Distributive Lattices from Graphs

VI Jornadas de Matemática Discreta y Algorítmica
Universitat de Lleida
21-23 de julio de 2008

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The Talk

Lattices from Graphs

Proving Distributivity: ULD-Lattices

Embedded Lattices and D-Polytopes
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Embedded Lattices and D-Polytopes
Definition. Given $G = (V, E)$ and $\alpha : V \rightarrow \mathbb{IN}$. An $\alpha$-orientation of $G$ is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all $v$. 
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- Reverting directed cycles preserves $\alpha$-orientations.
Definition. Given $G = (V, E)$ and $\alpha : V \to \mathbb{IN}$. An $\alpha$-orientation of $G$ is an orientation with $\text{outdeg}(v) = \alpha(v)$ for all $v$.

- Reverting directed cycles preserves $\alpha$-orientations.

Theorem. The set of $\alpha$-orientations of a planar graph $G$ has the structure of a distributive lattice.

- Diagram edge $\sim$ revert a directed essential/facial cycle.
Example 1: Spanning Trees

Spanning trees are in bijection with $\alpha_T$ orientations of a rooted primal-dual completion $\tilde{G}$ of $G$

- $\alpha_T(v) = 1$ for a non-root vertex $v$ and $\alpha_T(v_e) = 3$ for an edge-vertex $v_e$ and $\alpha_T(v_r) = 0$ and $\alpha_T(v_r^*) = 0$. 
Question. How does a change of roots affect the lattice?
Example 2: Matchings and $f$-Factors

Let $G$ be planar and bipartite with parts $(U, W)$. There is a bijection between $f$-factors of $G$ and $\alpha_f$ orientations:

- Define $\alpha_f$ such that $\text{indeg}(u) = f(u)$ for all $u \in U$ and $\text{outdeg}(w) = f(w)$ for all $w \in W$.

Example. A matching and the corresponding orientation.
Example 3: Eulerian Orientations

- Orientations with \( \text{outdeg}(v) = \text{indeg}(v) \) for all \( v \), i.e. \( \alpha(v) = \frac{d(v)}{2} \)
**Example 4: Schnyder Woods**

A plane triangulation with outer triangle $F = \{a_1, a_2, a_3\}$.

A coloring and orientation of the interior edges of $G$ with colors $1, 2, 3$ is a Schnyder wood of $G$ iff

- Inner vertex condition:

- Edges $\{v, a_i\}$ are oriented $v \rightarrow a_i$ in color $i$. 
Digression: Schnyder’s Theorem

The incidence order $P_G$ of a graph $G$

**Theorem [Schnyder 1989].**

A Graph $G$ is planar $\iff \dim(P_G) \leq 3$. 
Schnyder Woods and 3-Orientations

**Theorem.** Schnyder wood and 3-orientation are in bijection.

**Proof.**
- All edges incident to \( a_i \) are oriented \( \rightarrow a_i \).
  Prf: \( G \) has \( 3n - 9 \) interior edges and \( n - 3 \) interior vertices.
- Define the path of an edge:
- The path is simple (Euler), hence, ends at some \( a_i \).
Theorem. The set of Schnyder woods of a plane triangulation $G$ has the structure of a distributive lattice.
A Dual Construction: c-Orientations

- Reorientations of directed cuts preserve flow-difference (\#forward arcs $-$ \#backward arcs) along cycles.

Theorem [Propp 1993]. The set of all orientations of a graph with prescribed flow-difference for all cycles has the structure of a distributive lattice.

- Diagram edge $\sim$ push a vertex ($\neq v_\dagger$).
Theorem [Khuller, Naor and Klein 1993].
The set of all integral flows respecting capacity constraints \((\ell(e) \leq f(e) \leq u(e))\) of a planar graph has the structure of a distributive lattice.

\[ 0 \leq f(e) \leq 1 \]

- Diagram edge \(\sim\) add or subtract a unit of flow in ccw oriented facial cycle.
\( \Delta \)-Bonds

\[ G = (V, E) \] a connected graph with a prescribed orientation.

With \( x \in \mathbb{Z}^E \) and \( C \) cycle we define the circular flow difference

\[
\Delta_x(C) := \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).
\]

With \( \Delta \in \mathbb{Z}^C \) and \( \ell, u \in \mathbb{Z}^E \) let \( B_G(\Delta, \ell, u) \) be the set of \( x \in \mathbb{Z}^E \) such that \( \Delta_x = \Delta \) and \( \ell \leq x \leq u \).
The Lattice of \( \Delta \)-Bonds

**Theorem** [Felsner, Knauer 2007].
\( \mathcal{B}_G(\Delta, \ell, u) \) is a distributive lattice.
The cover relation is vertex pushing.
\(B_G(\Delta, l, u)\) is the set of \(x \in \mathbb{R}^E\) such that

- \(\Delta_x = \Delta\) (circular flow difference)
- \(l \leq x \leq u\) (capacity constraints).

**Special cases:**

- \(c\)-orientations are \(B_G(\Delta, 0, 1)\)
  \((\Delta(C) = |C^+| - c(C))\).
- Circular flows on planar \(G\) are \(B_{G^*}(0, l, u)\)
  \((G^* \text{ the dual of } G)\).
- \(\alpha\)-orientations.
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**ULD Lattices**

**Definition.** [Dilworth]
A lattice is an upper locally distributive lattice (ULD) if each element has a unique minimal representation as meet of meet-irreducibles, i.e., there is a unique mapping $x \rightarrow M_x$ such that

- $x = \bigwedge M_x$ (representation.) and
- $x \neq \bigwedge A$ for all $A \subseteq M_x$ (minimal).

$$0 = a \land e = \bigwedge \{a, b, c, d, e\}$$
**Proposition.**

A lattice it is ULD and LLD $\iff$ it is distributive.
A coloring of the edges of a digraph is a \textit{U-coloring} iff

- arcs leaving a vertex have different colors.
- completion property:

\begin{align*}
A \quad \rightarrow \quad \text{completion property:} \quad \rightarrow \quad \text{completion property:}
\end{align*}

\textbf{Theorem.}
A digraph $D$ is acyclic, has a unique source and admits a \textit{U-coloring} $\iff D$ is the diagram of an ULD lattice.

$\iff$ Unique 1.
Examples of U-colorings
Examples of U-colorings

- Chip firing game with a fixed starting position (the source), colors are the names of fired vertices.

- $\Delta$-bond lattices, colors are the names of pushed vertices. (Connected, unique 0).
More Examples

Some LLD lattices with respect to inclusion order:

- Subtrees of a tree (Boulaye ’67).
- Convex subsets of posets (Birkhoff and Bennett ’85).
- Convex subgraphs of acyclic digraphs (Pfaltz ’71).
  
  (C is convex if with x, y all directed (x, y)-paths are in C).
- Convex sets of an abstract convex geometry, this is an universal family of examples (Edelman ’80).
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Embedded Lattices

A $\mathcal{U}$-coloring of a distributive lattice $L$ yields a cover preserving embedding $\phi : L \to \mathbb{Z}^{\#\text{colors}}$. 
A $U$-coloring of a distributive lattice $L$ yields a cover preserving embedding $\phi : L \rightarrow \mathbb{Z}^{\#\text{colors}}$.

In the case of $\Delta$-bond lattices there is a polytope $P = \text{conv}(\phi(L))$ in $\mathbb{R}^{n-1}$ such that

$$\phi(L) = P \cap \mathbb{Z}^{n-1}$$

- This is a special property:
**Definition.** A polytope $P$ is a $D$-polytope if with $x, y \in P$ also $\max(x, y), \min(x, y) \in P$.

- A $D$-polytope is a (infinite!) distributive lattice.
- Every subset of a $D$-polytope generates a distributive lattice in $P$. E.g. Integral points in a $D$-polytope are a distributive lattice.
D-Polytopes

Remark. Distributivity is preserved under

• scaling
• translation
• intersection

Theorem. A polytope $P$ is a $D$-polytope iff every facet inducing hyperplane of $P$ is a $D$-hyperplane, i.e., closed under $\max$ and $\min$. 
**Theorem.** An hyperplane is a $D$-hyperplane iff it has a normal $e_i - \lambda_{ij} e_j$ with $\lambda_{ij} \geq 0$.

$(\Leftarrow) \lambda_{ij} e_i + e_j$ together with $e_k$ with $k \neq i, j$ is a basis. The coefficient of $\max(x, y)$ is the $\max$ of the coefficients of $x$ and $y$.

$(\Rightarrow)$ Let $n = \sum_i a_i e_i$ be the normal vector. If $a_i > 0$ and $a_j > 0$, then $x = a_j e_i - a_i e_j$ and $y = -x$ are in $n^\perp$ but $\max(x, y)$ is not.
Consider $\ell, u \in \mathbb{Z}^m$ and a $\Lambda$-weighted network matrix $N_\Lambda$ of a connected graph. (Rows of $N_\Lambda$ are of type $e_i - \lambda_{ij}e_j$ with $\lambda_{ij} \geq 0$.)

- **[Strong case, $\text{rank}(N_\Lambda) = n]$**
  The set of $p \in \mathbb{Z}^n$ with $\ell \leq N_\Lambda^\top p \leq u$ is a distributive lattice.

- **[Weak case, $\text{rank}(N_\Lambda) = n - 1]$**
  The set of $p \in \mathbb{Z}^{n-1}$ with $\ell \leq N_\Lambda^\top(0, p) \leq u$ is a distributive lattice.
A Second Graph Model for D-Polytopes

(Rows of \( N_\Lambda \) are of type \( e_i - \lambda_{ij}e_j \) with \( \lambda_{ij} \geq 0 \).)

**Theorem [Felsner, Knauer 2008].**
Let \( Z = \ker(N_\Lambda) \) be the space of \( \Lambda \)-circulations. The set of \( x \in \mathbb{Z}^m \) with

- \( \ell \leq x \leq u \) (capacity constraints)
- \( \langle x, z \rangle = 0 \) for all \( z \in Z \)
  (weighted circular flow difference).

is a distributive lattice \( D_G(\Lambda, \ell, u) \).

- Lattices of \( \Delta \)-bonds are covered by the case \( \lambda_{ij} = 1 \).
The Strong Case

For a cycle $C$ let

$$\gamma(C) := \prod_{e \in C^+} \lambda_e \prod_{e \in C^-} \lambda_e^{-1}.$$ 

A cycle with $\gamma(C) \neq 1$ is strong.

**Proposition.** $\text{rank}(N_{\Lambda}) = n$ iff it contains a strong cycle.

**Remark.** 
$C$ strong $\implies$ there is no circulation with support $C$. 
A fundamental basis for the space of $\Lambda$-circulations:

- Fix a 1-tree $T$, i.e, a unicyclic set of $n$ edges. With $e \not\in T$ there is a circulation in $T + e$
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Further topic: D-polytopes and optimization.
Conclusion

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- U-colorings yield pretty proves for UL-distributivity and distributivity.

- D-polytopes are related to generalized network matrices.

Finally: Don’t forget Schnyder’s Theorem.
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The End