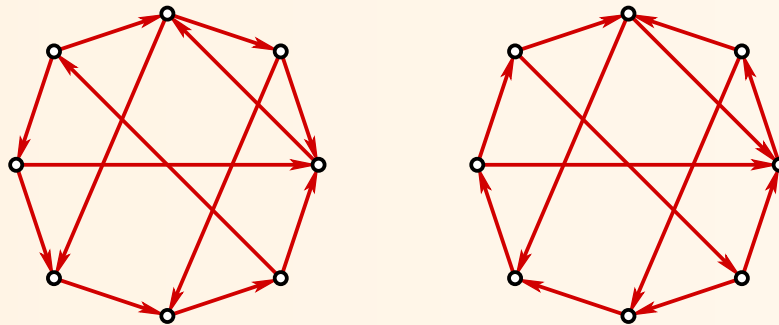
Lattices

Counting III: Random Sampling

alpha-Orientations

Definition. Given $G = (V, E)$ and $\alpha : V \rightarrow \mathbb{N}$.
An α -orientation of G is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all v .

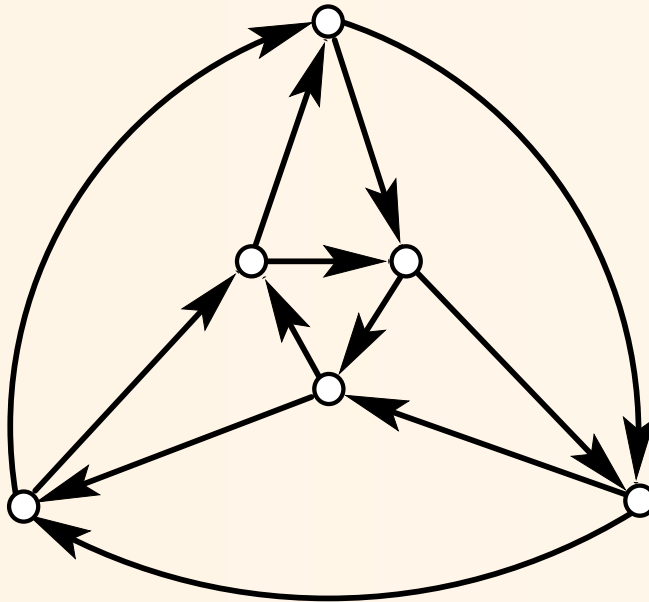
Example.



Two orientations for the same α .

Example 1: Eulerian Orientations

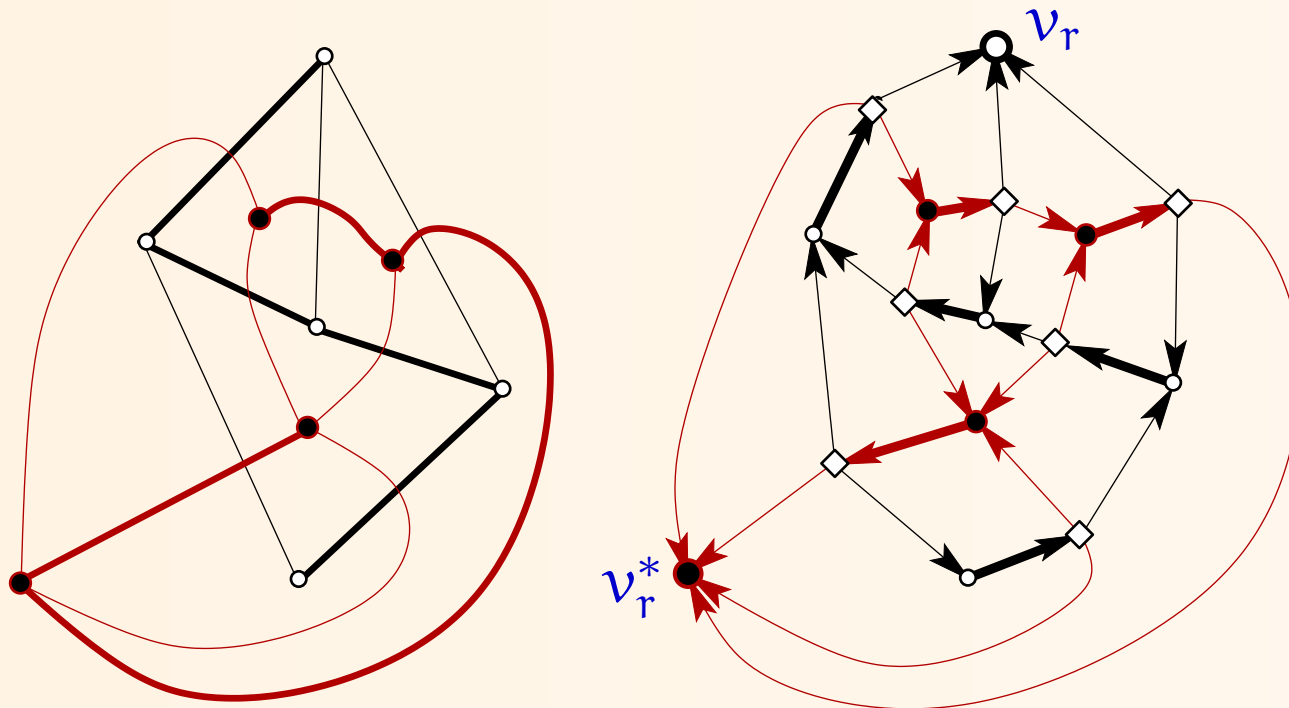
- Orientations with $\text{outdeg}(v) = \text{indeg}(v)$ for all v ,
i.e. $\alpha(v) = \frac{d(v)}{2}$



Example 2: Spanning Trees of Planar Graphs

G a planar graph. Spanning trees of G are in bijection with α_T orientations of a rooted primal-dual completion \tilde{G}

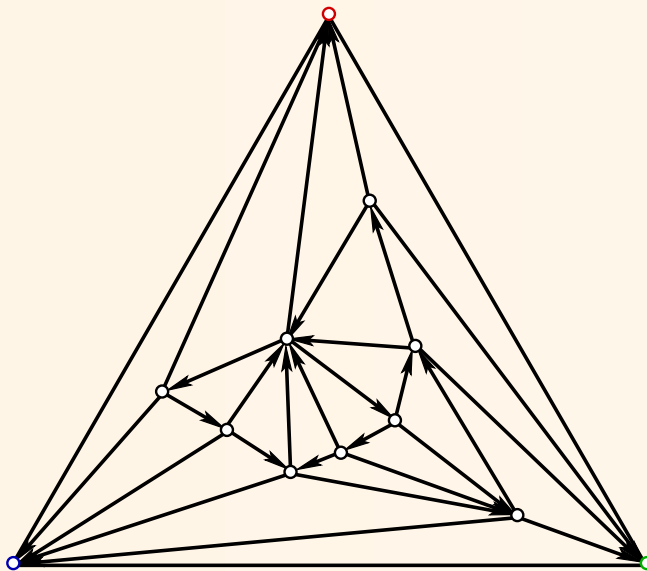
- $\alpha_T(v) = 1$ for a non-root vertex v and $\alpha_T(v_e) = 3$ for an edge-vertex v_e and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$.



Example 3: 3-Orientations

G a planar triangulation, let

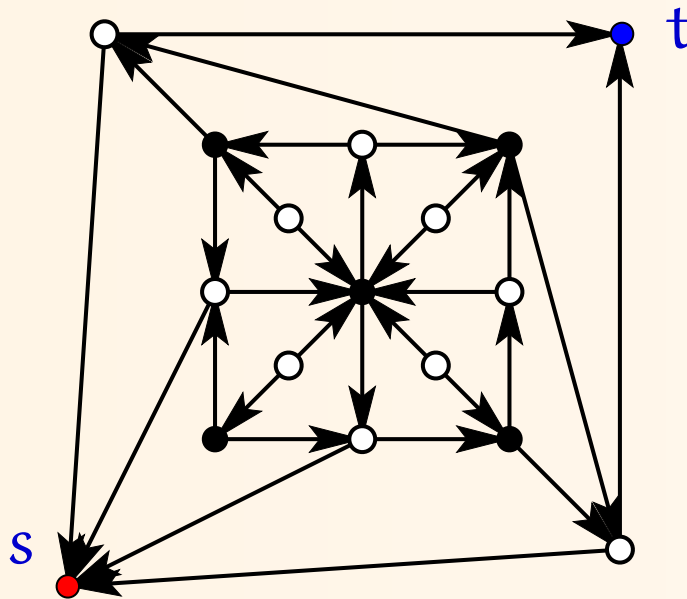
- $\alpha(v) = 3$ for each inner vertex and $\alpha(v) = 0$ for each outer vertex.



Example 4: 2-Orientations

G a planar quadrangulation, let

- $\alpha(v) = 0$ for an opposite pair of outer vertices and $\alpha(v) = 2$ for each other vertex.



Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

Lattices

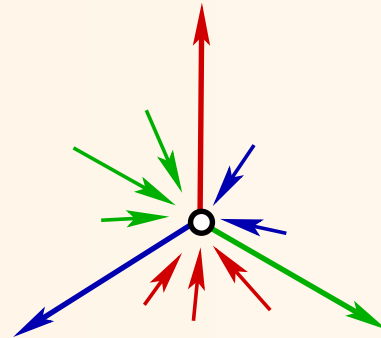
Counting III: Random Sampling

Schnyder Woods

$G = (V, E)$ a plane triangulation,
 $F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors 1, 2, 3 is a **Schnyder wood** of G iff

- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \rightarrow a_i$ in color i .

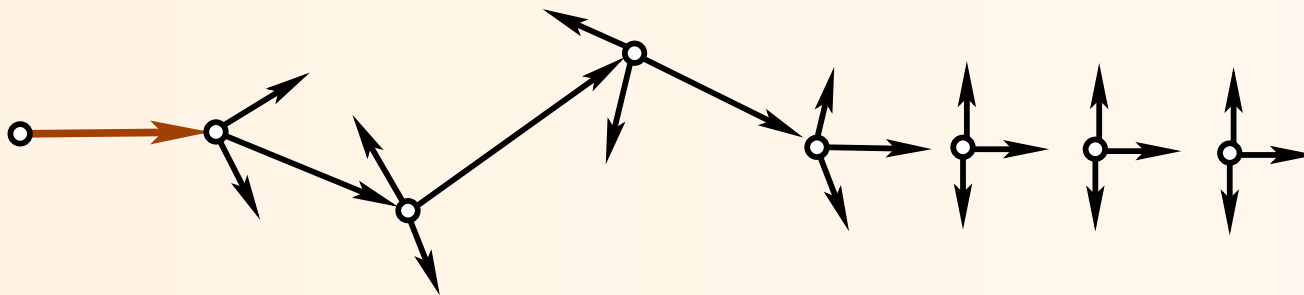
Schnyder Woods and 3-Orientations

Theorem.

Schnyder woods and 3-orientations are equivalent.

Proof.

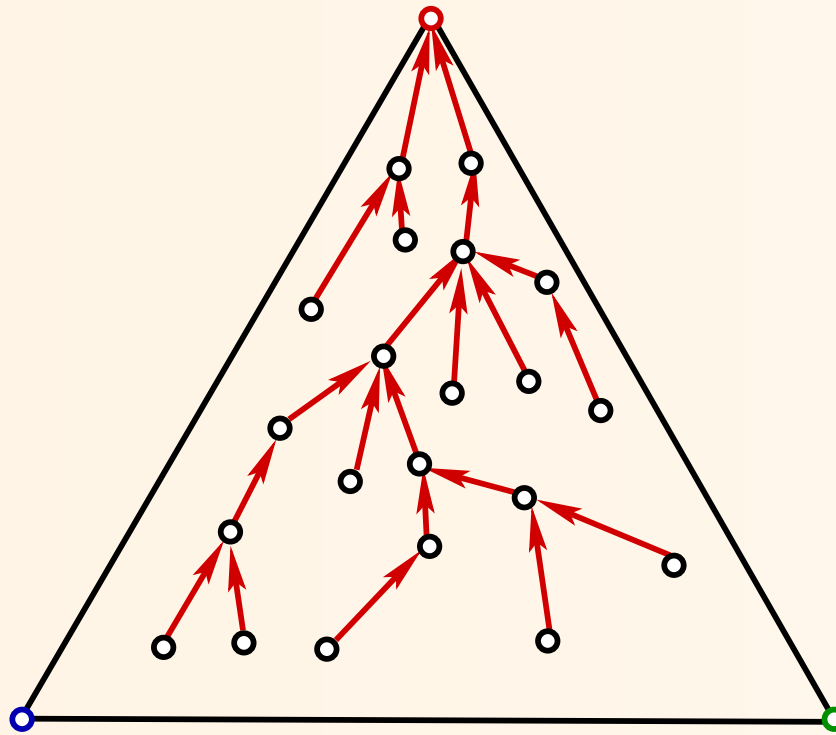
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some α_i .

Schnyder Woods - Trees

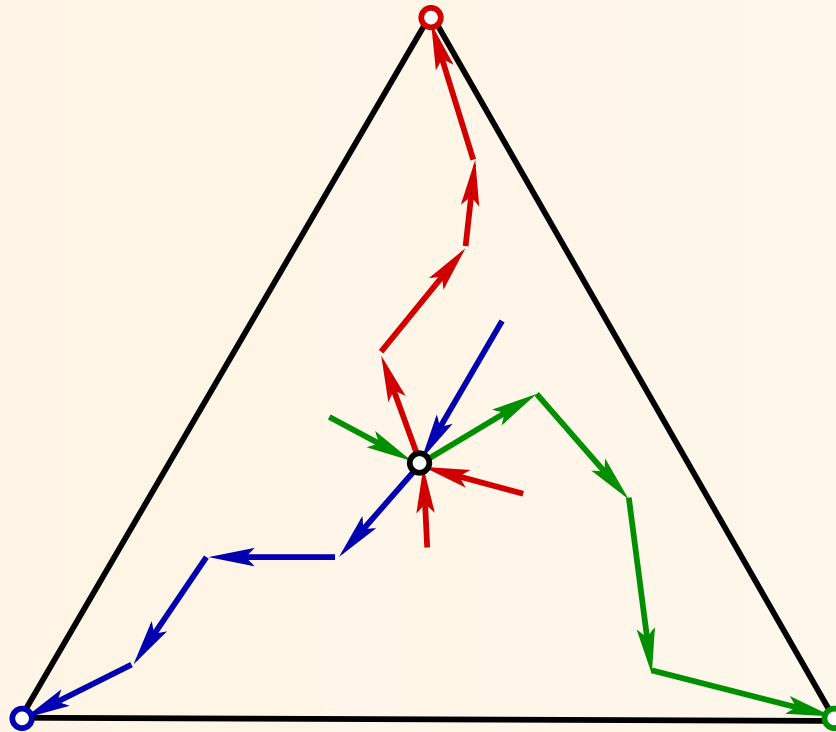
- The set T_i of edges colored i is a tree rooted at a_i .



Proof. Path $e \longrightarrow a_i$ is unique (again Euler).

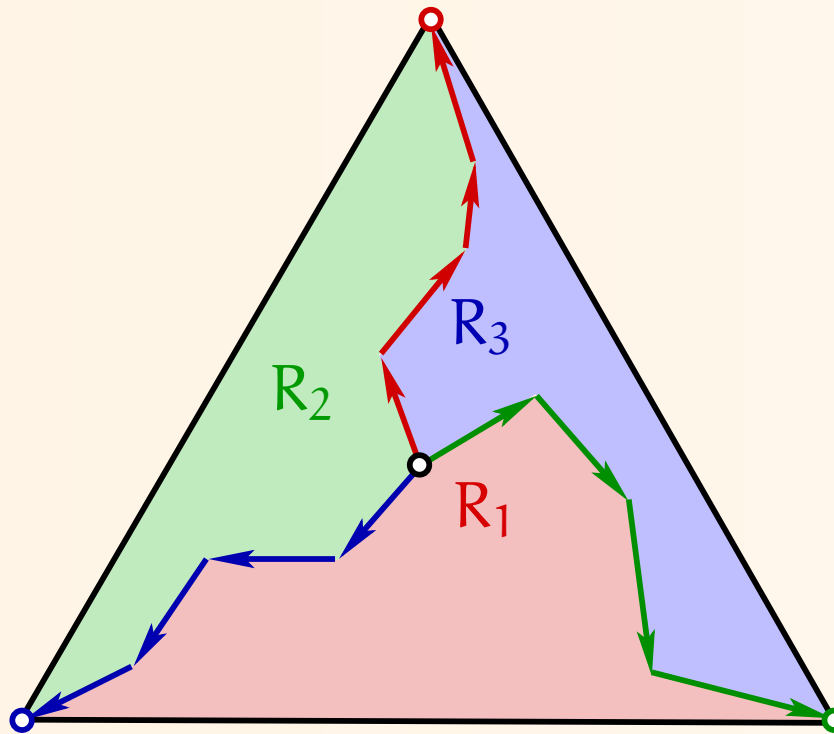
Schnyder Woods - Paths

- Paths of different color have at most one vertex in common.



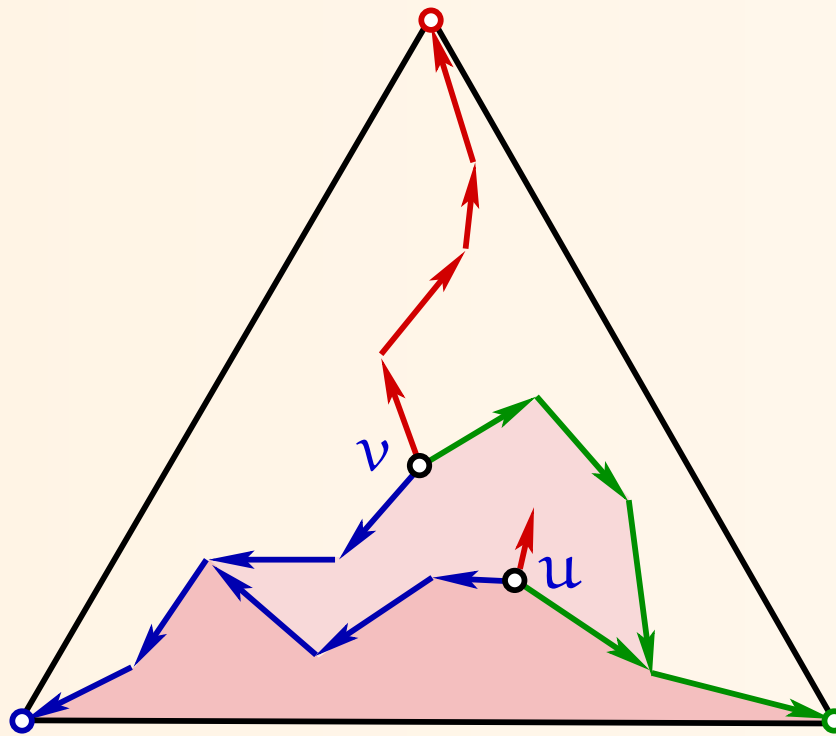
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



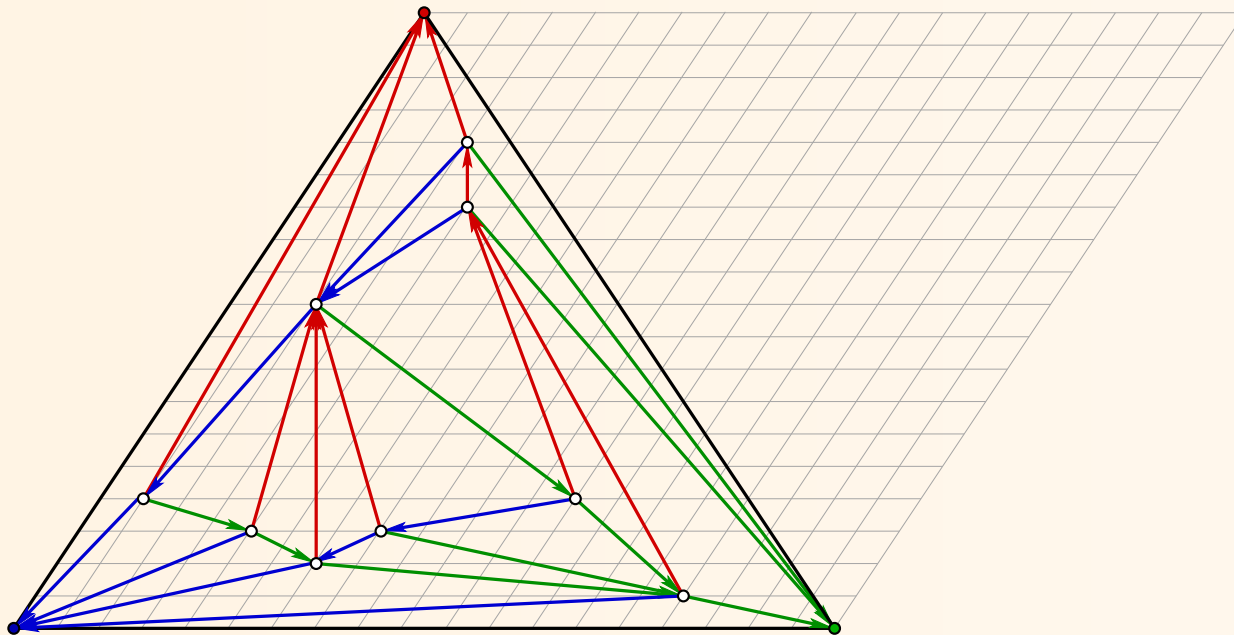
Schnyder Woods - Regions

- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.



Grid Drawings

The count of faces in the **green** and **red** region yields two coordinates (v_g, v_r) for vertex v .



\Rightarrow straight line drawing on the $2n - 5 \times 2n - 5$ grid.

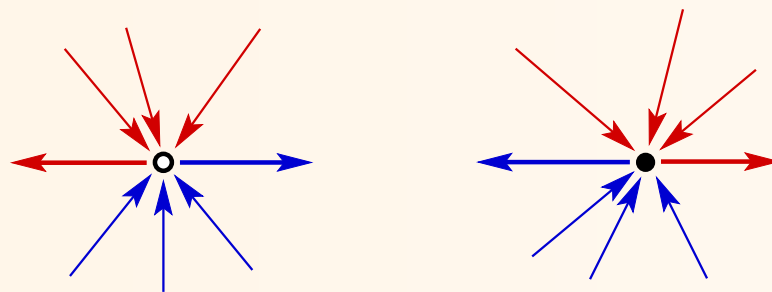
Separating Decompositions

$G = (V, E)$ a plane quadrangulation,

$F = \{a_0, x, a_1, y\}$ the outer face.

A coloring and orientation of the interior edges of G with colors $0, 1$ is a **separating decomposition** of G iff

- Inner vertex condition:



- Edges incident to a_0 and a_1 are oriented $v \rightarrow a_i$ in color i .

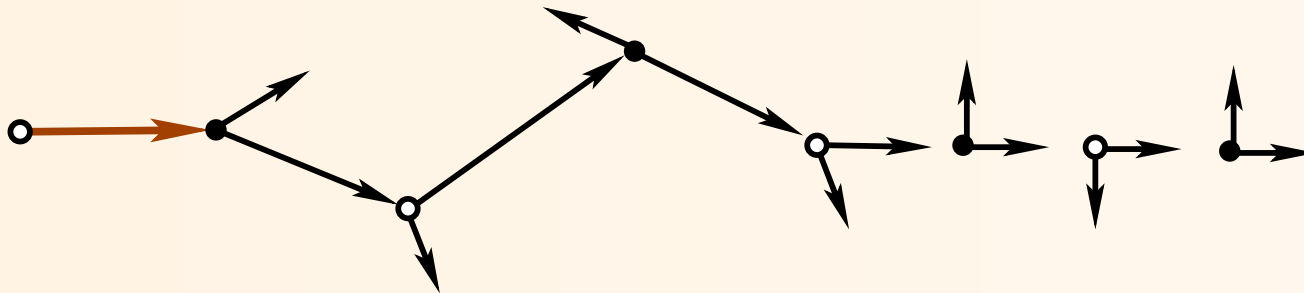
Separating Decompositions and 2-Orientations

Theorem.

Separating decompositions and 2-orientations are equivalent.

Proof.

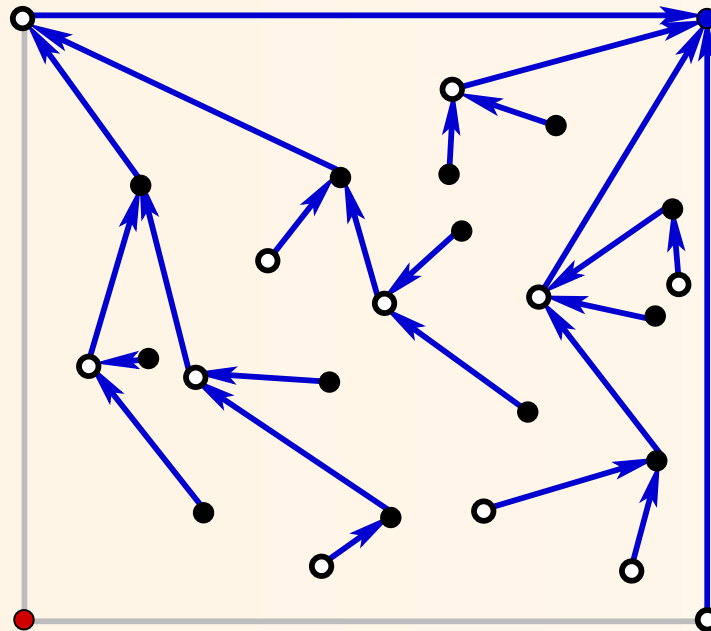
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i .

Separating Decompositions - Trees

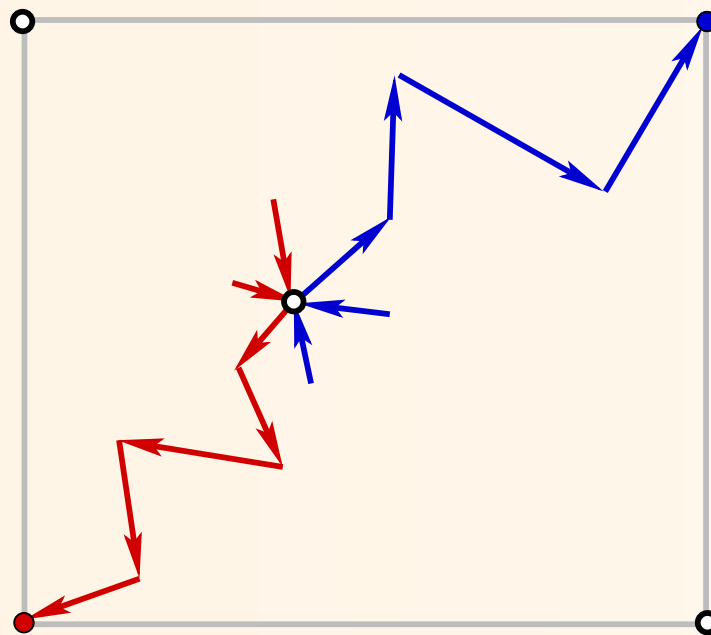
The set T_i of edges colored i is a tree rooted at a_i .



Proof. Path $e \longrightarrow a_i$ is unique.

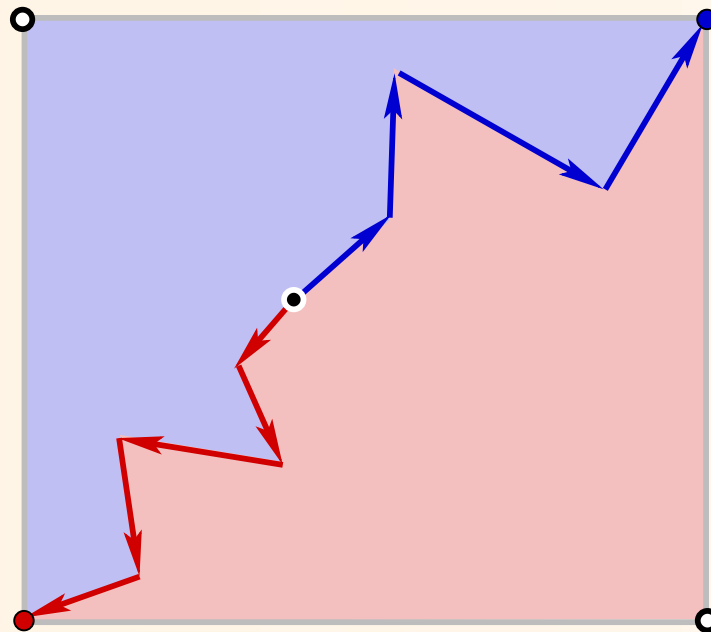
Separating Decompositions - Paths

- Paths of different color have at most one vertex in common.



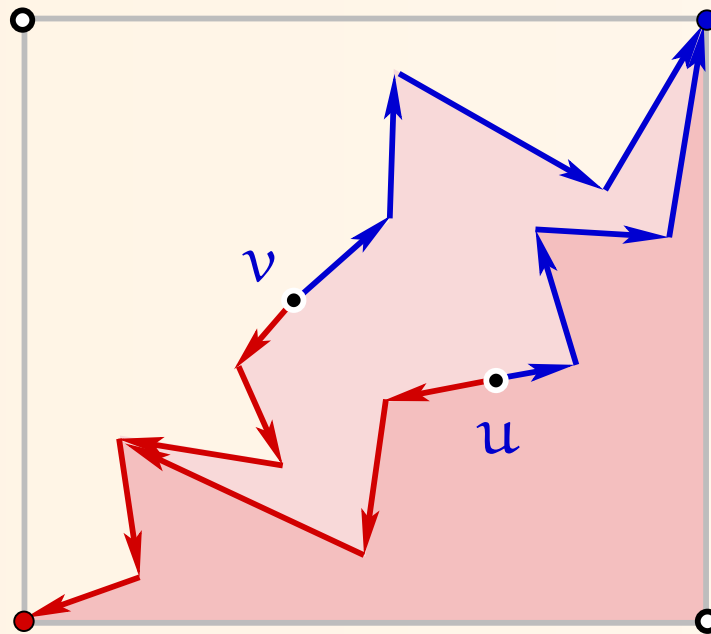
Separating Decompositions - Regions

- Every vertex has two distinguished regions.



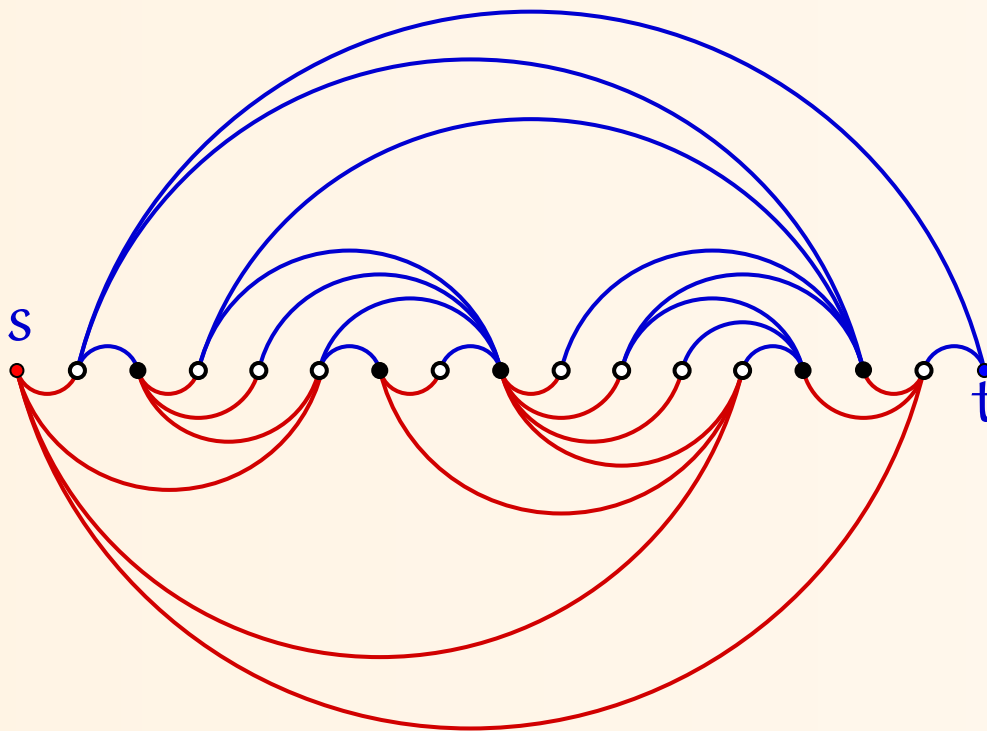
Separating Decompositions - Regions

- If $u \in R_0(v)$ then $R_0(u) \subset R_0(v)$.



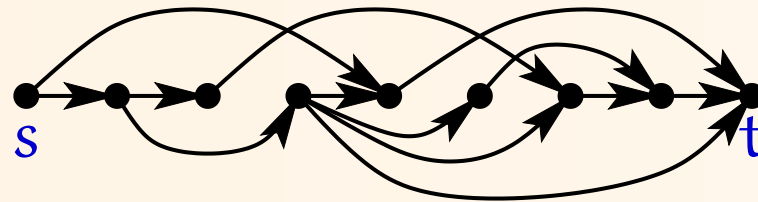
2-Book Embedding

The count of faces in the **red** region yields a number ν_r for vertex $v \neq s, t$.



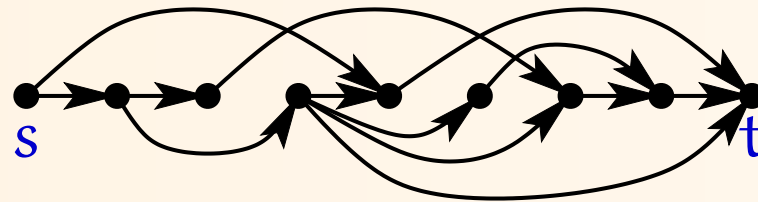
Bipolar Orientations

Definition. A **bipolar orientation** is an acyclic orientation with a unique source s and a unique sink t .

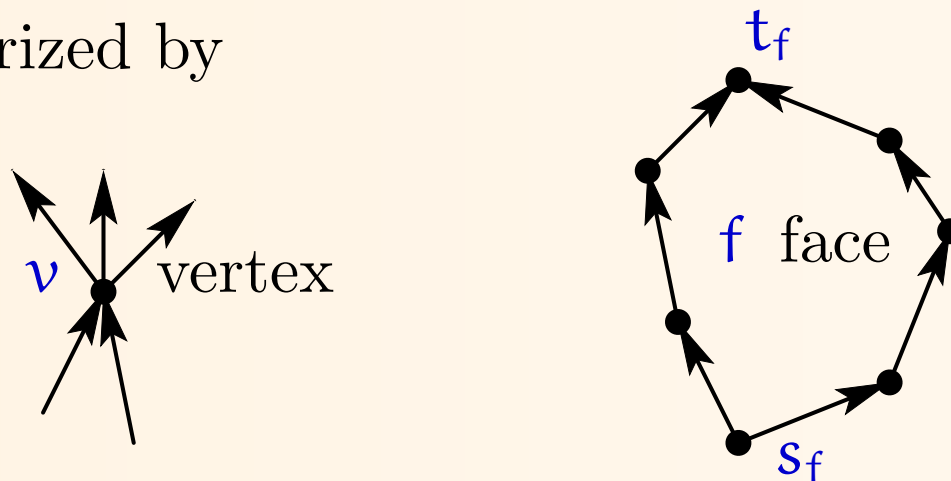


Bipolar Orientations

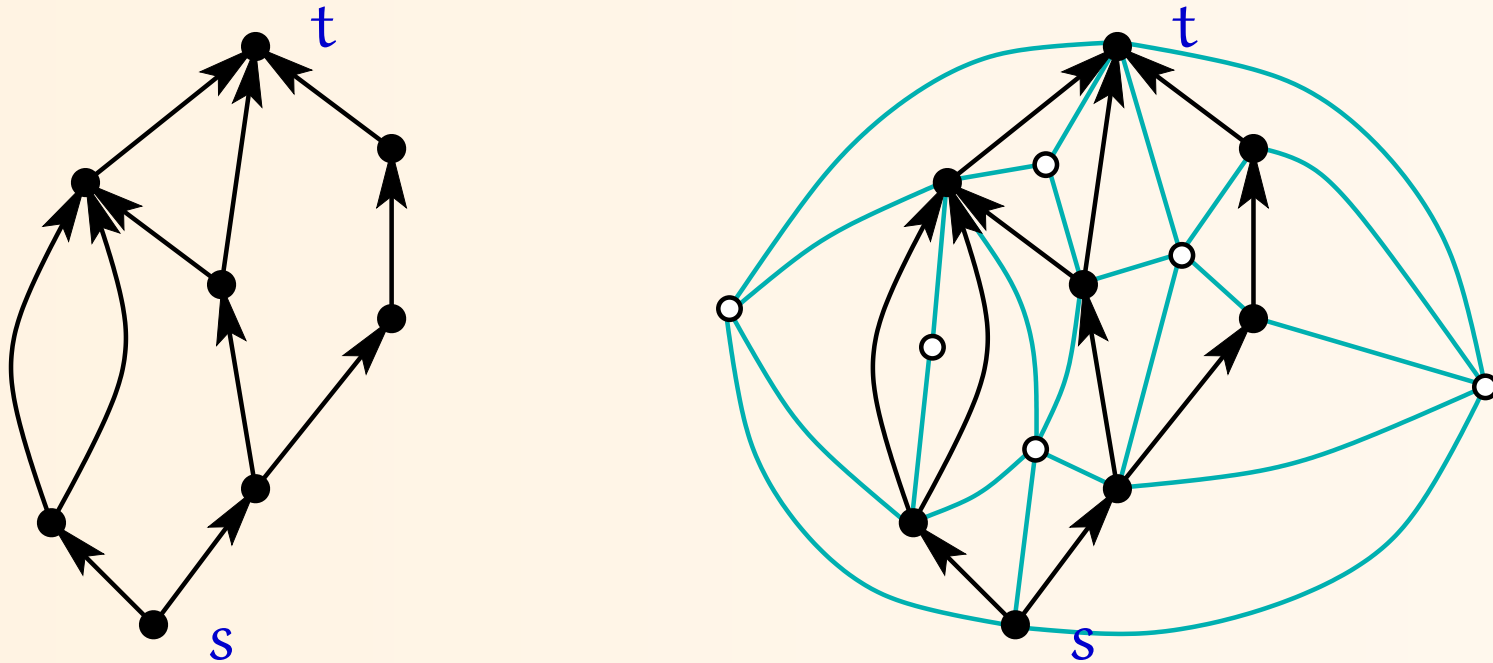
Definition. A **bipolar orientation** is an acyclic orientation with a unique source s and a unique sink t .



Plane bipolar orientations with s and t on the outer face are characterized by

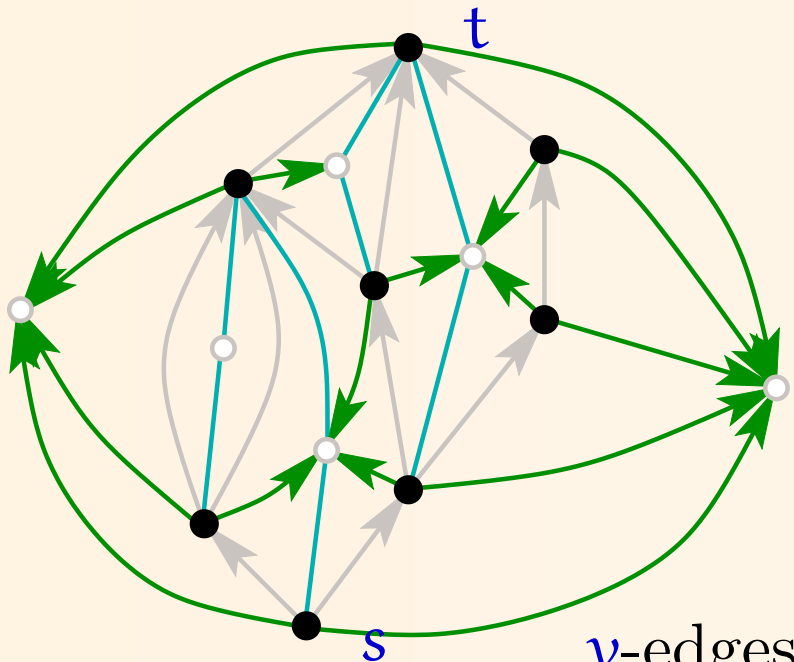


Plane Bipolar Orientations and 2-Orientations

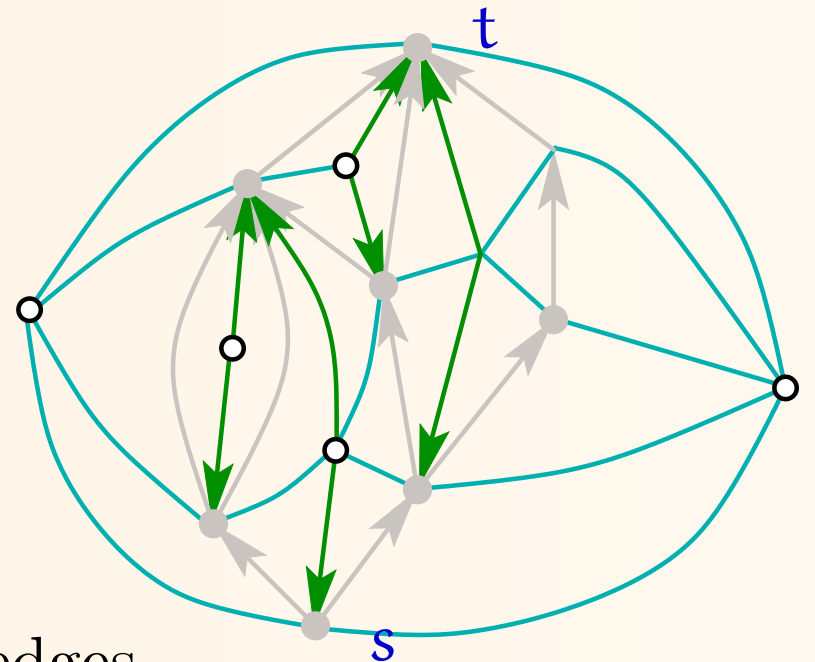


A plane bipolar orientation and its angular map.

Orienting the Angular Map



v-edges



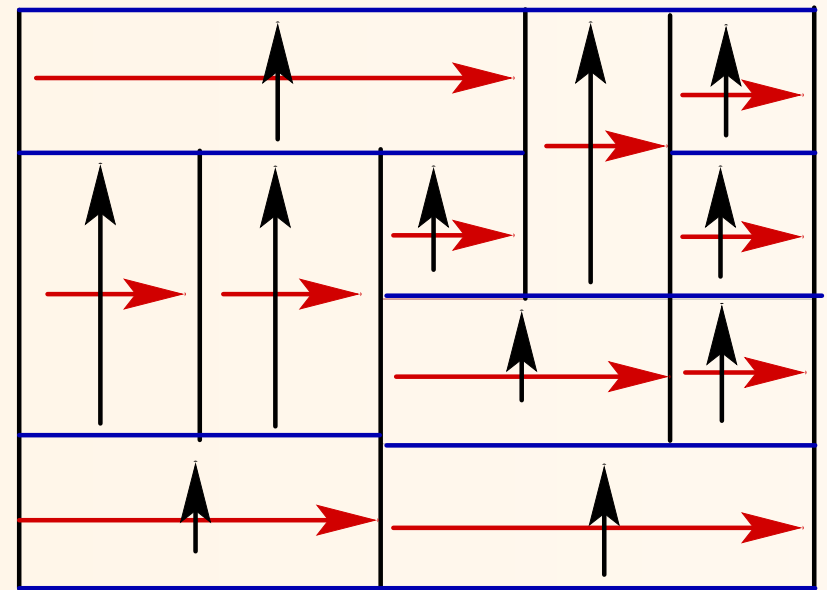
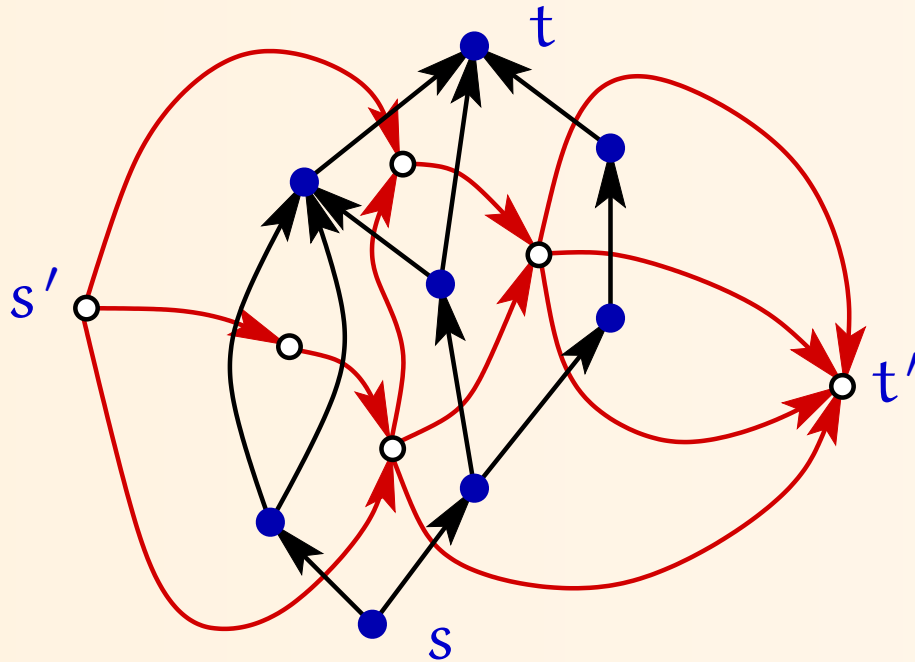
f-edges



Angular edges oriented by vertices and faces.

Plane Bipolar Orientations and Rectangular Layouts

A plane bipolar orientation and its dual orientation yield a rectangular layout (visibility representation).



coordinates from longest paths

Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

Lattices

Counting III: Random Sampling

How Many?

Let G be a plane graph and $\alpha : V \rightarrow \mathbb{N}$.

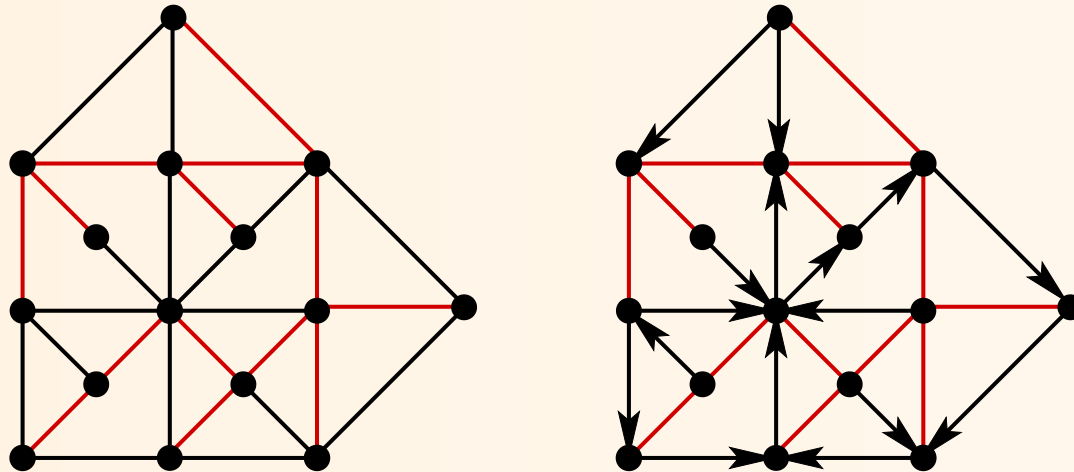
How many α -orientations can G have?

How Many?

Let G be a plane graph and $\alpha : V \rightarrow \mathbb{N}$.

How many α -orientations can G have?

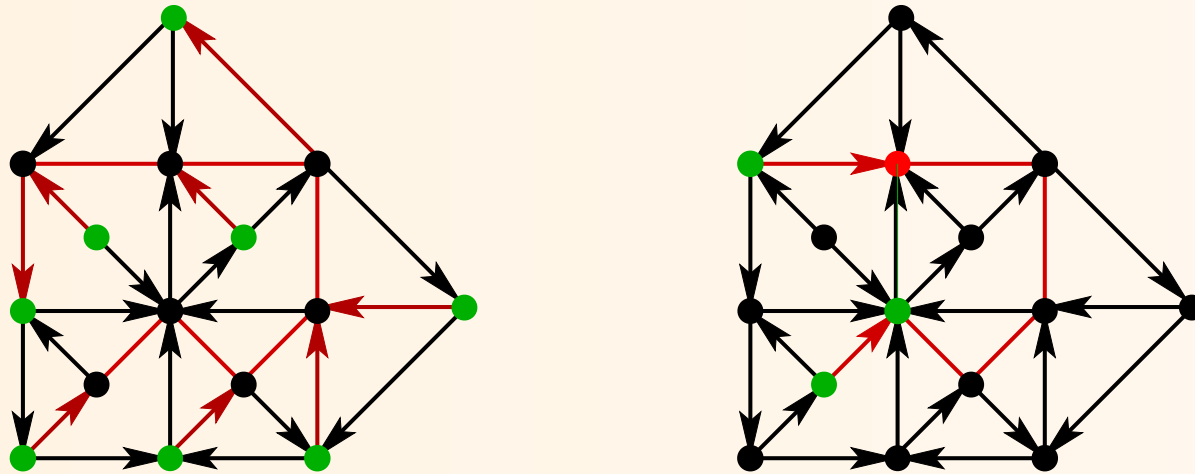
Choose a spanning tree T of G and orient the edges not in T randomly.



Towards an Upper Bound

If at all the orientation on $G - T$ is uniquely extendible.

$\alpha \equiv 2$



\implies there are at most $2^{m-(n-1)}$ α -orientations.

Improve on one color

An orientation can be extended only if $\text{outdeg}(v) \in \{\alpha(v), \alpha(v) - 1\}$ for all v .

Let I be an independent set of size $\geq \frac{n}{4}$ (4CT)

Choose a tree T such that $I \subset \text{leaves}(T)$.

Each $v \in I$ can independently obstruct extendability.

There are $\binom{d(v)-1}{\alpha(v)} + \binom{d(v)-1}{\alpha(v)-1} = \binom{d(v)}{\alpha(v)} \leq \binom{d(v)}{\lfloor d(v)/2 \rfloor}$ good choices for the orientations of edges at v .

The Result

Since

$$\text{Prob}(d(v) = \alpha(v)) \leq \frac{1}{2^{d(v)-1}} \binom{d(v)}{\lfloor d(v)/2 \rfloor} \leq \frac{3}{4}$$

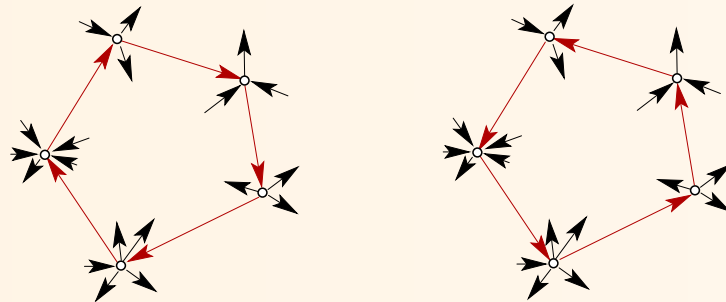
we conclude:

Theorem. The number of α -orientations of a plane graph on n vertices is at most

$$2^{m-n} \left(\frac{3}{4}\right)^{n/4} \leq 2^{2n} \left(\frac{3}{4}\right)^{n/4} \approx 3.73^n$$

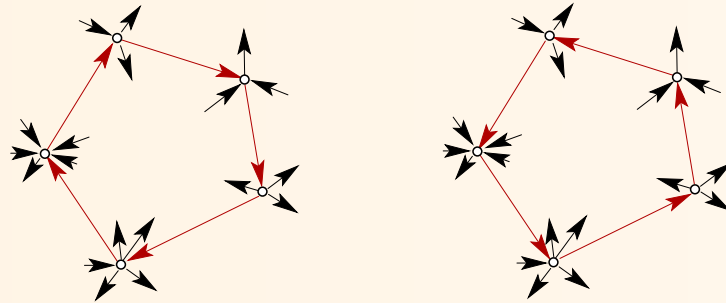
Towards a Lower Bound

Observation. Flipping cycles preserves α -orientations.



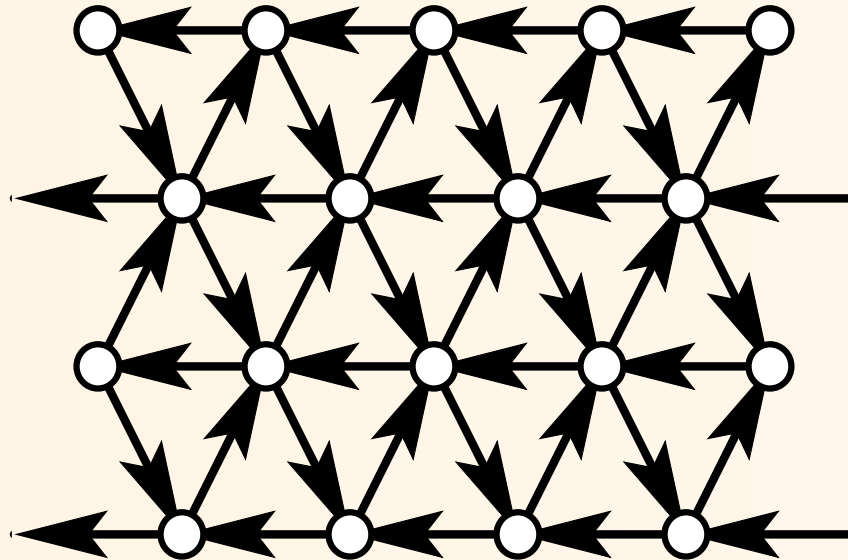
Towards a Lower Bound

Observation. Flipping cycles preserves α -orientations.

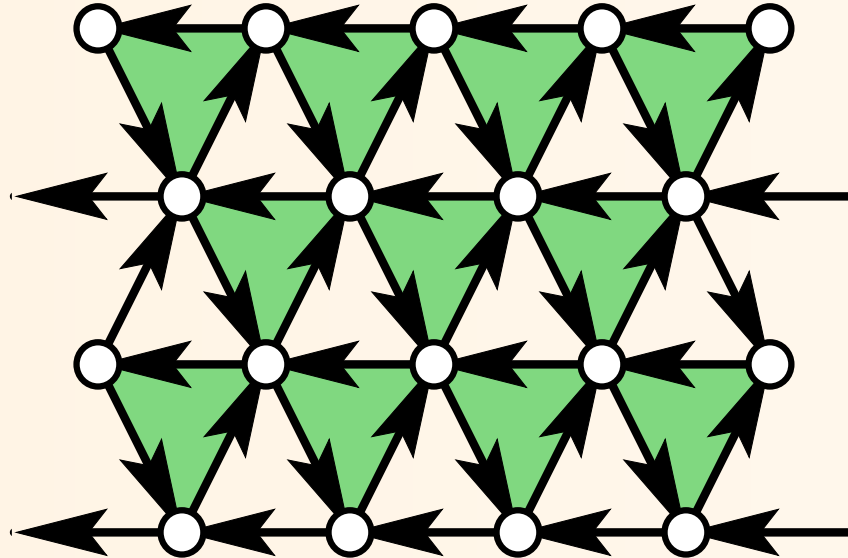


We show that there are **many** 3-orientation of the triangular lattice

The Initial Orientation

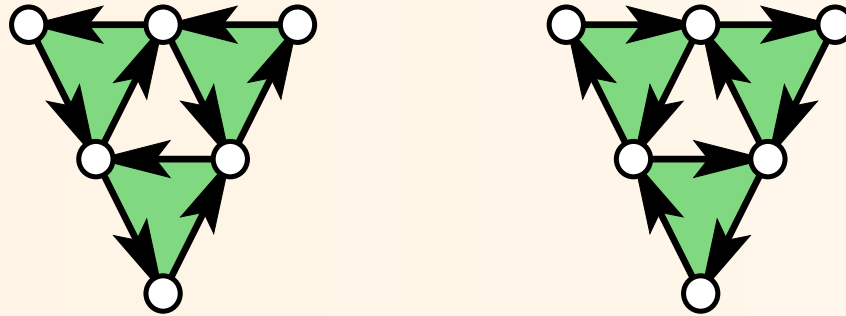


The Initial Orientation



Any subset of the green triangles can be flipped.

Green and White Flips



If 0 or 3 of the green neighbors are flipped a white triangle can be flipped.

using Jensen's ineq. \implies

$$\# \text{ 3-orientations} \geq 2^{\#f\text{-green}} \mathbf{E}(2^{f\text{-white-flippable}}) \geq$$

$$2^n 2^{\mathbf{E}(f\text{-white-flippable})} = 2^n 2^{\frac{2}{8}\#f\text{-white}} = 2^{\frac{5}{4}n} = 2.37^n$$

Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

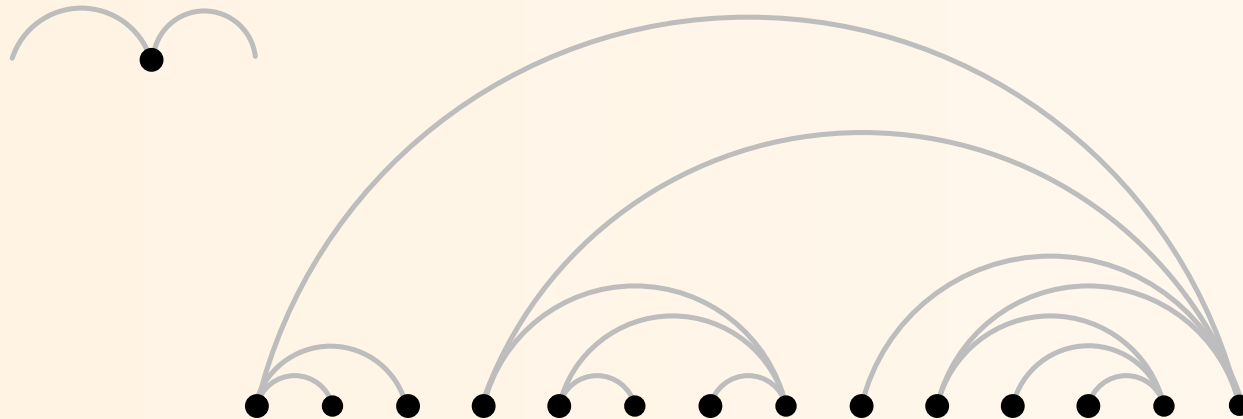
Lattices

Counting III: Random Sampling

Alternating Layouts of Trees

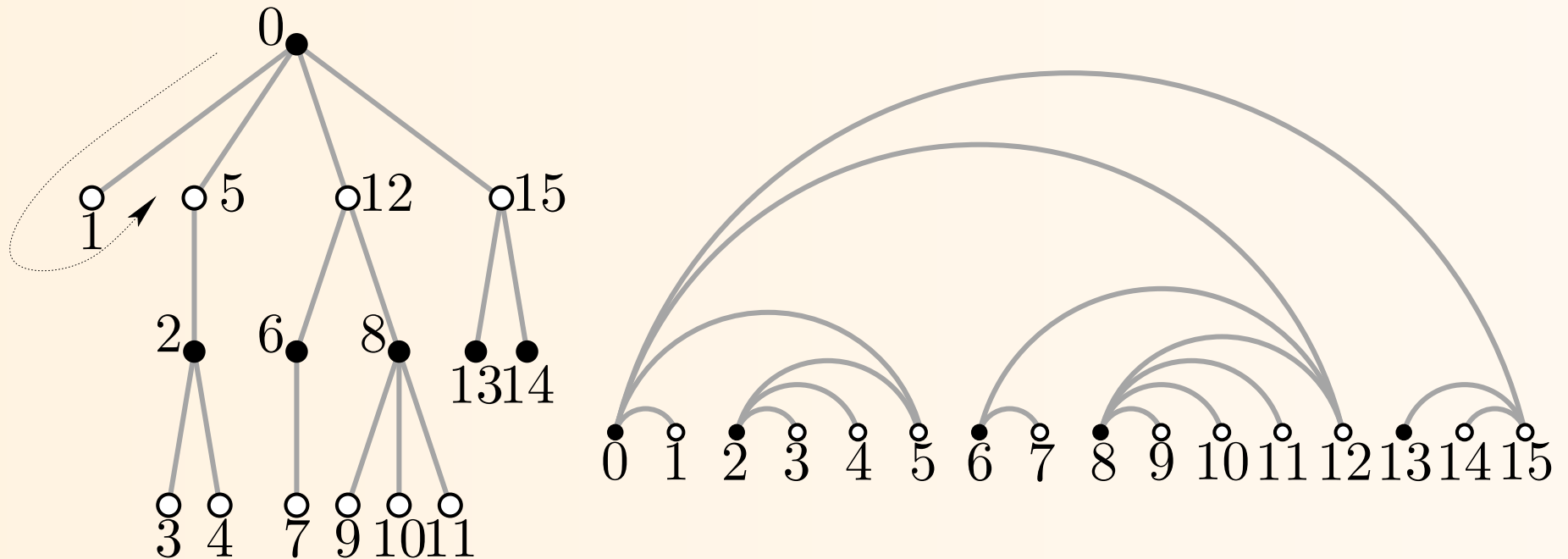
Definition. A numbering of the vertices of a tree is alternating if it is a 1-book embedding with no **double-arc**.

double arc



Alternating Layouts of Trees

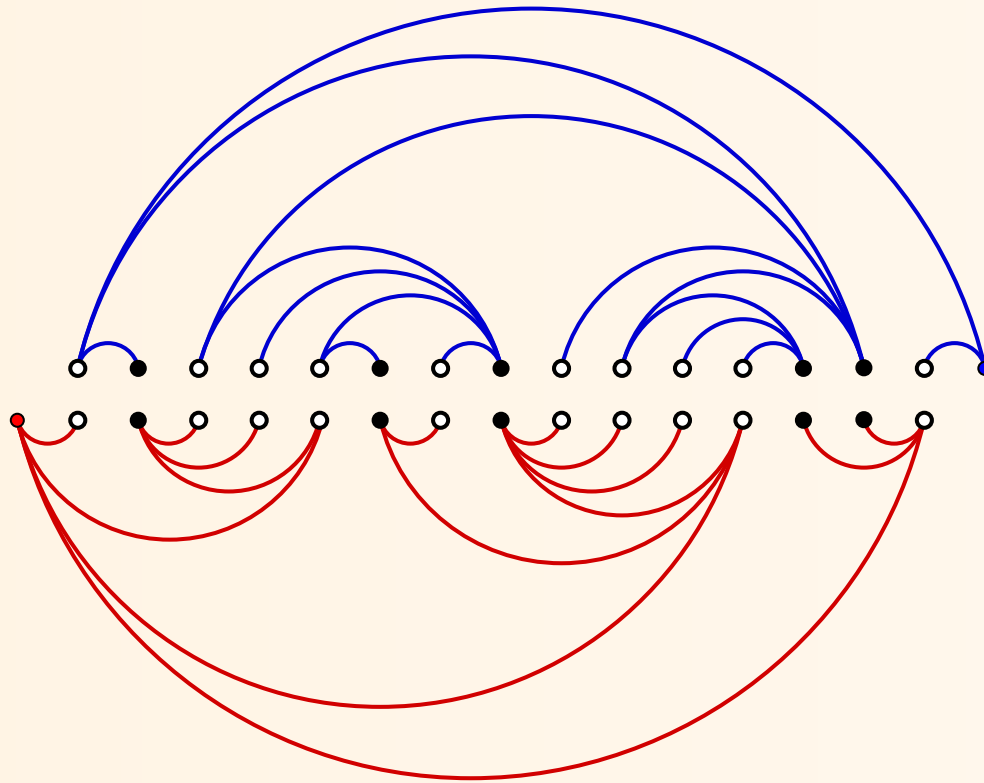
Proposition. A rooted plane tree has a unique alternating layout with the root as leftmost vertex.



Label black vertices at first visit, white vertices at last visit.

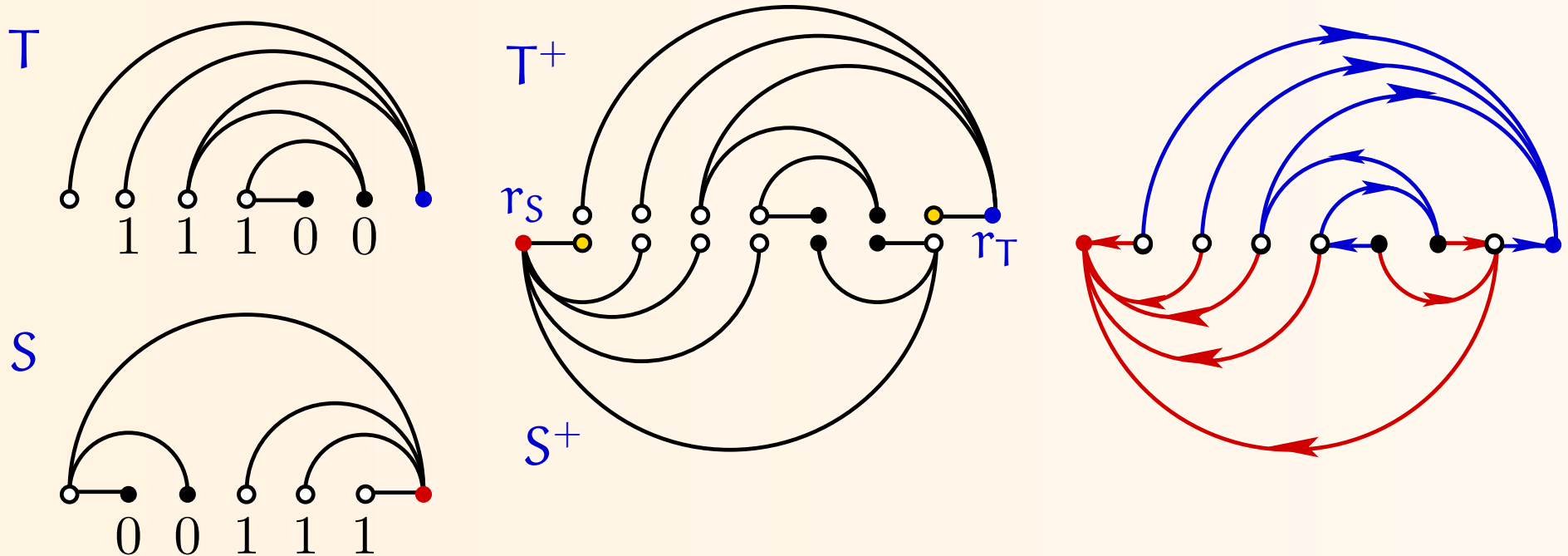
Separating Decompositions and Alternating Trees

Proposition. The 2-book embedding induced by a separation decomposition splits into two alternating trees.



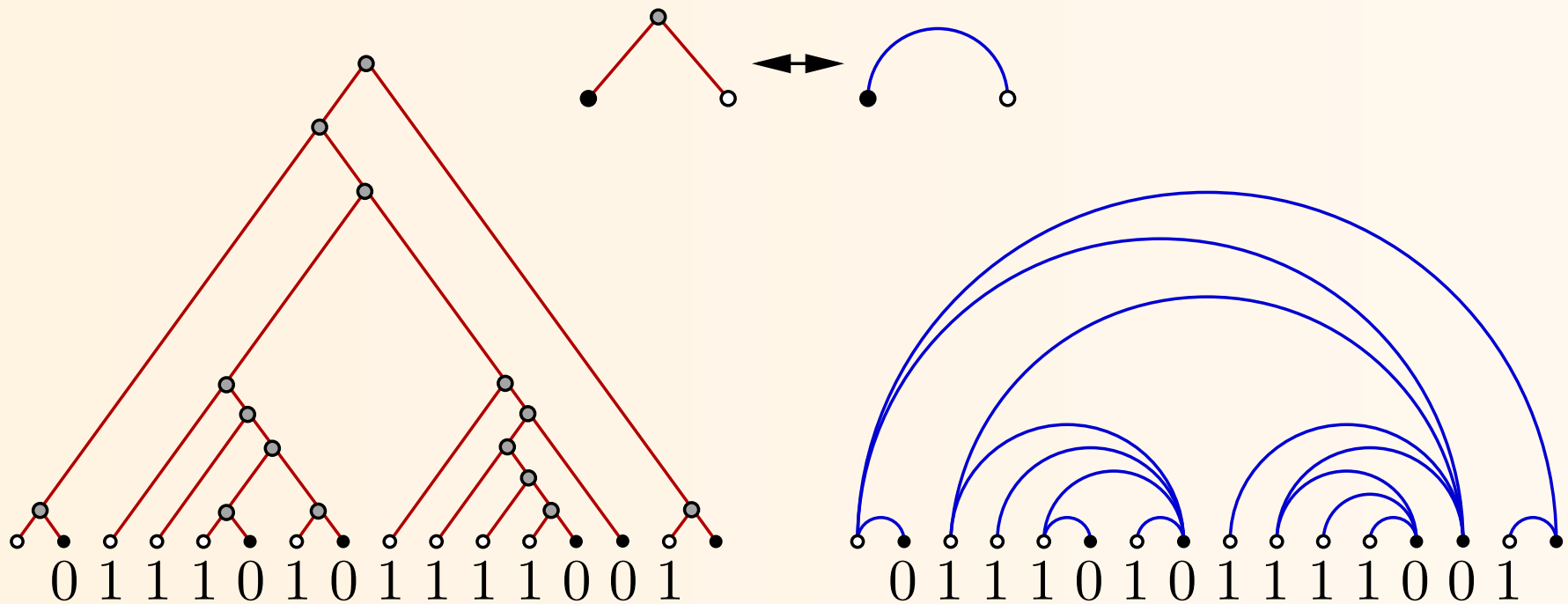
A Bijection

Theorem. There is a bijection between pairs (S, T) of alternating trees on n vertices with reverse fingerprints and separating decompositions of quadrangulations with $n + 2$ vertices.



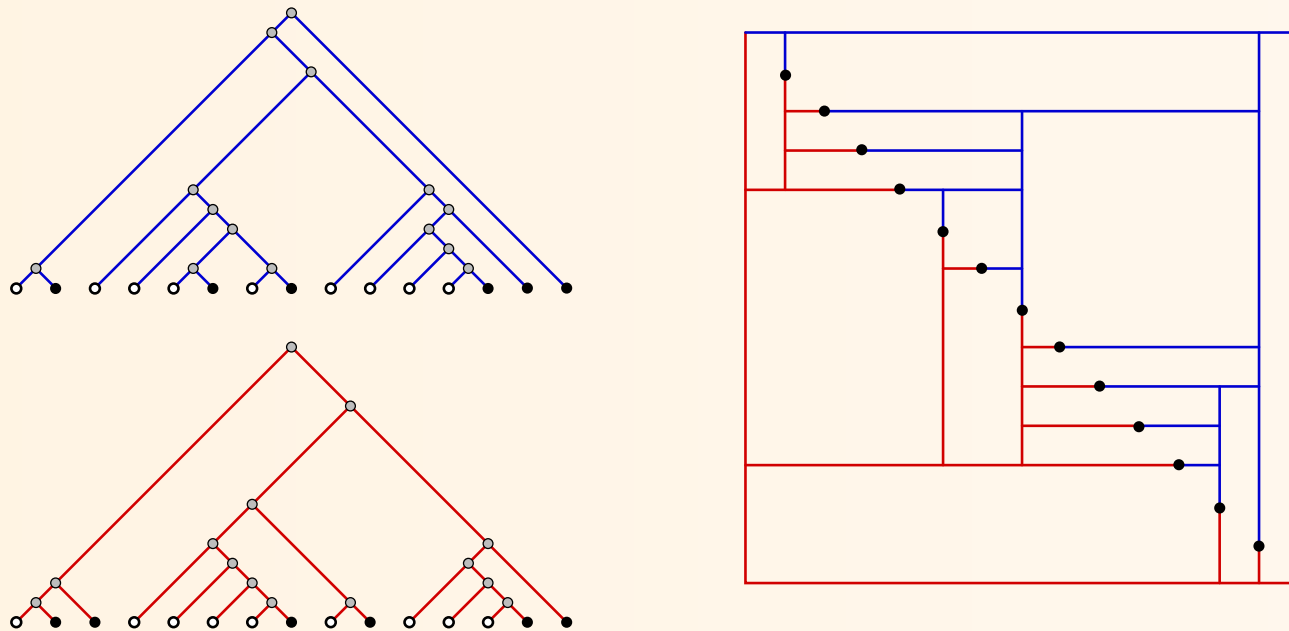
Alternating and Full Binary Trees

Proposition. There is bijection between alternating and binary trees that preserves fingerprints.



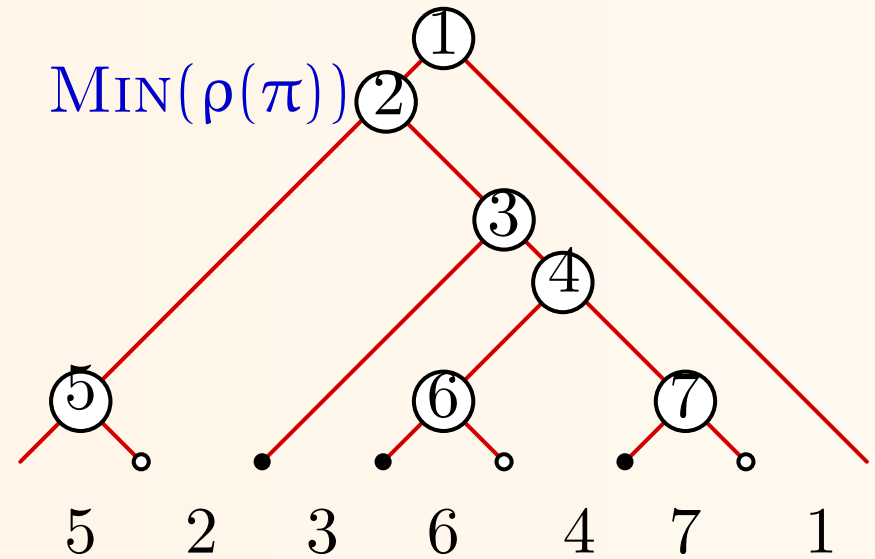
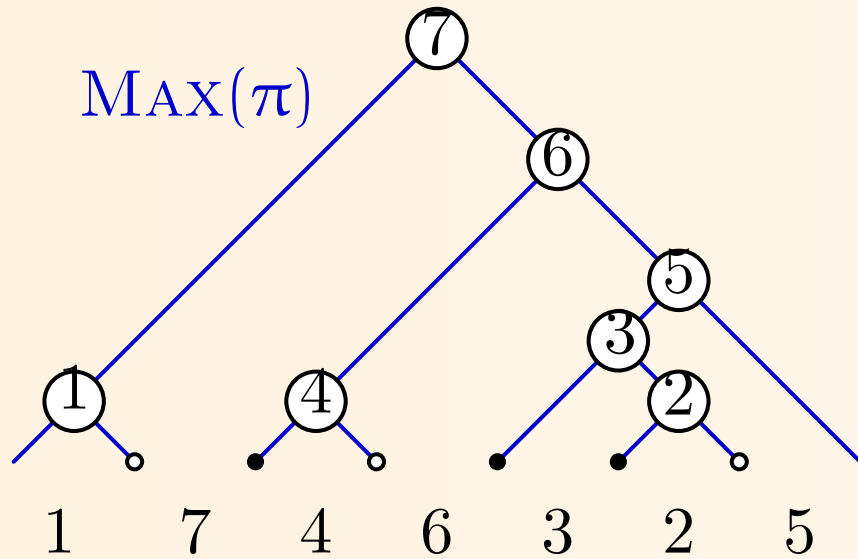
Rectangular Dissections

Theorem. There is a bijection between pairs (S, T) of binary trees with n leaves and reverse fingerprints and rectangular dissections* of the square based on $n - 2$ diagonal points.



*This *is* again the rectangular layout associated to the bipolar orientation.

Permutations and Trees



Proposition. For a permutation π of $[n - 1]$ the pair $(\text{MAX}(\pi), \text{MIN}(\bar{\pi}))$ is a pair of binary trees with n leaves and reverse fingerprints.

Baxter Permutations

Definition. A permutation is **Baxter** if it avoids the pattern $3 - 14 - 2$ and $2 - 41 - 3$.

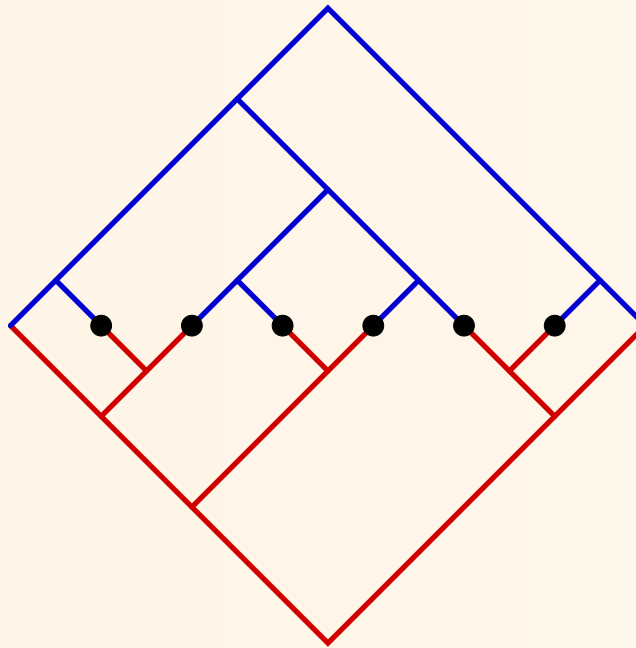
Example: A non-Baxter permutation with a $2 - 41 - 3$ pattern

$$\pi = 6, 3, 8, 7, 2, 9, 1, 5, 4$$

Theorem. The mapping $\pi \rightarrow (\text{MAX}(\pi), \text{MIN}(\bar{\pi}))$ is bijection between Baxter permutations of $[n-1]$ and binary trees with n leaves and reverse fingerprints, i.e., rectangular dissections of the square based on $n - 2$ diagonal points.

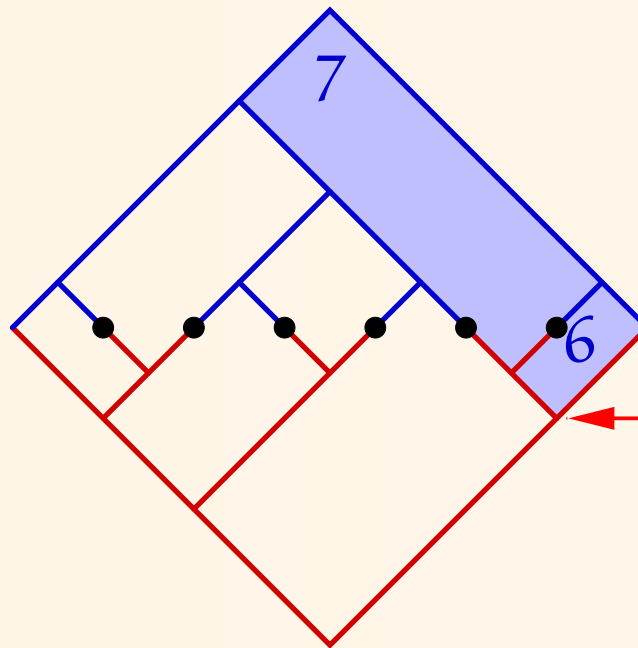
Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.



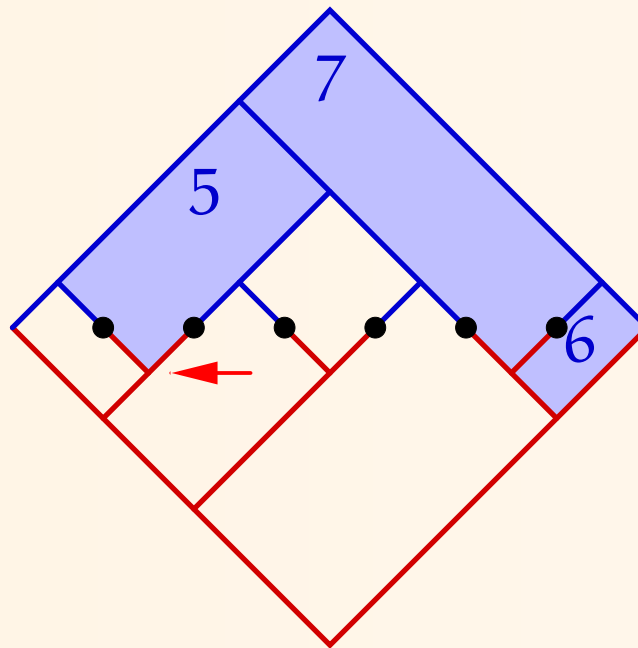
Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.



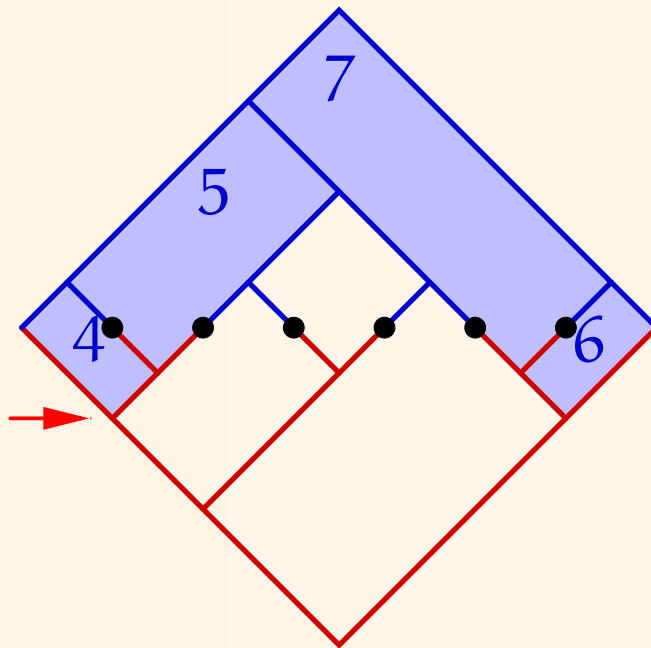
Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.



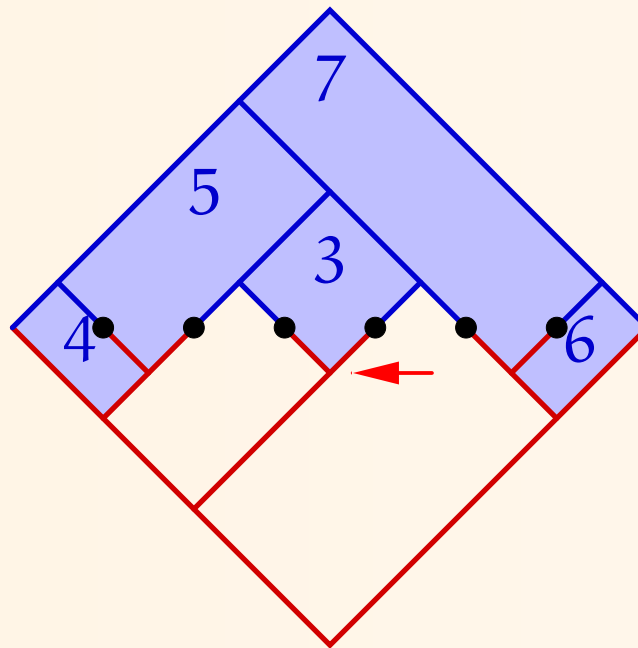
Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.



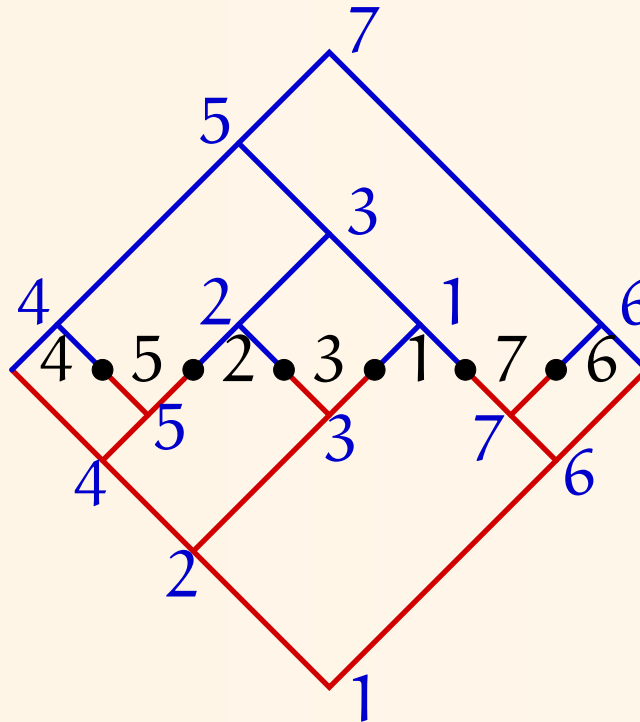
Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.



Constructing the Permutation

Rule: If the south-corner of $R(k)$ is a \times , i.e., a left child in tree T , then $R(k-1)$ is the next-left, otherwise, next-right.

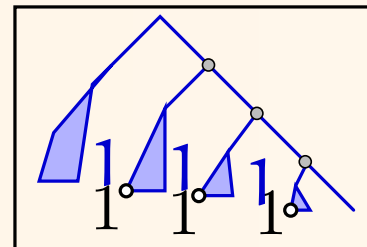
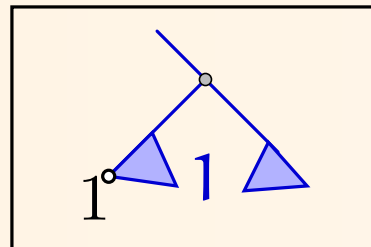


Encoding a Binary Tree, Cont.

α : Fingerprint including the left extreme leaf.

β : Inner nodes in in-order represented by 0 (left) and 1 (right) with the root being a 1.

Lemma. $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \beta_i$ and $\sum_{i=1}^k \alpha_i \geq \sum_{i=1}^k \beta_i$



Lemma. The tree can be reconstructed.

Proof. The minimal k with $\sum_{i=1}^k \alpha_i = \sum_{i=1}^k \beta_i$ and $\sum_{i=1}^{k+1} \alpha_i > \sum_{i=1}^{k+1} \beta_i$ determines the position of the root.

Counting Binary Trees

Proposition. The number of binary trees with $i + 1$ left leaves and $j + 1$ right leaves equals the number of nonintersecting lattice paths α' and β' where:

$$\alpha' : (0, 1) \rightarrow (j, i + 1)$$

$$\beta' : (1, 0) \rightarrow (j + 1, i)$$

From the Lemma of Gessel Viennot we deduce that their number is

$$\det \begin{pmatrix} \binom{j+i}{j} & \binom{j+i}{j-1} \\ \binom{j+i}{j+1} & \binom{j+i}{j} \end{pmatrix} = \frac{1}{i+j+1} \binom{i+j+1}{j} \binom{i+j+1}{j+1}$$

This is the **Narayana number** $N(i + j + 1, j)$.

Counting Baxter

Theorem. The number of Baxter permutations of $[n - 1]$, separating decompositions and 2-orientations on $n + 2$ vertices, rectangular dissections on $n - 2$ diagonal points is given by

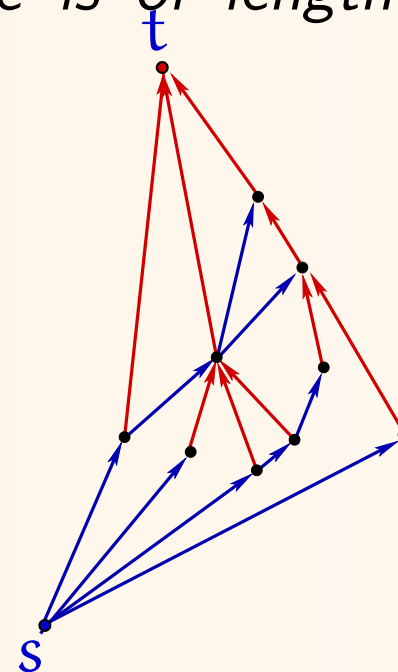
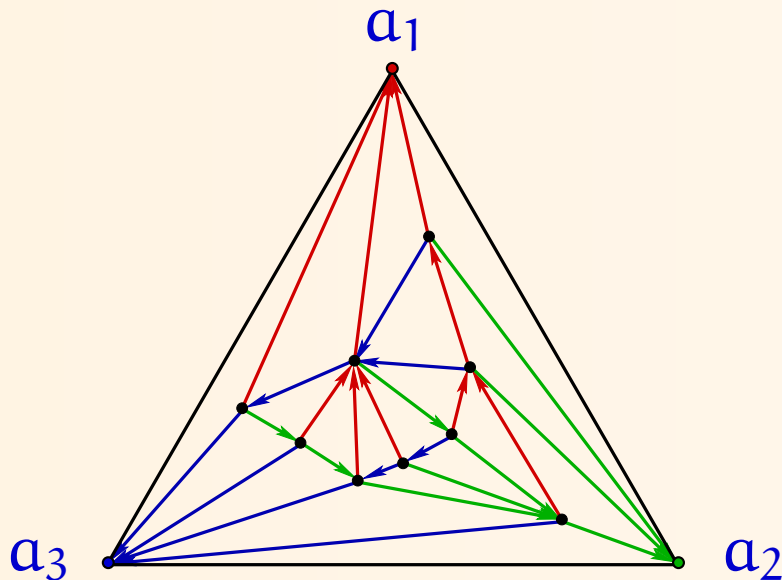
$$\sum_{i=0}^{n-2} \frac{2n!(n-1)!(n-2)!}{i!(i+1)!(i+2)!(n-i)!(n-i-1)!(n-i-2)!} =$$

$$\frac{2}{n(n-1)^2} \sum_{i=0}^{n-2} \binom{n}{i} \binom{n}{i+1} \binom{n}{i+2}$$

Schnyder Woods and Bipolar Orientations

Proposition. There is a bijection between Schnyder woods with $n + 3$ vertices and bipolar orientations with $n + 2$ vertices and the special property:

★ *The right side of every bounded face is of length two.*



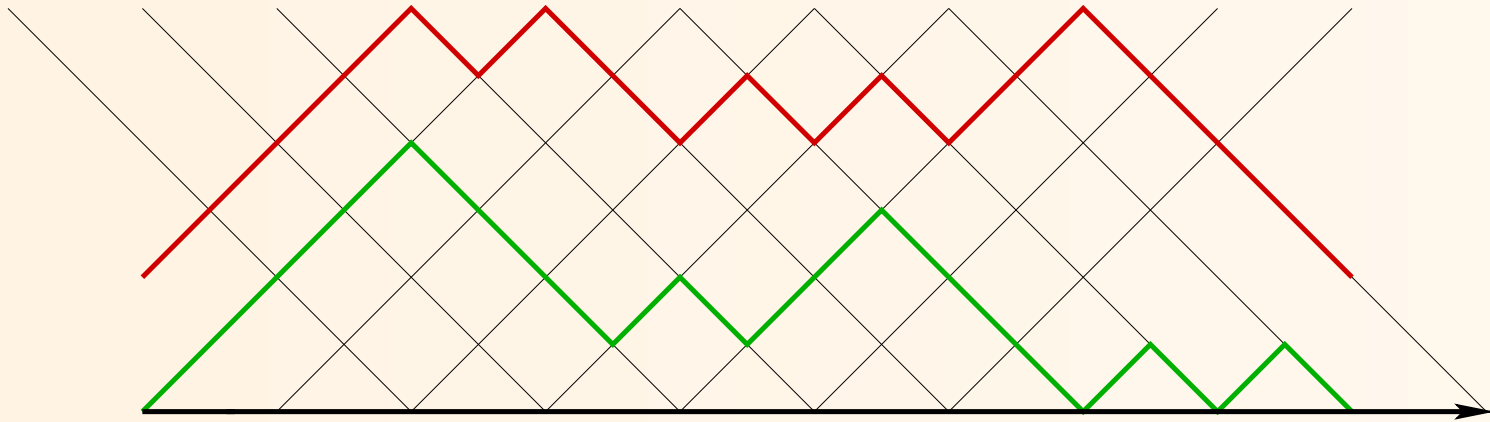
Special Property

Let T^b and T^r be the blue and red tree corresponding to a Schnyder wood. From (\star') we get some crucial properties of the fingerprint and the bodyprints of the trees:

Fact. 1. The extended fingerprint $1 + \alpha$ is a Dyck word; in symbols $(01)^n \leq_{\text{dom}} 1 + \alpha$.

Fact. 2. The fingerprint uniquely determines the bodyprint of the blue tree, precisely $\overline{\beta^b} = 1 + \alpha$.

Schnyder Woods and Dyck Path



Theorem [Bonichon].

The number of Schnyder woods on plane triangulations on $n + 3$ vertices equals the pairs of non-crossing Dyck-path of length $2n$ which is $C_{n+2}C_n - C_{n+1}^2$.

Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

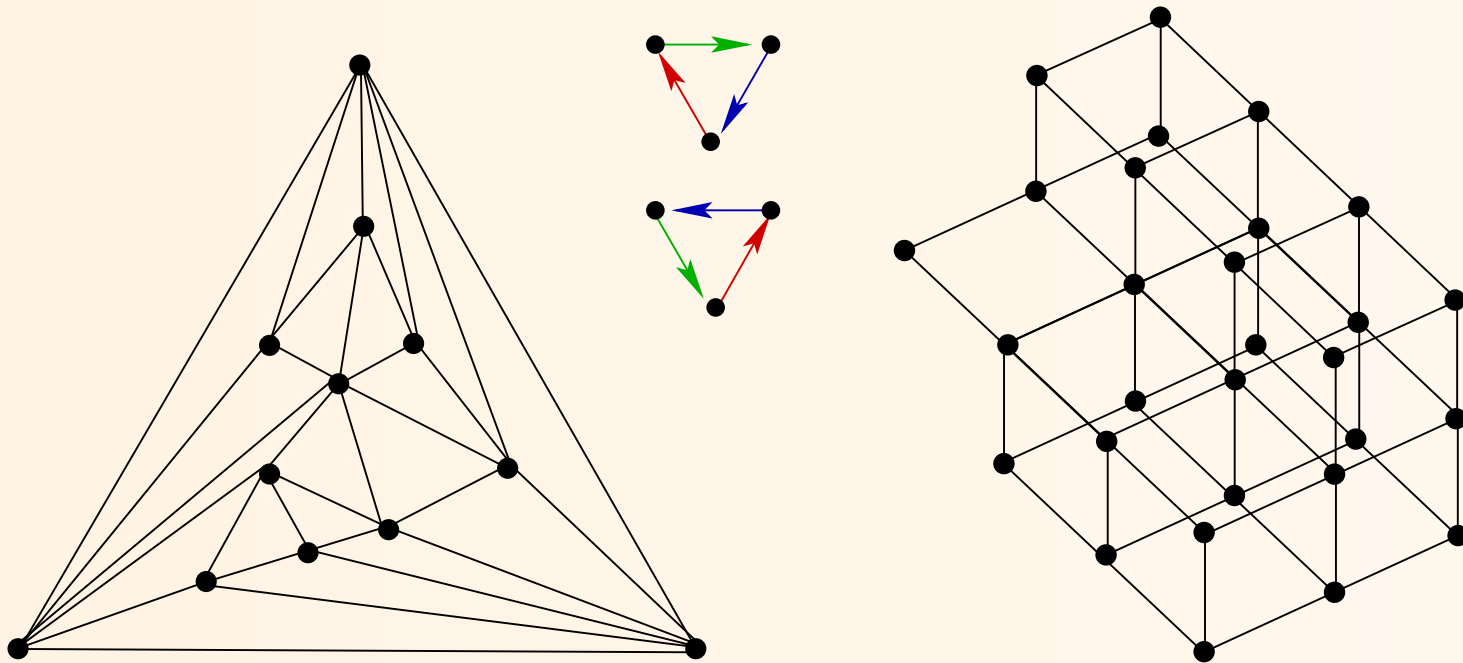
Lattices

Counting III: Random Sampling

Distributive Lattices

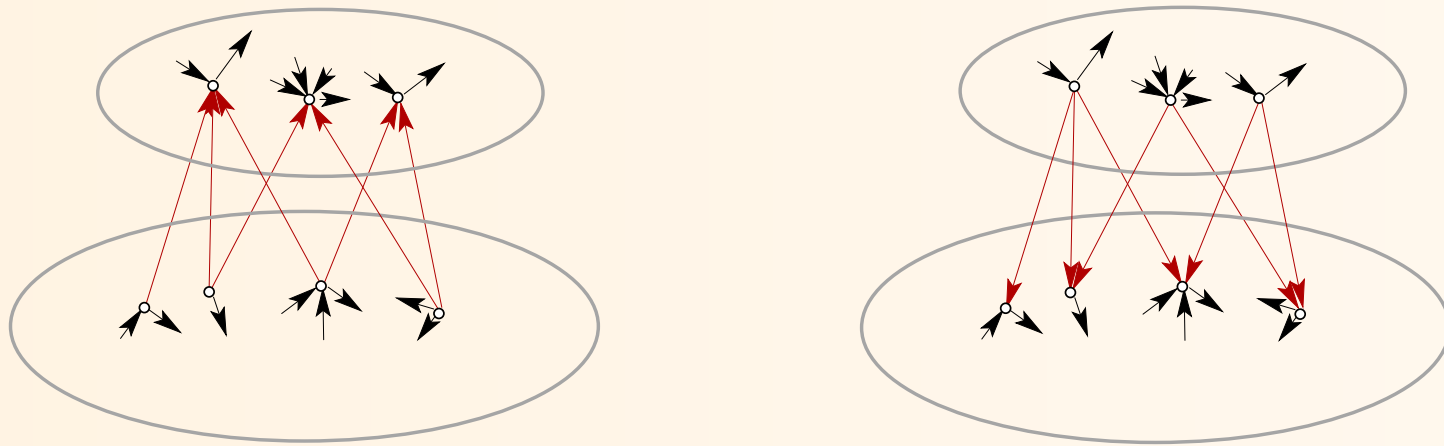
Theorem. The set of α -orientations of a planar graph G has the structure of a distributive lattice.

Example.



A Dual Construction

- Reorientations of directed cuts preserve flow-differences along cycles.



Theorem [Propp 1993].

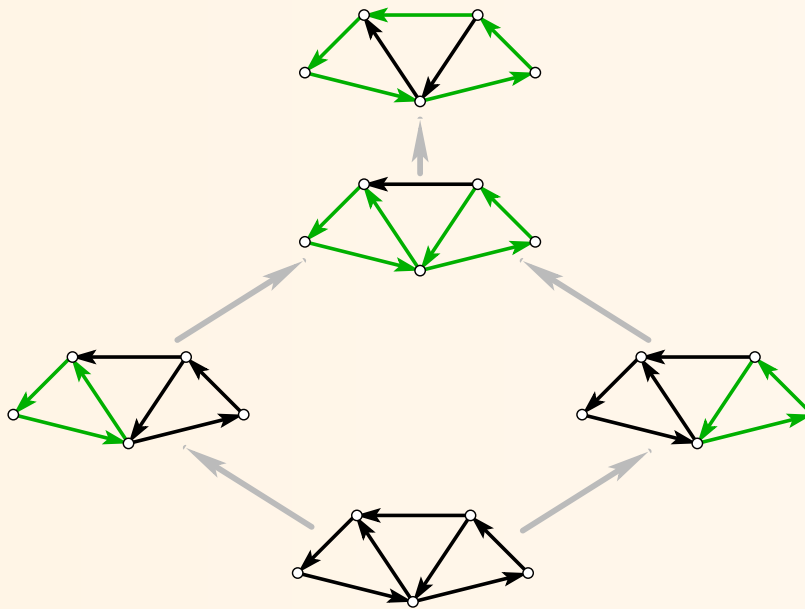
The set of all orientations of a graph G with prescribed flow-differences for all cycles has the structure of a distributive lattice.

Circulations in Planar Graphs

Theorem [Khuller, Naor and Klein 1993].

The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.

$$0 \leq f(e) \leq 1$$



- Diagram edge \sim add or subtract a unit of flow in ccw oriented facial cycle.

Δ -Bonds

$G = (V, E)$ a connected graph with a prescribed orientation.

With $x \in \mathbb{Z}^E$ and C cycle we define the circular flow difference

$$\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^C$ and $\ell, u \in \mathbb{Z}^E$ let $\mathcal{B}_G(\Delta, \ell, u)$ be the set of $x \in \mathbb{Z}^E$ such that $\Delta_x = \Delta$ and $\ell \leq x \leq u$.

Theorem [Felsner, Knauer 2007]. $\mathcal{B}_G(\Delta, \ell, u)$ is a distributive lattice. The cover relation is vertex pushing.

Δ -Bonds as Generalization

$\mathcal{B}_G(\Delta, \ell, \mathbf{u})$ is the set of $\mathbf{x} \in \mathbb{R}^E$ such that

- $\Delta_{\mathbf{x}} = \Delta$ (circular flow difference)
- $\ell \leq \mathbf{x} \leq \mathbf{u}$ (capacity constraints).

Special cases:

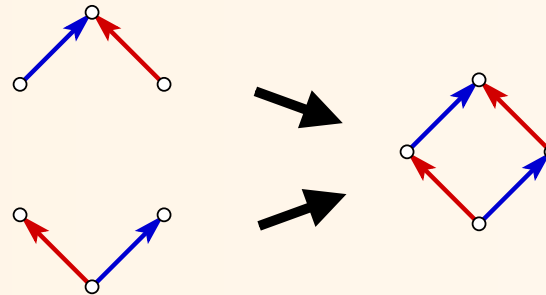
- \mathbf{c} -orientations are $\mathcal{B}_G(\Delta, 0, 1)$
($\Delta(C) = |C^+| - \mathbf{c}(C)$).
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, \mathbf{u})$
(G^* the dual of G).
- α -orientations.

Diagrams of Distributive Lattices: A Characterization

A coloring of the edges of a digraph is a **D-coloring** iff

- arcs leaving a vertex have different colors.

- completion property:



Theorem.

A digraph **D** is connected, acyclic and admits a **D-coloring**

\iff **D** is the diagram of a distributive lattice.

Topics

α -Orientations

Sample Applications

Counting I: Bounds

Counting II: Exact

Lattices

Counting III: Random Sampling

Counting and Sampling

Proposition. Counting α -orientations is #P-complete for

- planar maps with $d(v) = 4$ and $\alpha(v) \in \{1, 2, 3\}$ and
- planar maps with $d(v) \in \{3, 4, 5\}$ and $\alpha(v) = 2$.

Problem.

- Is counting 3-orientations in triangulations #P-complete?
- Is counting 2-orientations in quadrangulations #P-complete?

Approximate Counting

- Approximate counting and **almost uniform sampling** are polynomial equivalent for self-reducible problems.
- An attractive and widely used approach to almost uniform sampling is the **Markov Chain Monte Carlo** method (MCMC).
- The diagram walk on a distributive lattice is a particularly nice instance of MCMC. It allows exact uniform sampling via **Coupling From The Past** (CfP).

Approximate Counting II

Fact. The fully polynomial randomized approximation scheme for counting perfect matchings of bipartite graphs (Jerrum, Sinclair, and Vigoda 2001) can be used for approximate counting of α -orientations.

Approximate Counting II

Fact. The fully polynomial randomized approximation scheme for counting perfect matchings of bipartite graphs (Jerrum, Sinclair, and Vigoda 2001) can be used for approximate counting of α -orientations.

But what about the lattice walk?

Lattice Walks

Fact. The lattice walk is only **rarely** rapidly mixing, i.e., to a polynomial convergence to the uniform distribution.

Dyer, Frieze and Jerrum: Glauber dynamics is exponential for random bipartite graphs with min-degree ≥ 6 .

Lattice Walks

Fact. The lattice walk is only **rarely** rapidly mixing, i.e., to a polynomial convergence to the uniform distribution.

Dyer, Frieze and Jerrum: Glauber dynamics is exponential for random bipartite graphs with min-degree ≥ 6 .

In several situations where planarity plays a role rapid mixing could be proven:

- Monotone paths in the grid.
- Lozenge tilings of an $a \times b \times c$ hexagon.
- Domino tilings of a rectangle.

Lattice Walks for α -orientations

Theorem [Fehrenbach 03]. Sampling Eulerian orientations of simply connected patches of the quadrangular grid using the LW Markov chain is polynomial.

Theorem [Creed 05].

- Sampling Eulerian orientations of simply connected patches of the triangular grid using the LW Markov chain is polynomial.
- Sampling Eulerian orientations of patches of the triangular grid with holes using the LW Markov chain can be exponential.

Lattice Walks for α -orientations

Theorem [Fehrenbach 03]. Sampling Eulerian orientations of simply connected patches of the quadrangular grid using the LW Markov chain is polynomial.

Theorem [Creed 05].

- Sampling Eulerian orientations of simply connected patches of the triangular grid using the LW Markov chain is polynomial.
- Sampling Eulerian orientations of patches of the triangular grid with holes using the LW Markov chain can be exponential.

Problem.

We do not know much about sampling α -orientations.

THE END

THE END

Thank you.