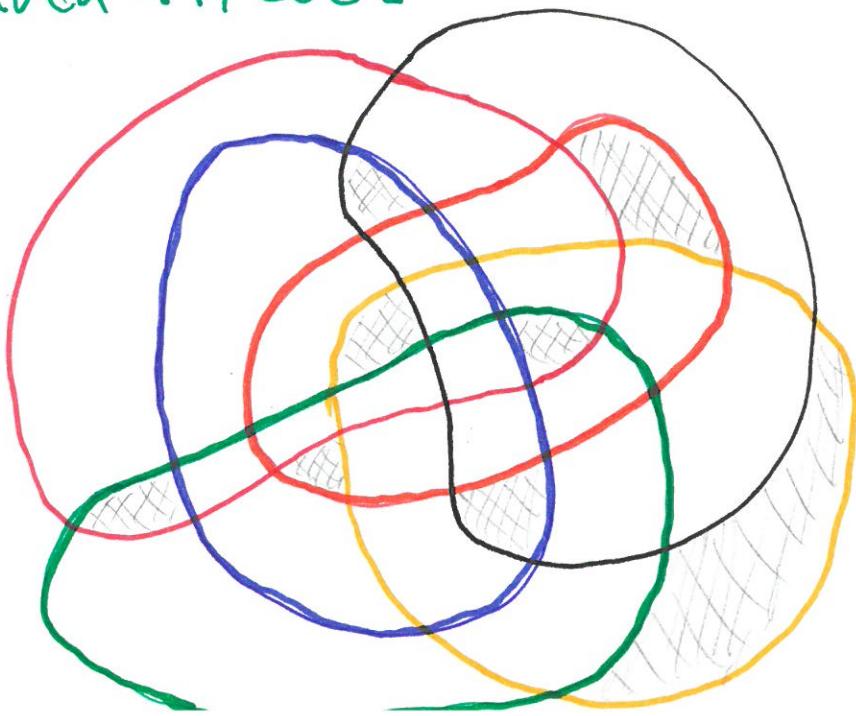


# COMBINATORICS OF PSEUDOCIRCLE ARRANGEMENTS

TCS seminar at JU  
March 17, 2021

Stefan Felsner

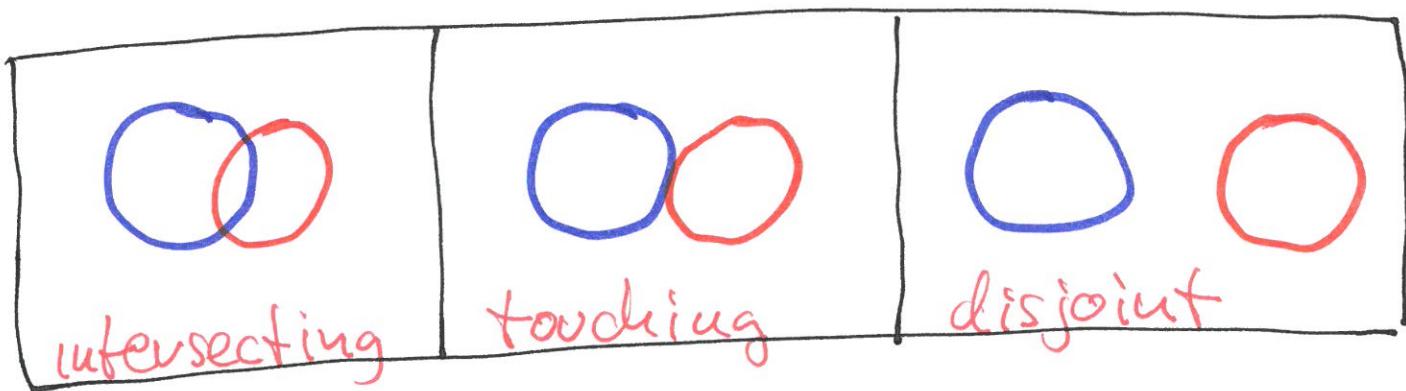
Based on work with  
Manfred Schreder  
and several  
others



Pseudocircle: A closed curve in the plane or on the sphere

Arrangement of pseudocircles:

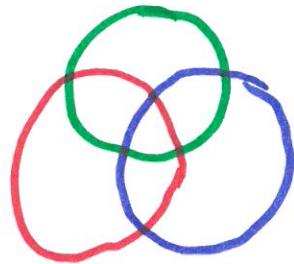
A collection of pseudocircles such that any two intersect as circles would do



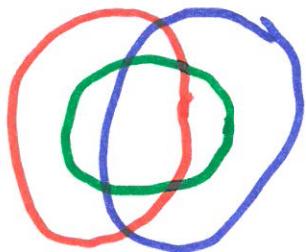
Arrangement is simple: No crossing  
of  $\geq 3$  pseudocircles in one point  
no touch

Arrangement is intersecting: Any  
two pseudocircles intersect

$n = 3$  There are two simple  
intersecting arrangements  
on  $S^2$



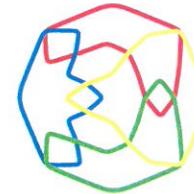
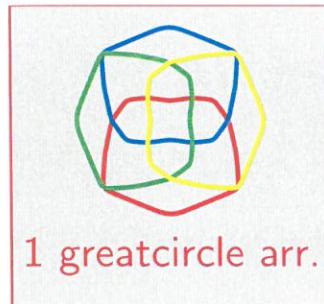
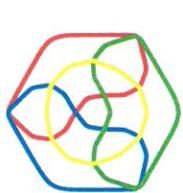
Krupp



Non-Krupp

$n=4$

## Enumeration



asymptotic

$\Theta(n^2)$

8 intersecting arrangements

21 connected arrangements

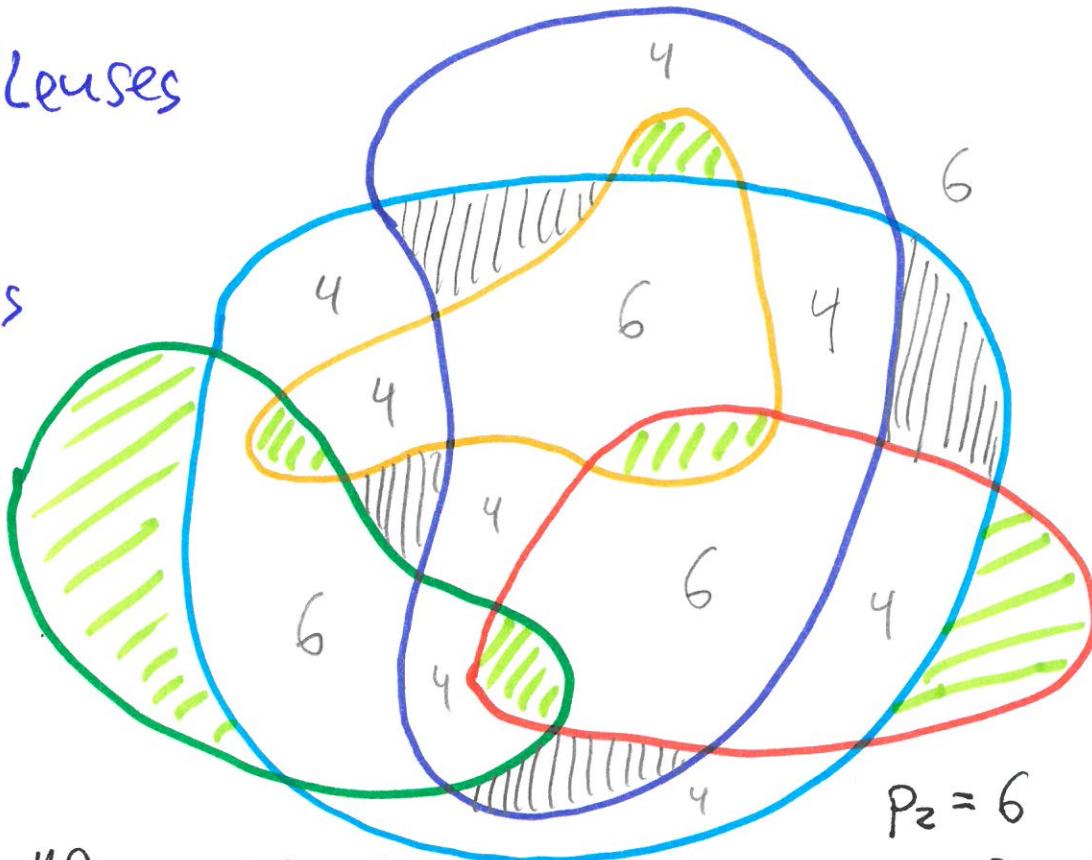
$n$	#arr simple intersections
3	2
4	8
5	278
6	145058
7	447905202

# Cells/faces of an arrangement

digons / lenses



triangles



$$n = 5$$

$$\# \text{faces} = n(n-1) + 2 = 22$$

$$\begin{aligned} P_2 &= 6 & P_3 &= 4 \\ P_4 &= 8 & P_6 &= 4 \end{aligned}$$

# Topics of this talk

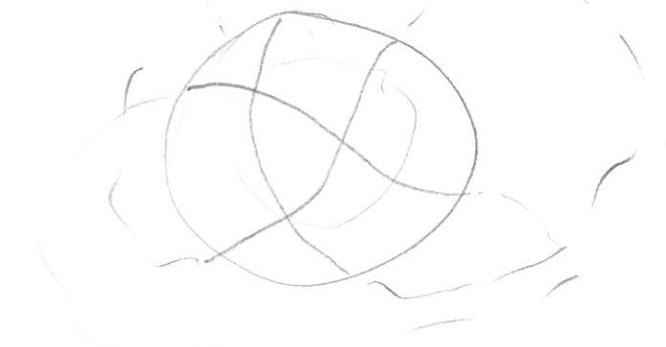
- Triangles and digons
- Circularizability
- Colorings

# Grünbaum Conjecture 1972

A arrangement of pseudocircles  
simple intersecting digon-free  
 $\Rightarrow \# \text{triangles} \geq 2n - 4$

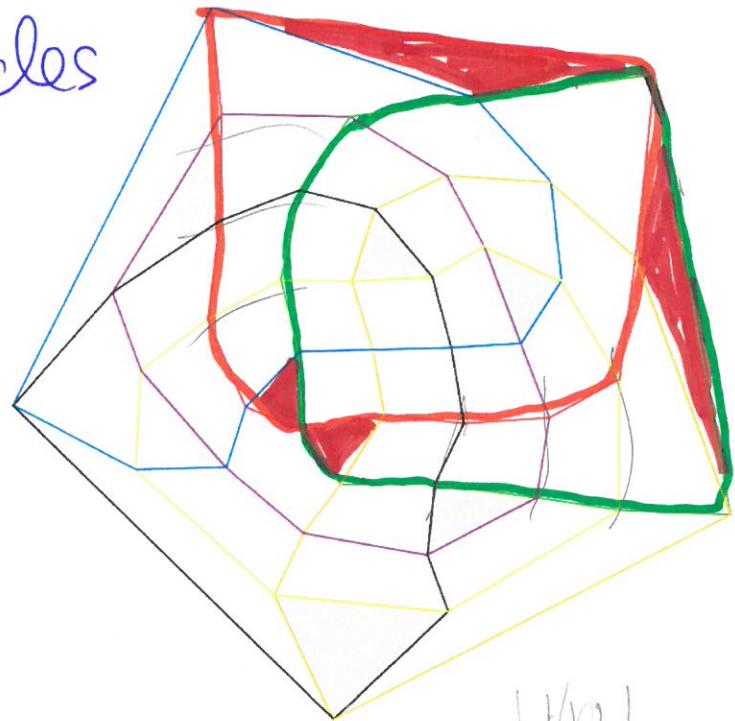
Pseudolines Euclidean plane

$$\# \text{triangles} \geq n - 2$$

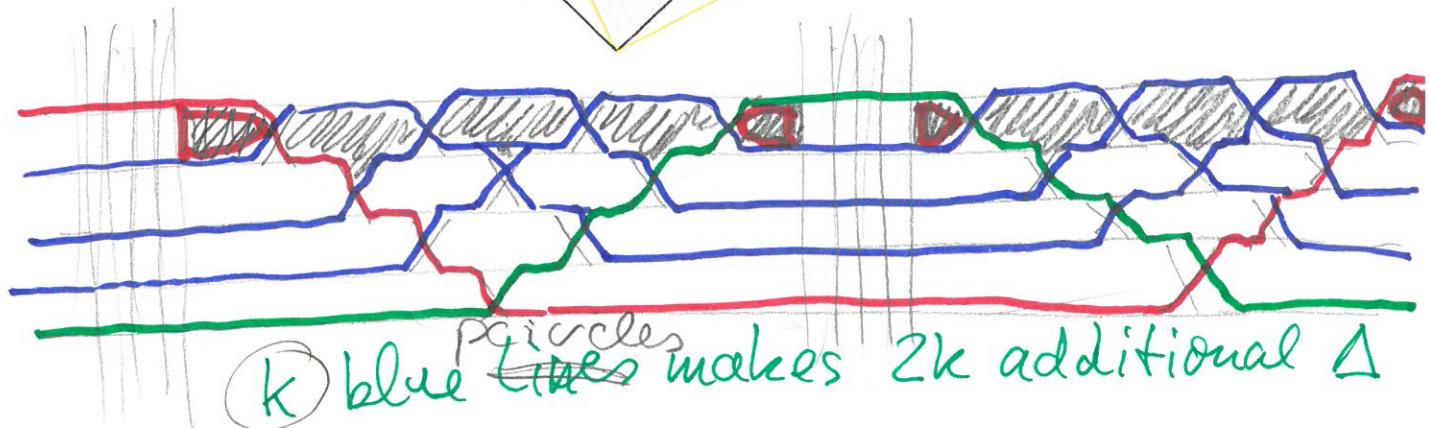


7 pseudocircles

10 triangles



Examples  
with  
 $2n-4$   
triangles  
exist.



# Sweeping arrangements of pseudocircles

Sucsyuk, Heeschberger 1989

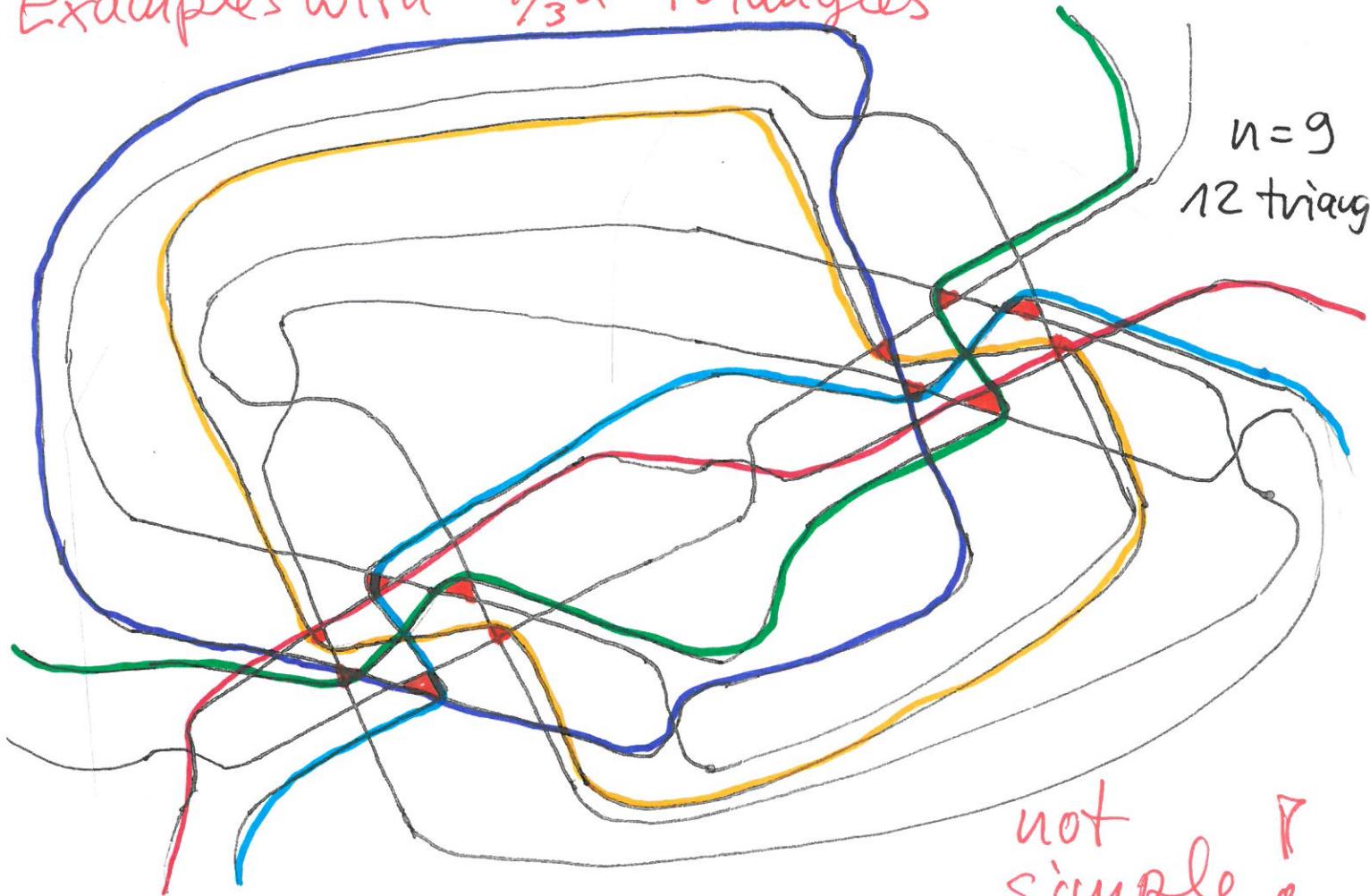
Every pseudocircle in an intersecting arrangement has  $\geq 2$  sweepable cells (triangle or digon) on each side.

Consequence:

digon-free  $\Rightarrow \geq 4$  triangles incident to each PC.  $\Rightarrow$

$\geq \frac{4}{3} n$  triangles

Examples with  $\frac{7}{3}n$  triangles



not  
simple!

Grünbaum Conjecture 1972

A arrangement of pseudocircles  
simple intersecting digon-free  
 $\Rightarrow \# \text{triangles} \geq 2n - 4$

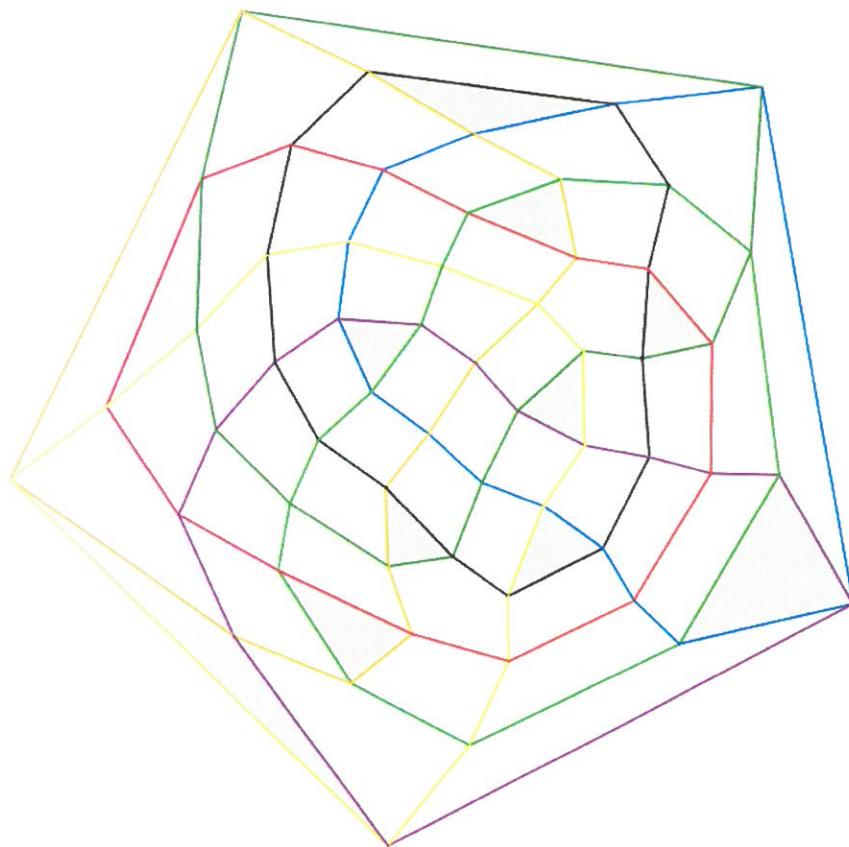
Lower bound  $\frac{4}{3}n$

In the non-simple case  $\frac{4}{3}n$  is tight.

Grünbaum's Conjecture is wrong

$$n = 8$$

$$\begin{aligned} \# \text{triangles} \\ = 11 \end{aligned}$$



# A larger example

$$n = 12$$

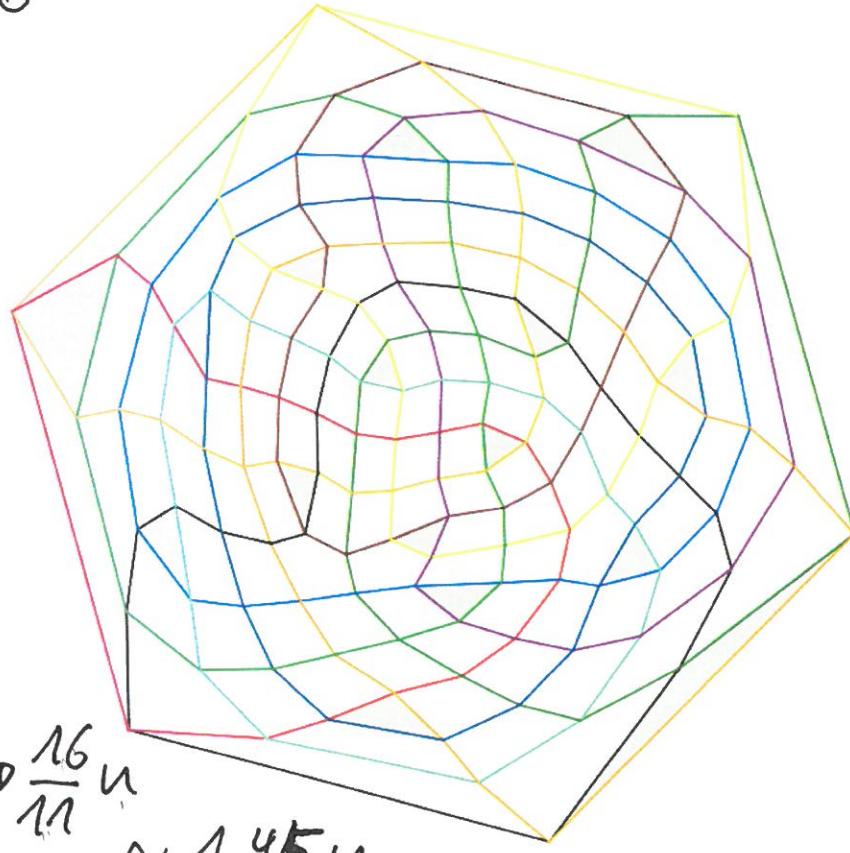
$$\# \text{triangl} = 16$$

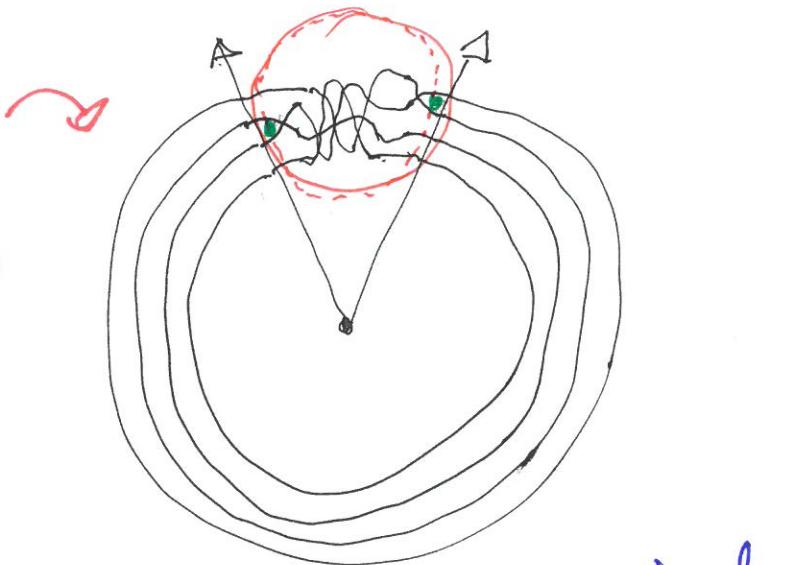
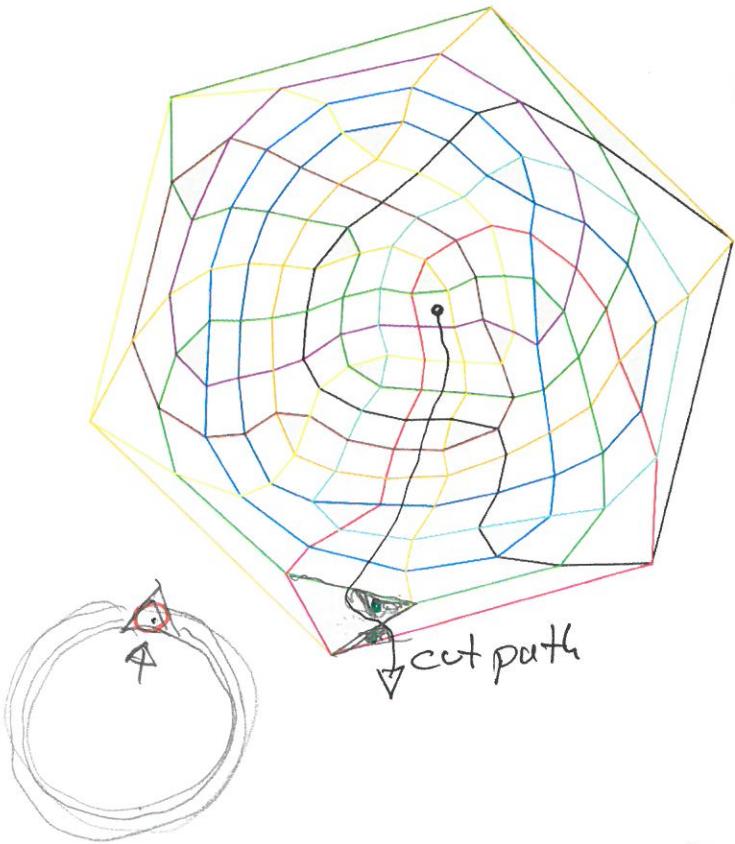
$$\#\Delta = \frac{4}{3} n$$

use it  
for a  
family

$$\text{with } P_3 \rightarrow \frac{16}{11} n$$

$\sim 1.45 n$





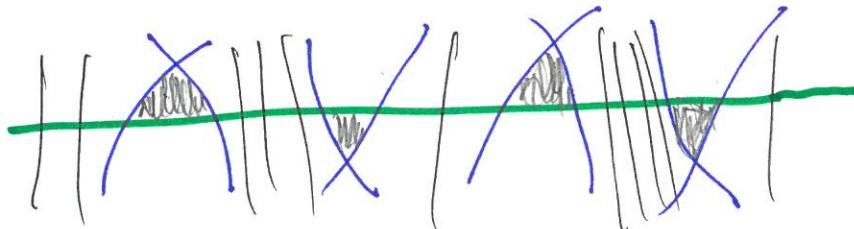
circle  
 $\downarrow$   
 replace 1 pseudo ~~triangle~~  
 in  $\Delta_{S-1}$  by 12 pcs  
 such that.

- kill 1 triang of  $\Delta_{S-1}$
- 2 boundary tr.
- 1 triang dest. by cut path

$$n_S = 11s + 1$$

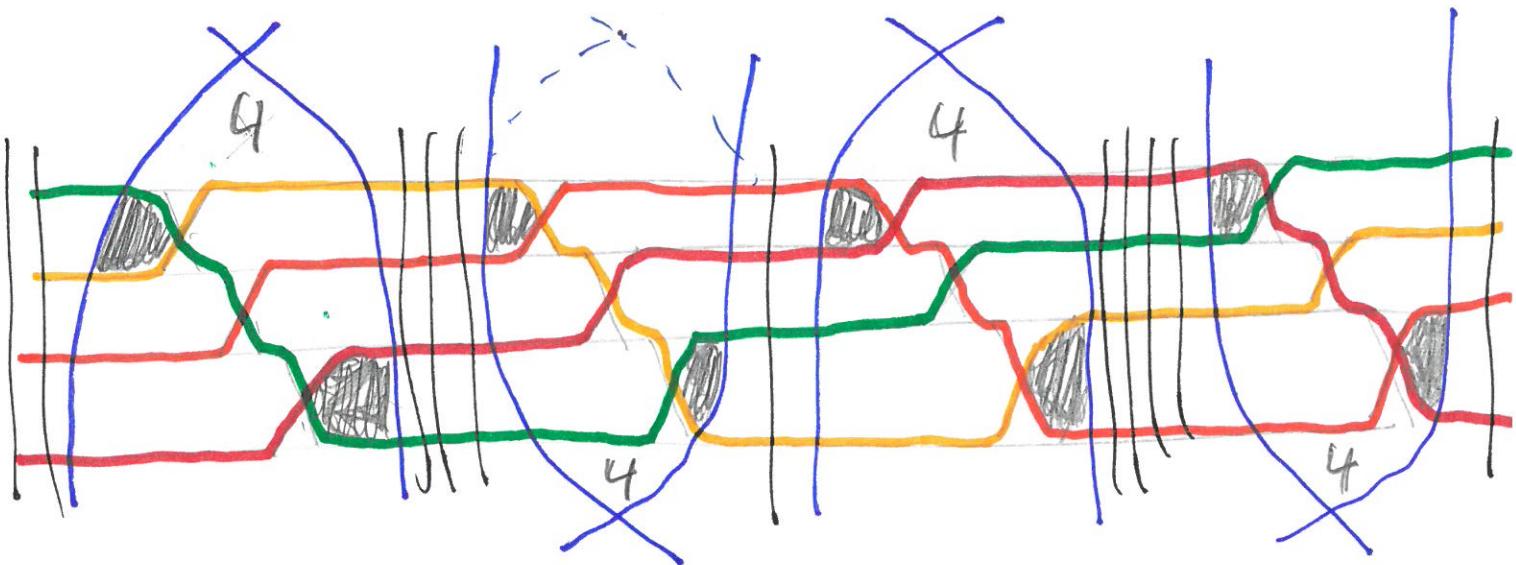
$$\#\Delta = 16s$$

# A new replacement strategy



a pseudocircle  
and 4 incident  
triangles

$$n \rightarrow n+3 \quad \# \Delta \rightarrow \# \Delta + 4$$



# Grünbaum Conjecture 1972

A arrangement of pseudocircles  
simple intersecting digon-free

$$\Rightarrow \# \text{triangles} \geq 2n - 4$$

Wrong  $\frac{4}{3}n$  is the true value

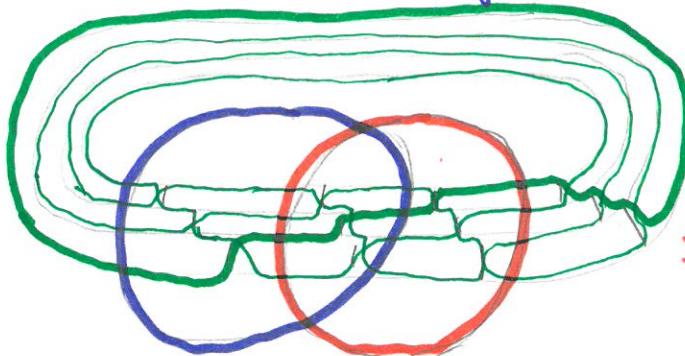
New conjecture:

A arrangement of circles  
simple intersecting digon free

$$\Rightarrow \# \text{triangles} \geq 2n - 4$$

# Support for the new conjecture

Let  $N_6^A$  be the arrangement obtained with new replacement from the Knapp



$$n = 6$$

$$\#A = 8$$

## Facts

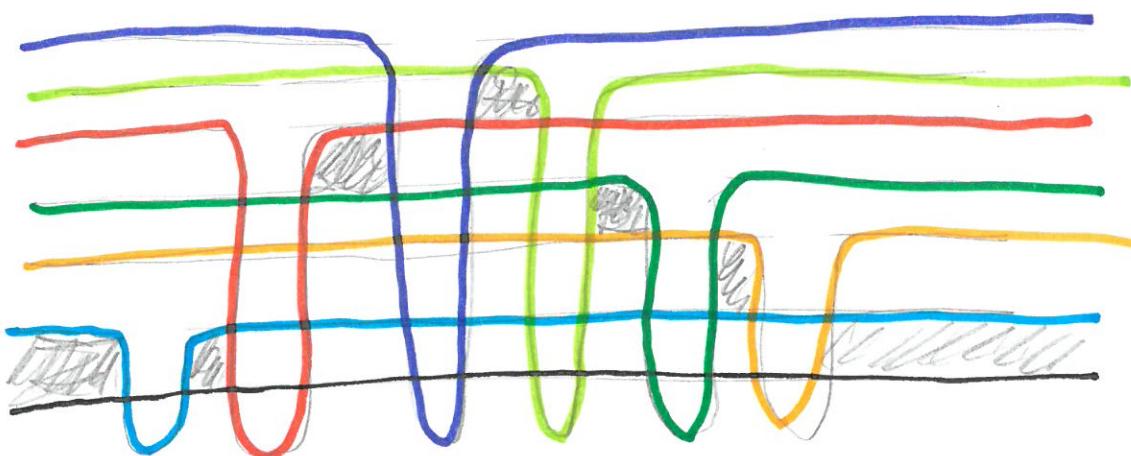
- $N_6^A$  is non-circulizable
- $N_6^A$  is a sub-arrangement to all known counterex. to Grünbaum's conj.

# Triangles and digons

Conjecture [Gr. 72]

A arrangement of pseudo circles  
simple + intersecting

$$\Rightarrow \# \Delta \geq n - 1$$

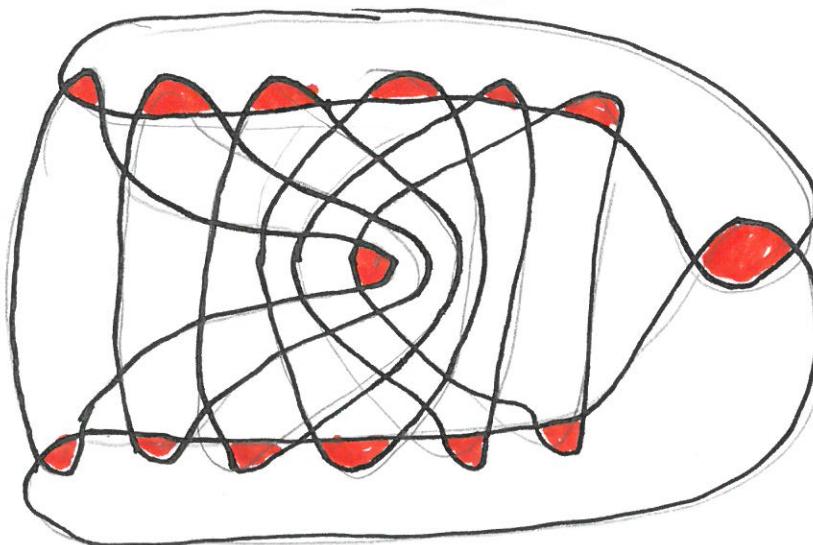


Sweeping  
implies  
 $\# \Delta \geq \frac{2}{3} n$

# Digons

Conjecture [Gr. 72]

A arrangement of pseudo circles  
simple + intersecting  
 $\Rightarrow \# \text{digons} \leq 2n - 2$



ANPPSS'03

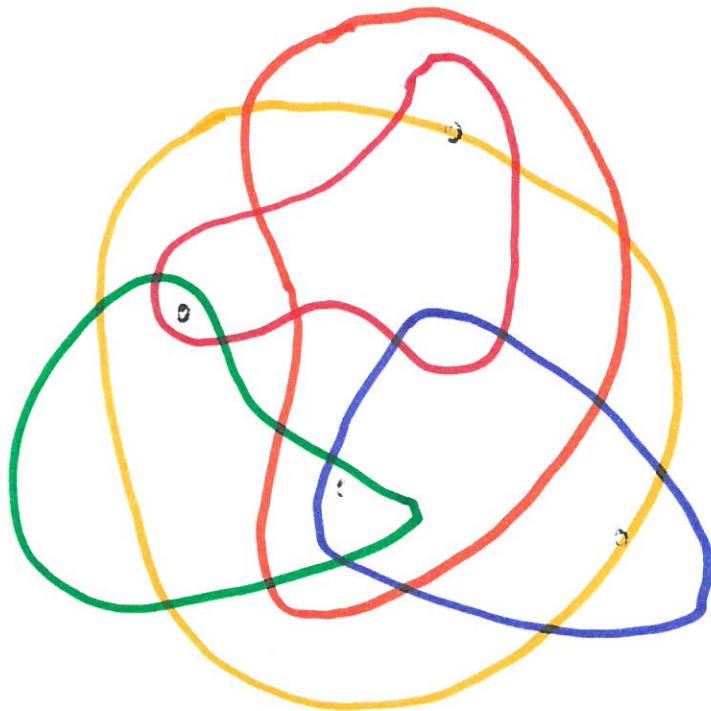
Lenses in arrangement  
of pseudo circles

- True for cylindrical  $\Delta$
- $\# \text{digons} = O(n)$

## Topics of this talk

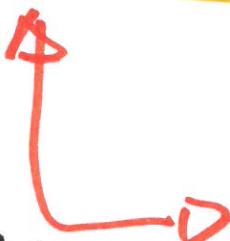
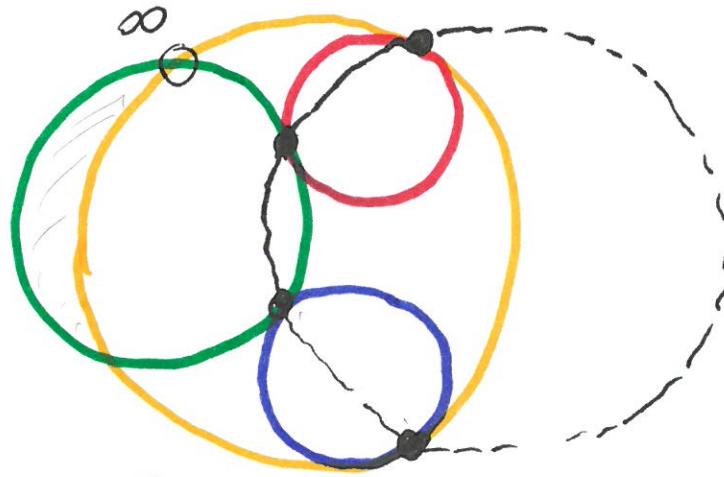
- Triangles and digons
- Circularizability
- Colorings

# Using an incidence theorem

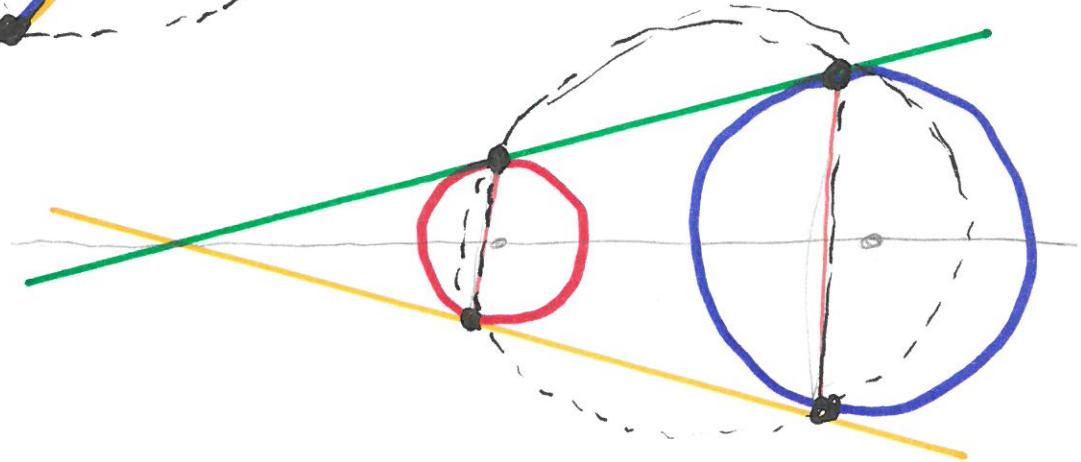


The unique  
non-circularizable  
intersecting  
arrangement  
of 5 pseudocircles

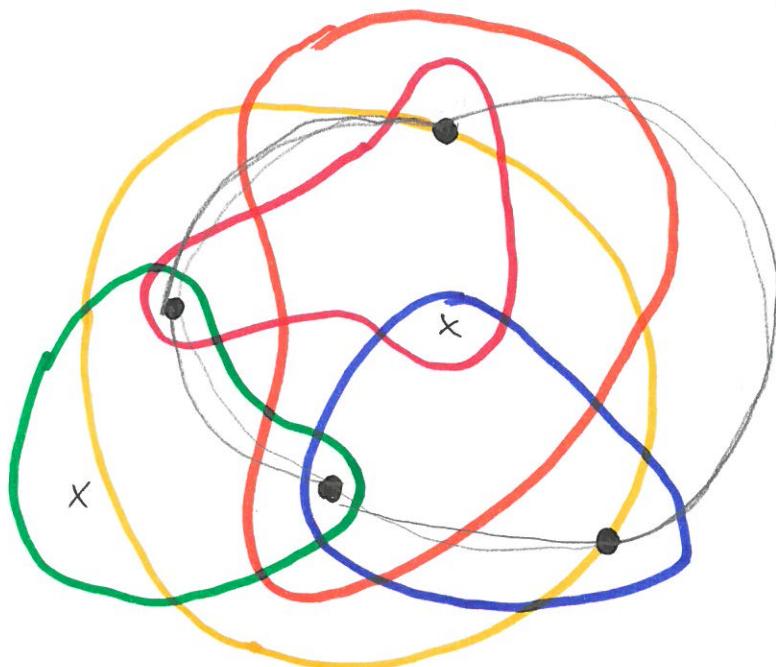
# The incidence theorem



Möbius  
transform



Assume a circle representation



Shrink

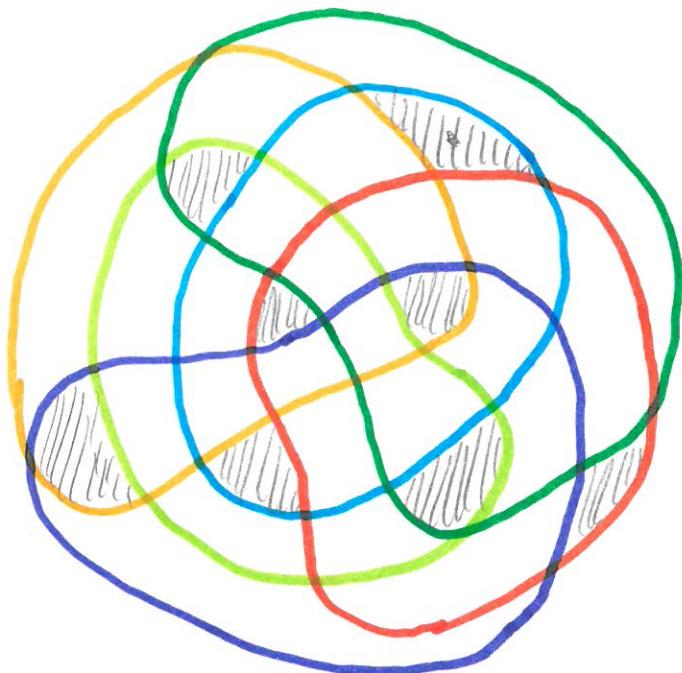
○ ○ ○

to make the  
black points touching

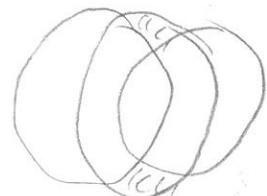
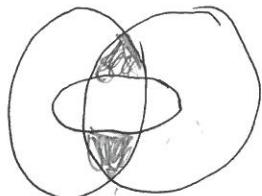
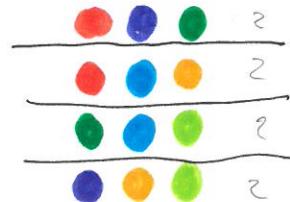
The circle  
guaranteed  
by the inc. thm.

has 4  
intersections  
with  $\odot$  ↴

# Non - Circularizability of $N_6^\Delta$

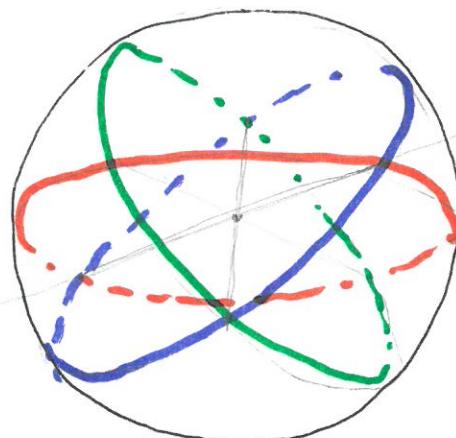


Observation:  
All triangles  
are non-Krapp

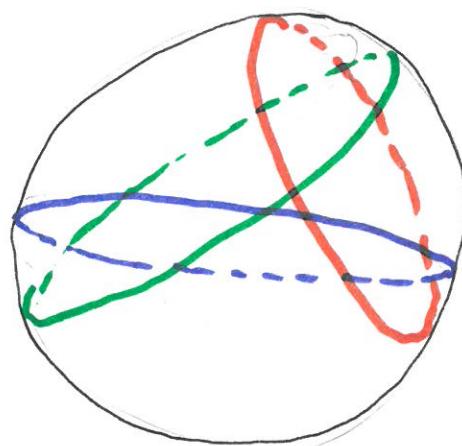


# Circle arrangements and hyperplanes

- Circle arrangement on sphere
- circles  $\rightarrow$  hyperplanes



Krupp



Non Krupp

## Shrinking the sphere

Possible events:

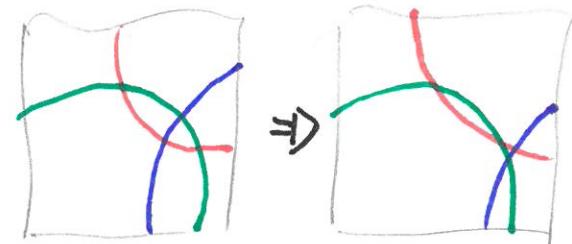
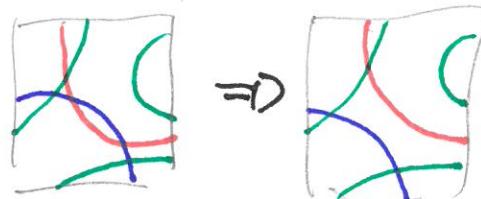
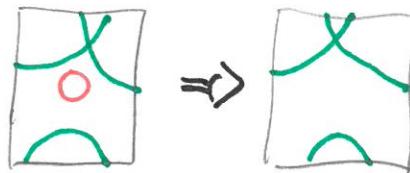
- S is Loosing contact to a hyperplane

keeping the arrangements of hyperplanes

- two circles loose contact

- a triangle flip

Krupp  $\rightarrow$  Non-Krupp



# Topics of this talk

- Triangles and digons
- Circularizability
- Colorings

# A coloring conjecture

Conj. [FHNS'98]

Every simple great circle arrangement is 3-colorable

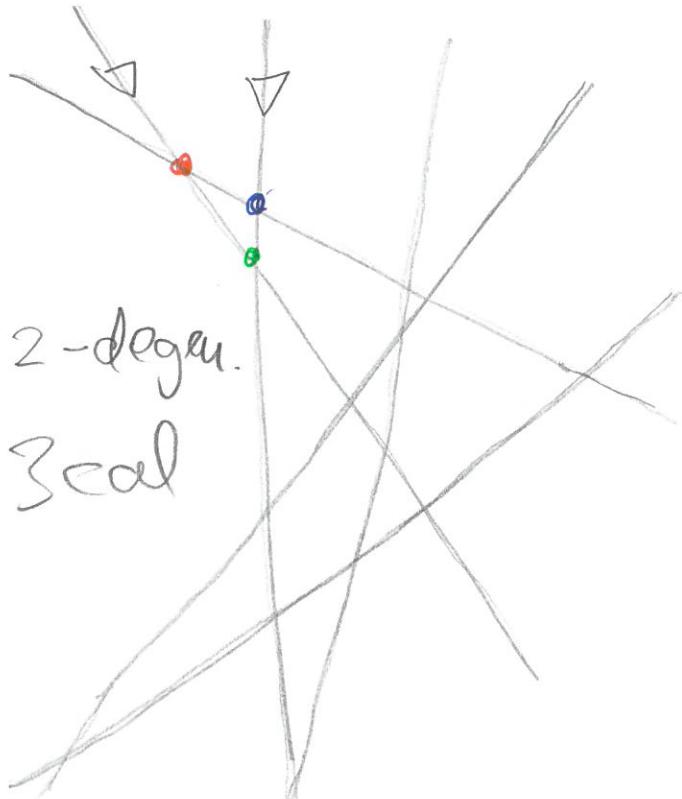
---

4colorability:

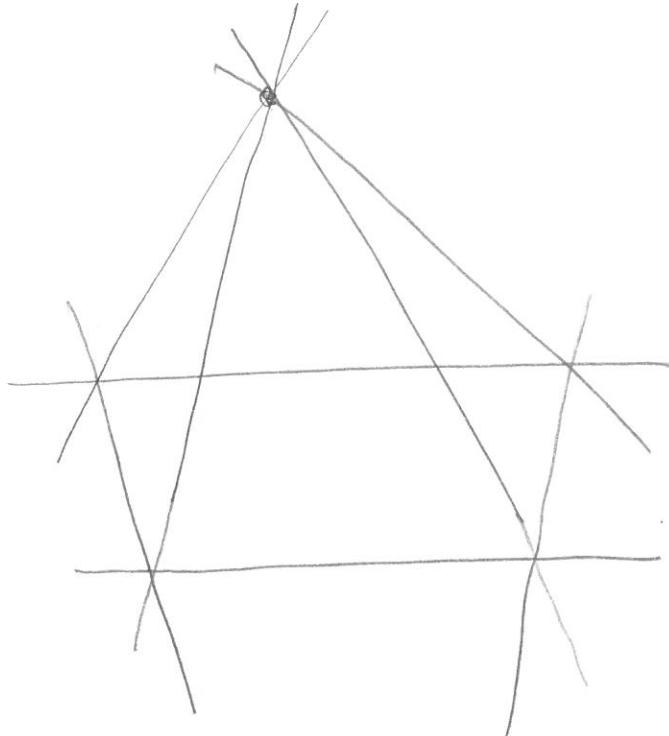
- 4regular
- planar

# Coloring Line arrangements

simple



non-simple



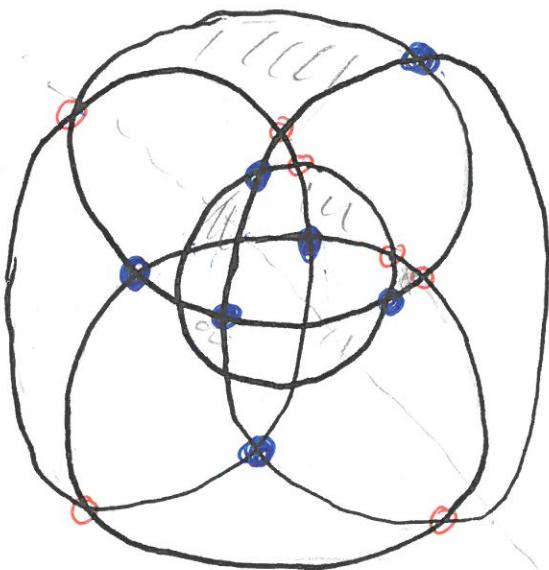
# Pseudo great-circle arrangements

Every triple of pseudo circles induces a Krupp



implies intersecting  
but intersecting is  
not enough

3 cal.



## Short Argument

Suppose 3col suffice: Every col class hits every triangle

Central 4gon has 2 diagonal vertices of same color

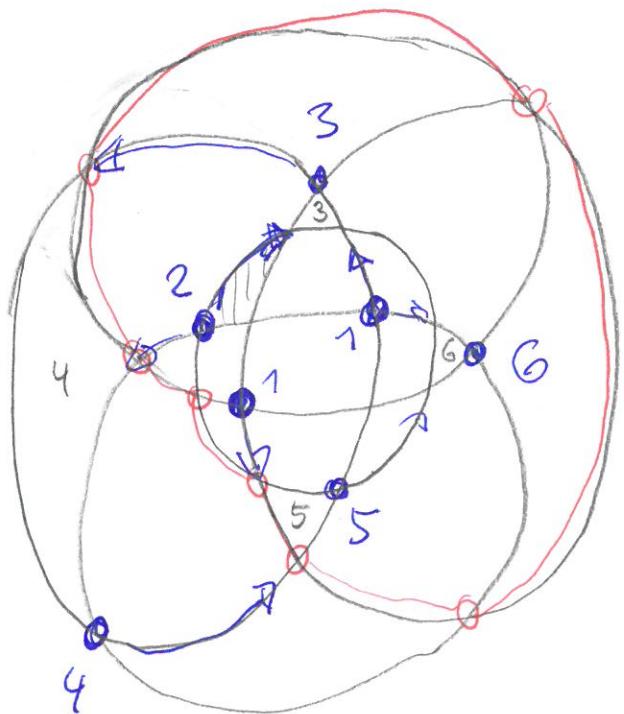
Symmetry 1,1

sym of shaded triang

2

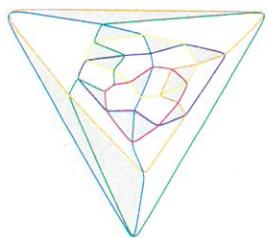
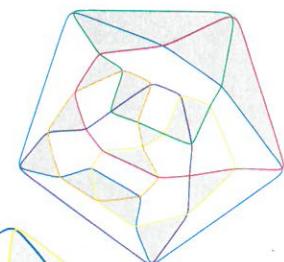
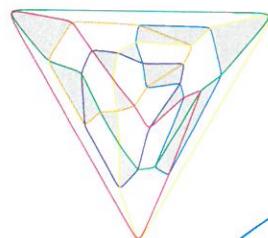
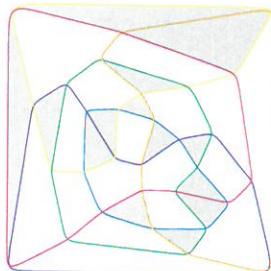
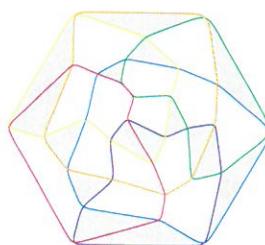
Now we get 3 4 5 6

Red: 7 cycle in the complement

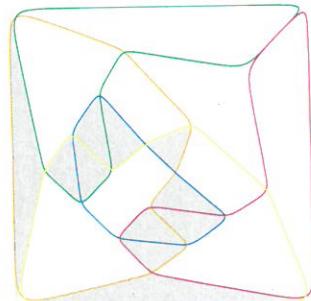
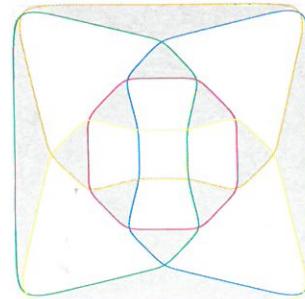


# Intersecting arrangements with $\chi = 4$

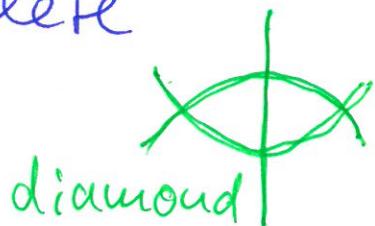
$n=6$



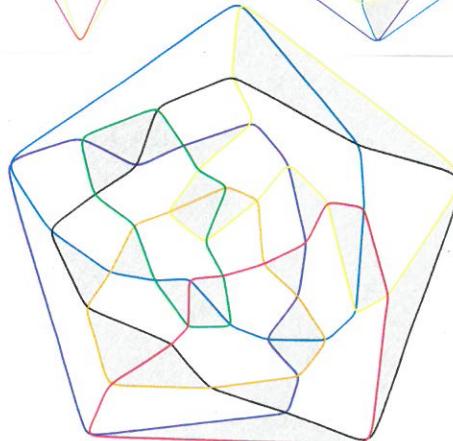
$n=5$



List is  
complete



$n=7$

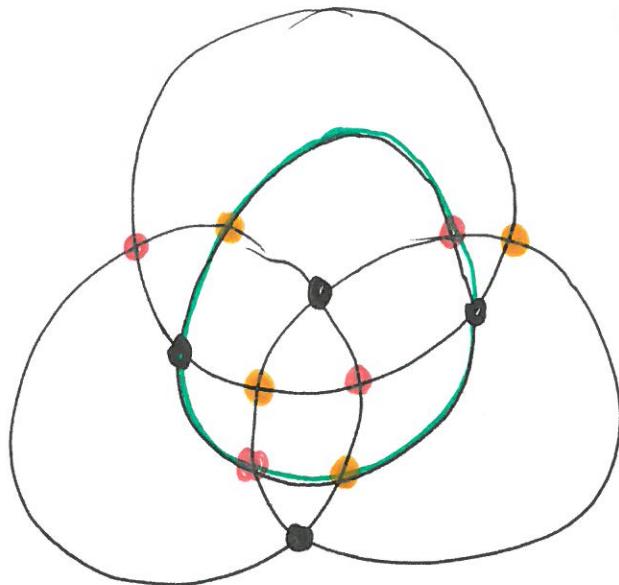


## New strong conjectures

Conj: Every diamond-free intersecting arrangement of  $n \geq 6$  pseudocircles is 3-colorable

Conj: Every sufficiently large simple intersecting arrangement of pseudocircles is 3-colorable

# Antipodal colorings of great circle - arr. equator

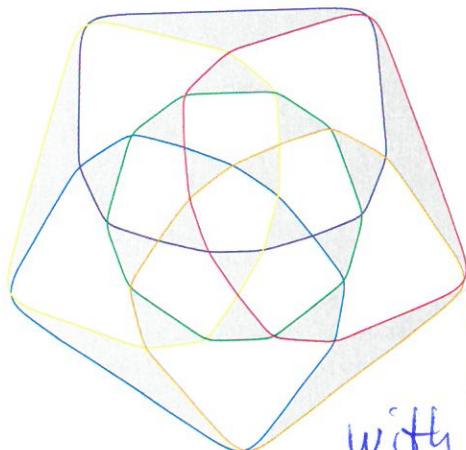


Antipodal  
3-colorings  
are 3-colorings  
of projective  
pseudo line arrangements.

## Antipodal colorings

All pseudo-great-circle arrangements  
with  $n = 7, 8, 9, 10, 11$  have antipodal 3-col.

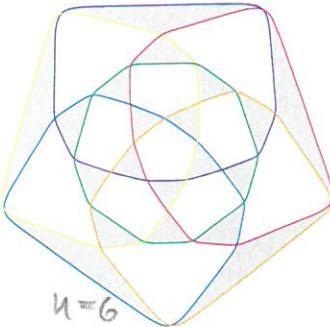
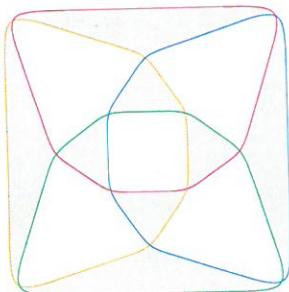
Conj: All pseudo great-circle arrangements  
with  $n \geq 7$  have antipodal 3-colorings



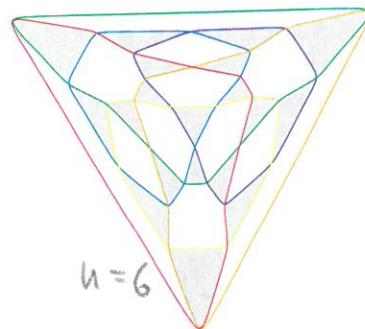
the  
only  $n=6$   
great-circle  
arrangements  
with no antipodal 3-colorings

# Triangle saturated arrangements

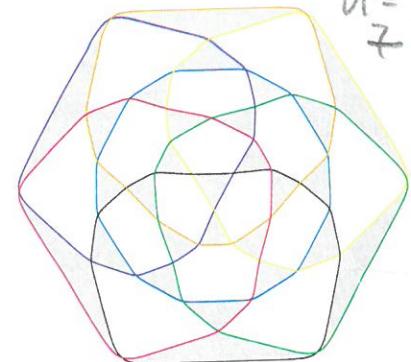
$n=4$



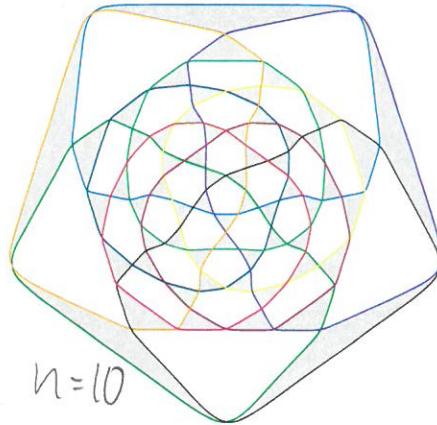
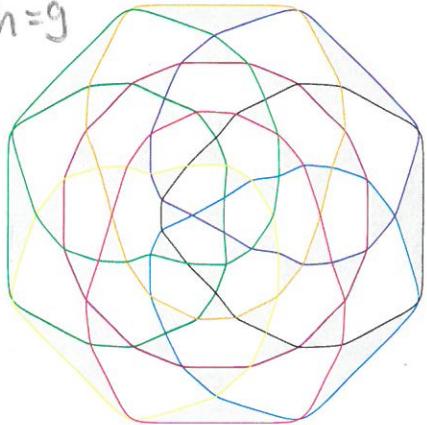
$n=6$



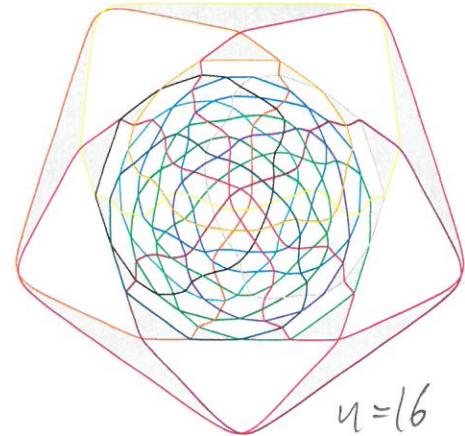
$n=7$



$n=9$

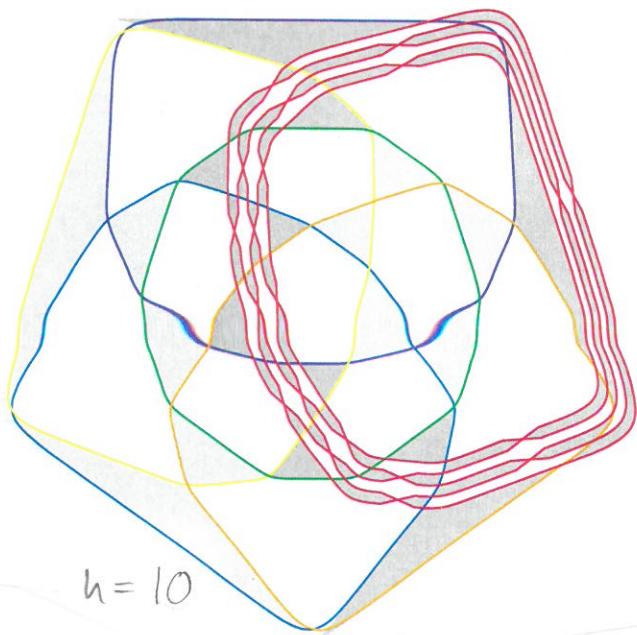
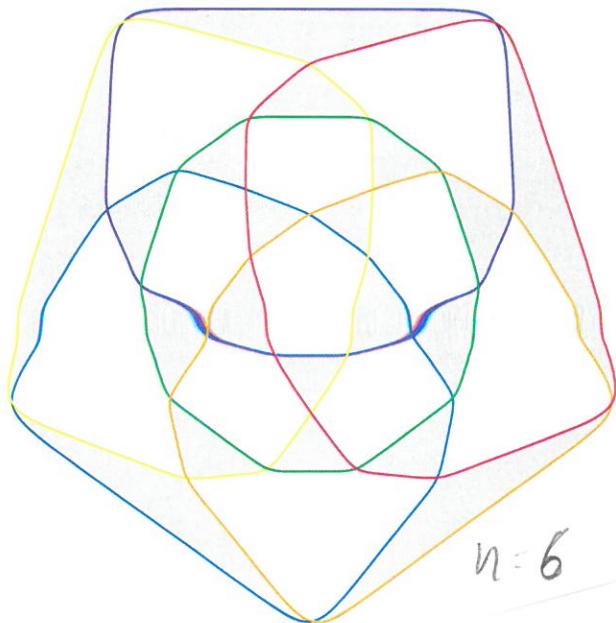
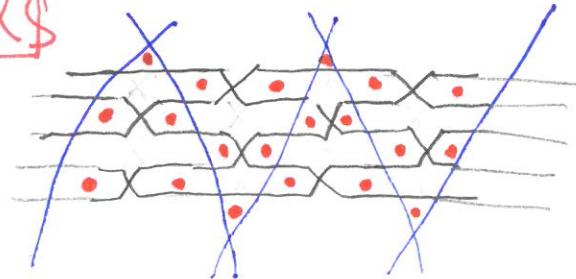
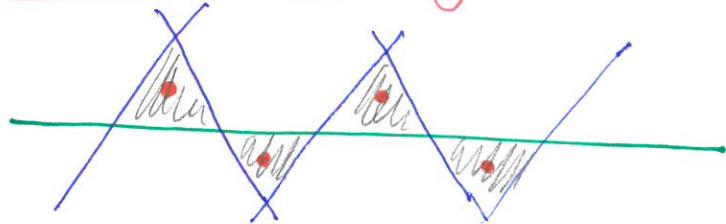


$n=10$

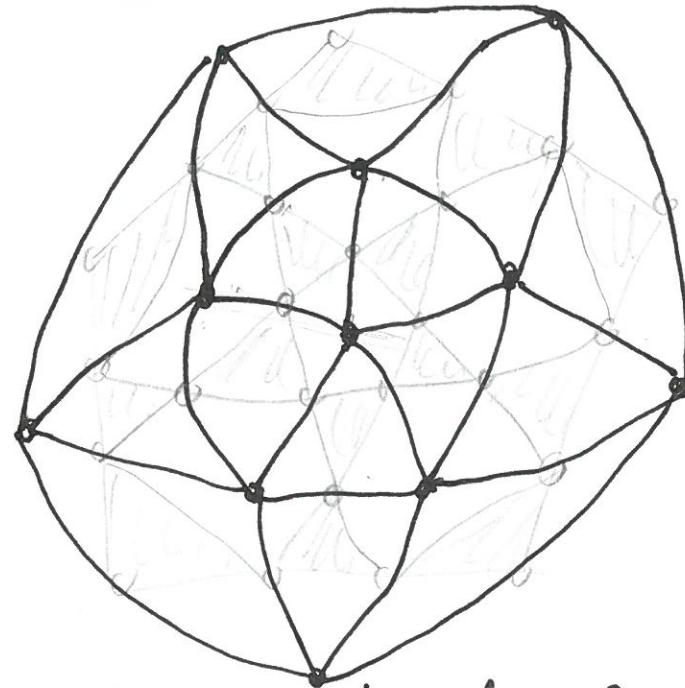
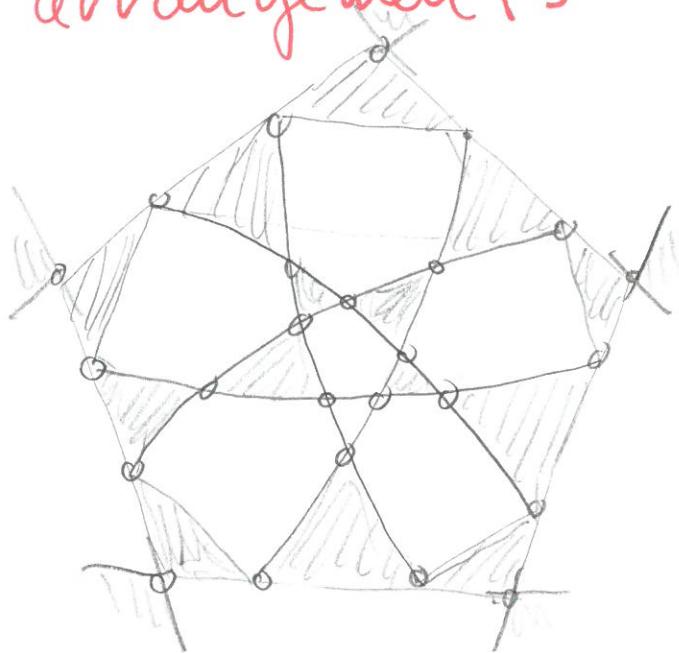


$n=16$

# The doubling method\$



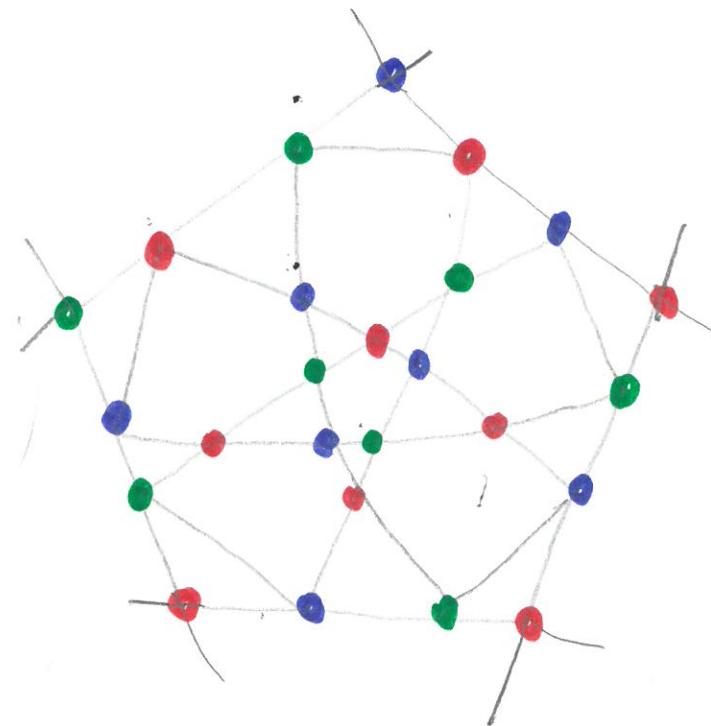
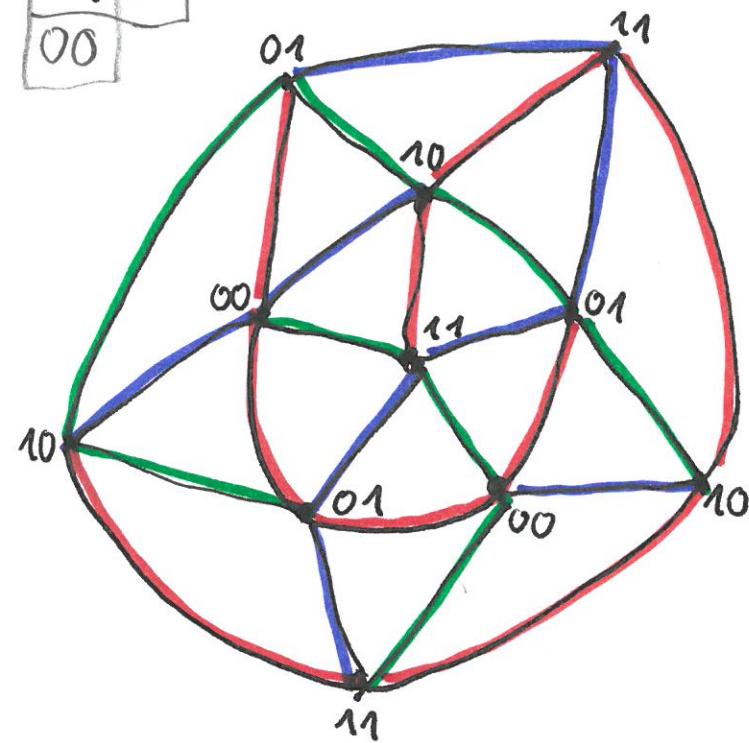
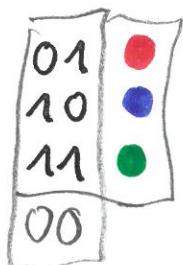
3-coloring triangle saturated arrangements



The white dual is  
a triangulation

Tait: Triangulations are 3-edge col.  
Rainbow

4CT  $\Rightarrow$  Tait  $\Rightarrow$  3 col satur. PCA

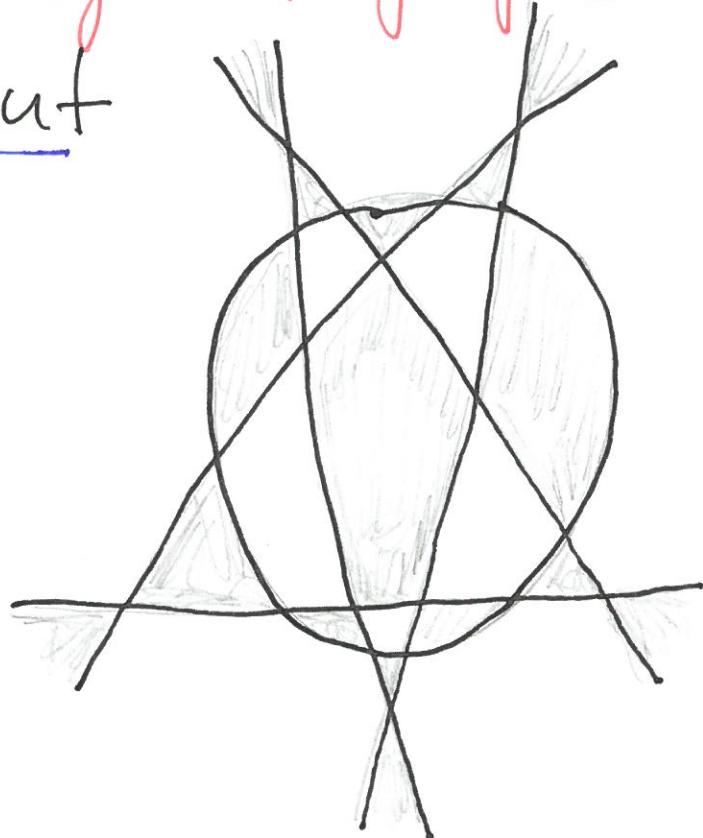


# 4-chromatic arrangement graphs

## The corona implant



A 5-gon  
in a 4-saturated  
arrangement



Double counting the corona arrangement.

- 4 regular  $\Rightarrow |E| = 2|V|$
- 4 black triangles 1 black pentagon  $|E| = 3\Delta + 5$   
 $\Rightarrow \Delta$  is odd.

I independent  $|I| = \alpha$

$X = \# \text{pairs } (v, f) \ v \in I, f \text{ black}$

- ▷ Each  $v$  incident to 2 faces  $\Rightarrow X = 2\alpha$
- ▷ Each triangle incid.  $\leq 1$  pentagon inc.  $\leq 2$   
 $\Rightarrow X \leq \Delta + 2$

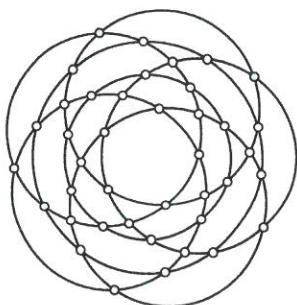
$$2\alpha \leq \Delta + 2 \quad \Delta \text{ odd} \Rightarrow 2\alpha \leq \Delta + 1$$

$$6\alpha \leq 3\Delta + 3 = 2|V| - 2$$

$$\Rightarrow \alpha < \frac{|V|}{3}$$

Conjecture.

The corona construction  
yields infinitely many  
 $\chi$ -critical arrangement graphs

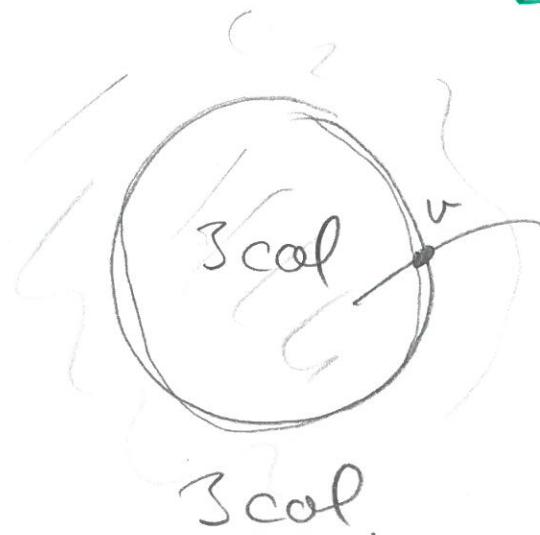


Koester's example  
of a 4 regular  
4 critical arrangement  
graph

# Fractional colorings

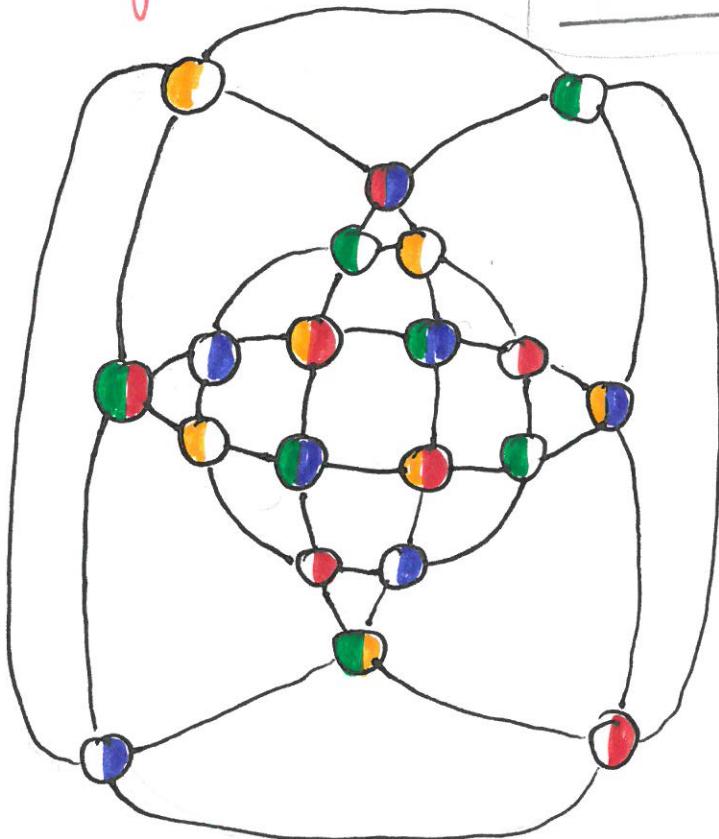
intersecting arrangements admit  
a  $(n-2)$ -coloring using  $3u$  colors  
in total

$$\chi_f(G_d) \leq \frac{3u}{n-2} = 3 - \frac{6}{n-2}$$



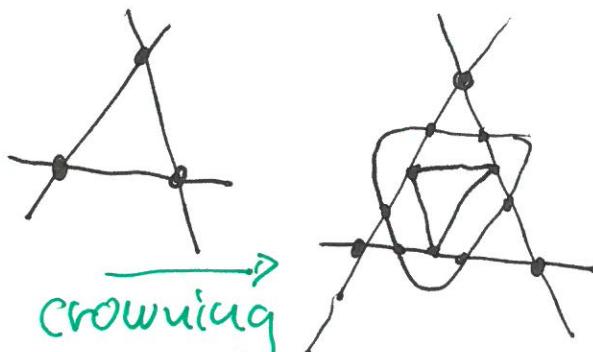
$n$  packets of 3 colors  
VC pseudocircle: find  
3coloring of int. and ext  
sweep!  
Each vertex  $v$  receives  $n-2$  colors

Graphs with  $\chi = 4$  and  $\chi_f = 3$



A b-coloring ( $b=2$ )  
with 6 colors.

$$\Rightarrow \chi_f = 3$$



Crowning preserves  
the properties  $\chi = 4, \chi_f = 3$

## List of conjectures

- Circle arr. simple inters digon-free  $\Rightarrow \# \Delta \geq 2n-4$
- Pseudocircles simple inters  $\Rightarrow \# \Delta \geq n-1$
- — || —  $\Rightarrow \# \text{digons} \leq 2n-2$
- Great pseudocircles simple  $\Rightarrow \chi = 3$
- Pseudocircles simple inters. diamond-free  $n \geq 6 \Rightarrow \chi = 3$
- Pseudocirc. simple inters n large  $\Rightarrow \chi = 3$
- Great pseudocircle simple  $n \geq 7$   $\Rightarrow$  antipodal 3-coloring