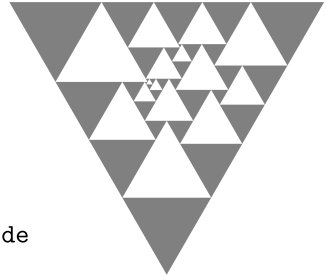
Contact Representations of Planar Graphs

Graduiertenkolleg MDS
TU Berlin
April 20., 2015



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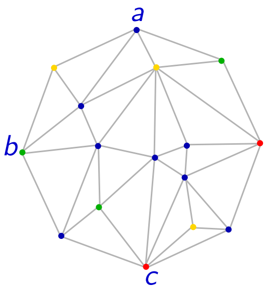
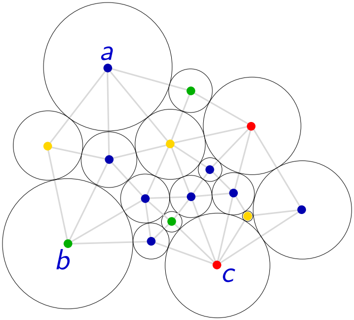
Overview

A Survey of Results and Problems

Straight Line Triangle Representations

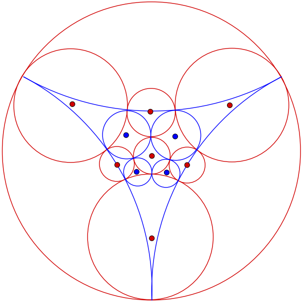
Discs

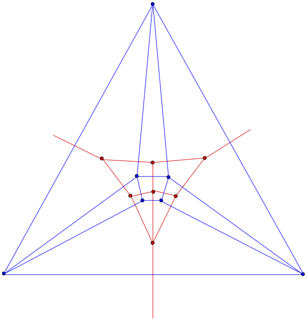


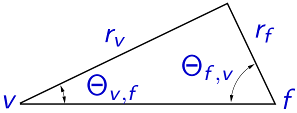


Theorem [Koebe 1935].
Planar graphs have contact representations with discs.

Discs (Primal-Dual)

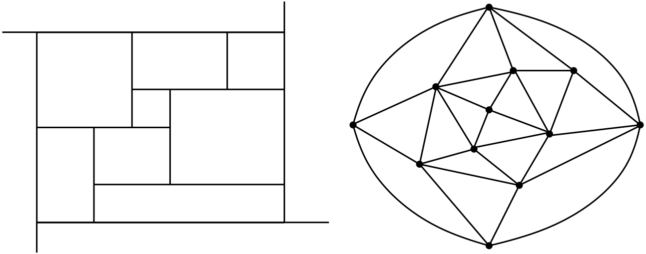






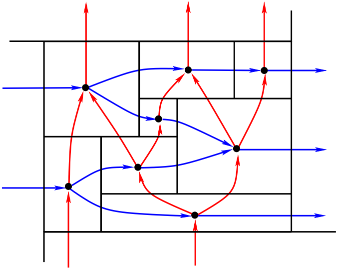
$$\sum_{f:v|f} \Theta_{v,f} = \pi \quad \text{for all } v.$$

Rectangles



Theorem [He 93]. 4-connected inner triangulations of a quadrangle have contact representations with rectangles.

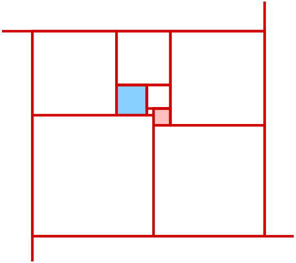
Rectangles

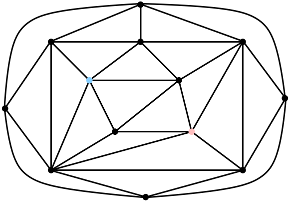


Theorem [He 93]. 4-connected inner triangulations of a quadrangle have contact representations with rectangles.

- transversal structure – laminar paths decomposition.

Squares

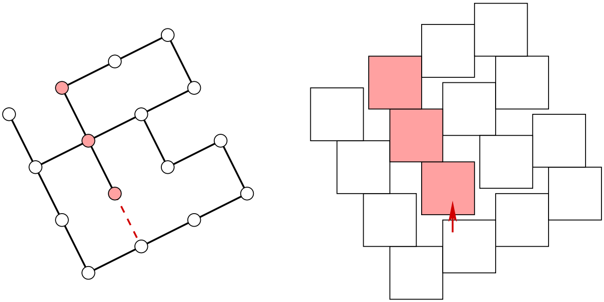




Theorem [Schramm 93].
5-connected inner triangulations of a quadrangle have contact representations with squares. The representation is unique.

- extremal length (Schramm) – blocking polyhedra (Lovász).

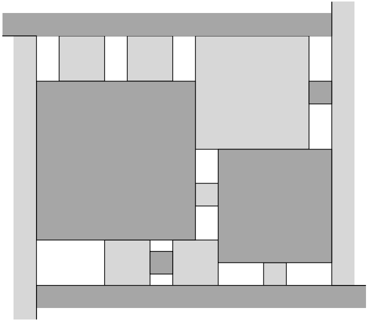
Unit Squares

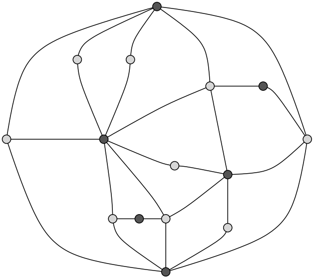


Theorem [Rahman 14]. Subgraphs of the square grid have contact representations with unit squares

- NP-complete to recognize the class USqCont (Kleist and Rahman).

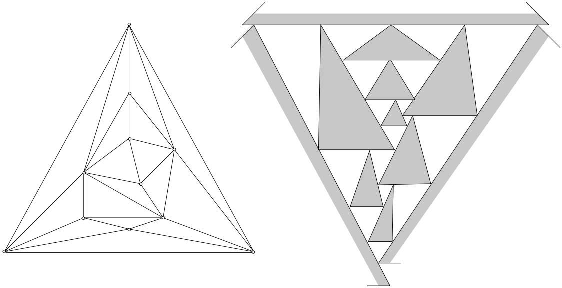
A Problem for Squares





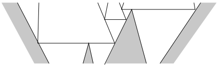
Conjecture. Every bipartite planar graph has a contact representations with squares.

Triangles

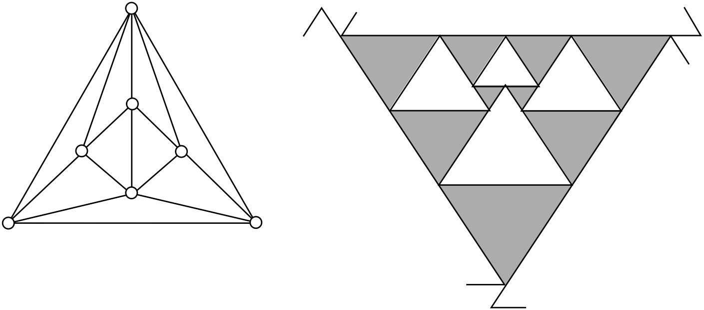


Theorem [de Fraysseix, Ossona de Mendez and Rosenstiehl 93 **].**
Triangulations have contact representations with triangles.

Construct along a good ordering of vertices
$T_1 + T_2^{-1} + T_3^{-1}$



Homothetic Triangles



Theorem [Gonçalves, Lévêque and Pinlou 10 **].**
4-connected triangulations have a contact representation with homothetic triangles.

Schramm's "Monster Packing Theorem" (1990)

G-L-P observe that the result follows from a corollary of Schramm's "Monster Packing Theorem".

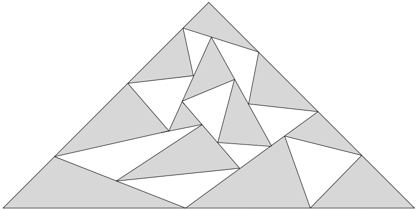
Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v. Then there is a contact representation of (a supergraph) of T with homothetic copies of the sets Q_v.

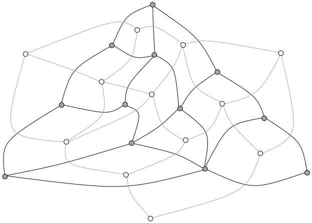
Remark. In general homothetic copies of the Q_v can degenerate to a point and thus induce additional edges.

Gonçalves et al. show that this is impossible if T is 4-connected.

It is also impossible if the Q_v have smooth boundary.

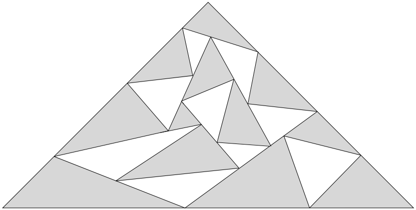
Triangles (Primal-Dual)

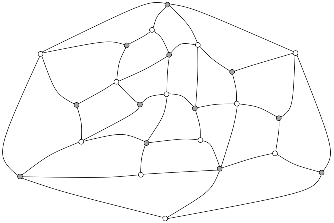




Theorem [Gonçalves, Lévêque and Pinlou 10 **].**
3-connected planar graphs have a primal-dual contact representation with triangles.

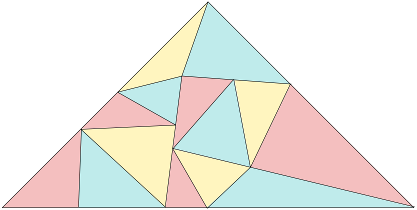
Triangles (Primal-Dual)

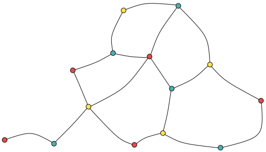




Theorem [Gonçalves, Lévêque and Pinlou 10 **].**
Angle graphs of 3-connected planar graphs have a touching triangle representation.

A Problem for Triangles

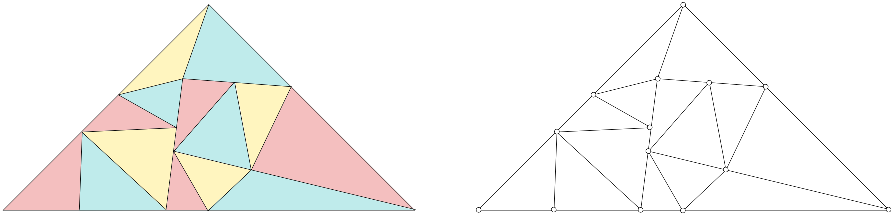




Problem. Which planar graphs have a touching triangle representation?

- We understand the quadrangulations in this class.

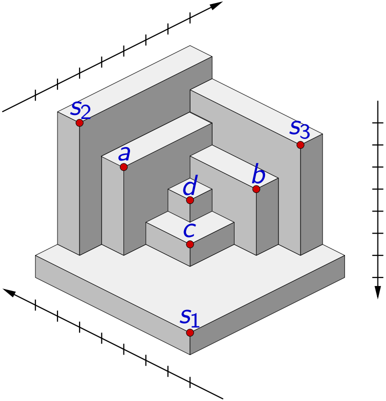
A 2nd Problem for Triangles

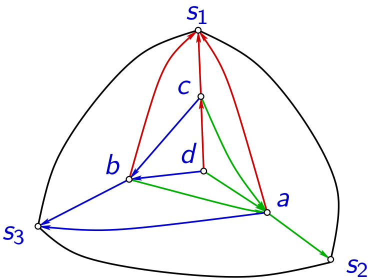


Problem. Which planar graphs have a straight line triangle representation (SLTR)?

- A characterization will be the topic in part **II**.

Axis-aligned Boxes in 3D

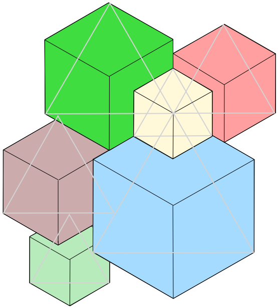




Theorem [Thomassen 86]. Planar graphs have (proper) contact representations with axis-aligned boxes in 3D.

- New proofs via Schnyder woods.

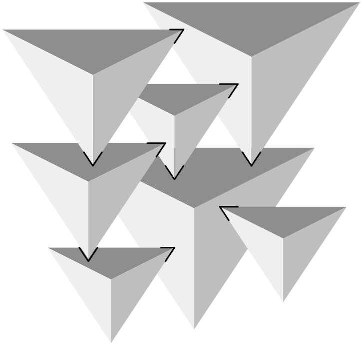
Axis-aligned Cubes in 3D



Theorem [Felsner and Francis 11]. Planar graphs have a contact representation with axis-aligned cubes in 3D.

- Based on homothetic triangles - contacts may degenerate.

A Problem for Tetrahedra

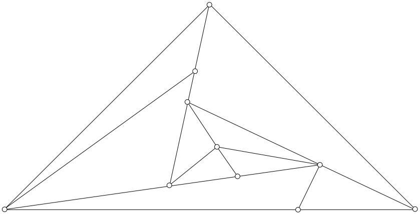


Problem. Which non-planar graphs have contact representations with contacts of homothetic tetrahedra.

Intermezzo

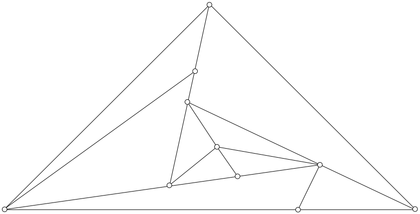
The SLTR Problem

Problem. Which planar graphs have a straight line triangle representation (SLTR)?



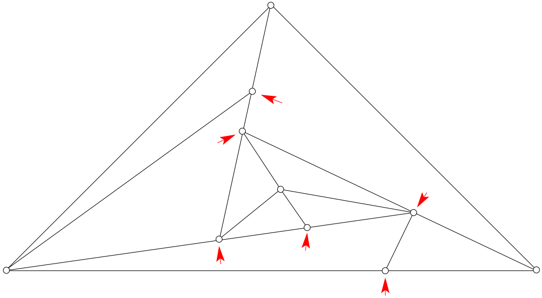
The SLTR Problem

Problem. Which planar graphs have a straight line triangle representation (SLTR)?



- Vertices of degree 2 cn be eliminated.
- Necessary: internally 3-connected.

Flat Angles



An SLTR induces a flat angle assignment (FAA).

(C_v) Each non-suspension vertex is assigned to at most one face.

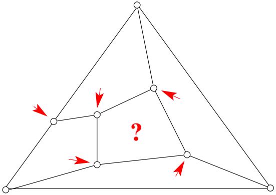
(C_f) Each face has $|f| - 3$ assigned vertices.

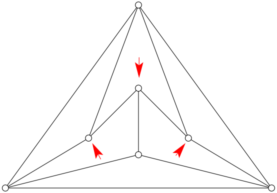
FAA Examples

(C_v) Each non-suspension vertex is assigned to at most one face.

(C_f) Each face has $|f| - 3$ assigned vertices.

Two negative examples:





Convex Corners

Observation. Each cycle of a SLTR has at least three convex corners.

Definition. Combinatorially convex corners of a cycle γ:

(K_1) Suspension vertices, or

(K_2) v not assigned has edge in outer side of γ, or

(K_3) v assigned to some outer face has edge in outer side of γ.

Combinatorially Convex Corners

Proposition. Geometrically convex corners of an SLTR are combinatorially convex of the associated FAA.

Additional condition:

(C_o) Each cycle has at least three combinatorially convex corners.

Definition. An FAA satisfying C_o is a good FAA (GFAA).

The Main Result

Theorem. A GFAA induces a SLTR.

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Remark. The drawback of this characterization is that we have no efficient way of deciding whether a graph has a FAA obeying condition C_o. — More on this later.

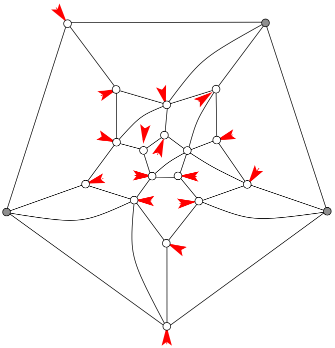
The Main Result

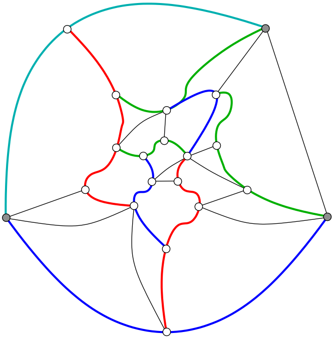
Theorem. A GFAA induces a SLTR.

Outline of the proof

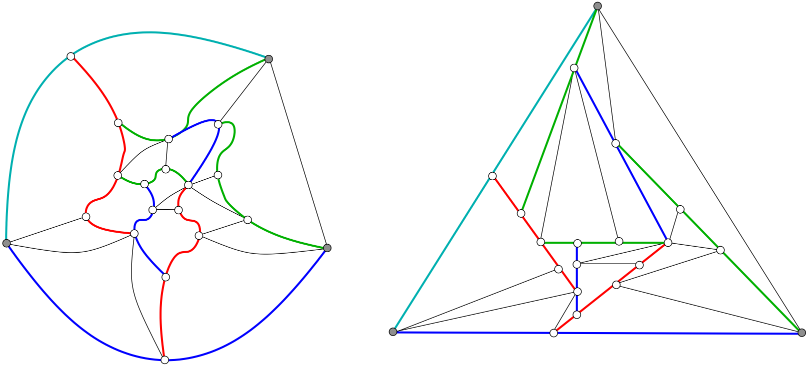
- Contact systems of pseudosegments (CSP).
- A system of linear equations for the stretchability of a CSP.
  - Discrete harmonic functions.
- Realizing the solution as a SLTR.

From FAA to CSP





The Stretching



The Stretching

Equations for the stretching:

- Fix coordinates for the outer triangle (suspension vertices).
- If v is assigned to the face between edges uv and vw choose $\lambda_v \in (0,1)$ and let:

$$x_v = \lambda_v x_u + (1-\lambda_v)x_w \quad \text{and} \quad y_v = \lambda_v y_u + (1-\lambda_v)y_w.$$

- If v is not assigned choose parameters $\lambda_{vu} > 0$ with $\sum_{u \in N(v)} \lambda_{vu} = 1$ and let:

$$x_v = \sum_{u \in N(v)} \lambda_{vu} x_u, \quad \text{and} \quad y_v = \sum_{u \in N(v)} \lambda_{vu} y_u.$$

Digression: Harmonic Functions

$G = (V, E)$ a strongly connected directed graph.

$\lambda : E \to \mathbf{R}_+$ weights such that $\sum_v \lambda_{uv} = 1$ for all $u \in V$.

A function $f : V \to \mathbf{R}$ is harmonic at u iff

$$f(u) = \sum_{v \in N^+(u)} \lambda_{uv} f(v).$$

A vertex where a function f is not harmonic is a pole of f.

Lemma. Every non-constant function has at least two poles.

- A pole where f attains its maximum resp. minimum value.

Harmonic Functions

Let $S \subseteq V$ with $|S| \geq 2$ and let $G = (V, E)$ be a directed graph such that each v has a directed path to some $s \in S$.

Proposition. For $\emptyset \neq S \subseteq V$ and $f_0 : S \to \mathbb{R}$, there is a unique function $f : V \to \mathbb{R}$ extending f_0 that is harmonic on $V \setminus S$.

- (Uniqueness) Assume $f \neq g$ are extensions, then $f - g$ is a non-zero extension of the 0-function on S – contradiction.
- (Existence)
  - The system has $|V|$ variables and $|V|$ linear equations.
  - Homogeneous system only has the trivial solution.
  - There is a solution for any right hand side f_0.

Applications of Harmonic Functions

- Random walks: Let $f_a(v)$ be the probability that a random walk hits $a \in S$ before it hits any other element of S. This function is harmonic in $v \notin S$.
- Electrical networks: Consider electrical flow in a network with a fixed potential $f_0(v)$ at vertices $v \in S$. The potential $f(v)$ is harmonic in $v \notin S$.
- Rubber band drawings: Fix the positions of each node $v \in S$ at a given point $f_0(v)$ of the real line, and let the remaining nodes find their equilibrium. The equilibrium position $f(v)$ is harmonic in $v \notin S$.

The Stretching

Equations for the stretching:

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A harmonic system $\implies$ unique solution.

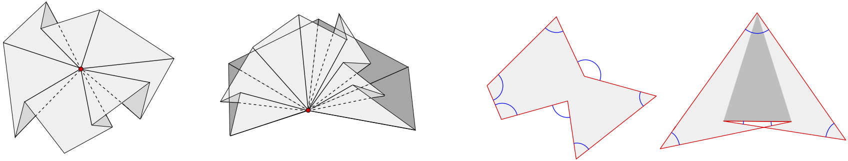
The SLTR

1. Pseudosegments become segments.
2. Convex outer face.
3. No concave angles.
4. No degenerate vertex.
   Degenerate: v together with 3 neighbors on a line.
   Use C_o and planarity.
5. Preservation of rotation systems.
   Next slide.
6. No crossings.
7. No degeneracy.
   No edges of length 0.
   Otherwise: degenerate vertex or crossing.

Preservation of Rotations

Let G have $b \geq 3$ boundary vertices.
Considering the smaller angle spanned by a pair of edges:

$$\sum_v \theta(v) \geq (|V| - b)2\pi + (b-2)\pi$$



$$\sum_f \theta(f) \leq \sum_f (|f| - 2)\pi = ((2|E| - b) - 2(|F| - 1))\,\pi$$

$\sum_v \theta(v) = \sum_f \theta(f)$ and the Euler-Formula imply equality.

Consequences & Applications

- Can efficiently check whether a FAA is good.
- Reprove: 3-connected planar graphs have a primal-dual contact representation with triangles.
- Can adapt C_f to have faces repr. by k-gons.
- Reprove a theorem about stretchability of contact systems of pseudosegments.

Contact Systems of Pseudosegments

Definition. A contact system of pseudosegments is stretchable if it is homeomorphic to a contact system of straight line segments.

Theorem [De Fraysseix & Ossona de Mendez 2005 **].**
A contact system Σ of pseudosegments is stretchable if and only if each subset $S \subseteq \Sigma$ of pseudosegments with $|S| \geq 2$, has at least 3 extremal points.

Contact Systems of Pseudosegments

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Definition. p is extremal for S if

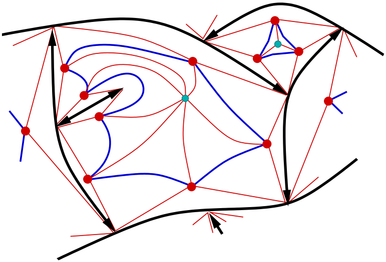
(E1) p is an endpoint of a pseudosegment in S, and

(E2) p is not interior to a pseudosegment in S, and

(E3) p is incident to the unbounded region of S.

Contact Systems of Pseudosegments

Extending a contact system Σ of pseudosegments to a graph G_Σ.



Proposition. If each subset $S \subseteq \Sigma$ of pseudosegments with $|S| \geq 2$, has at least 3 extremal points,

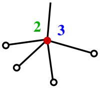
$\implies$ the intended FAA of G_Σ is good.

Intermezzo

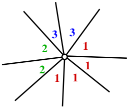
Schnyder Angle Labelings

Axioms for the 3-coloring of angles of a suspended 3-connected graph:

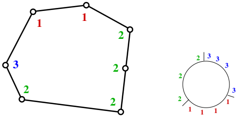
(A1) Angles at the half-edges:



(A2) Rule of vertices:



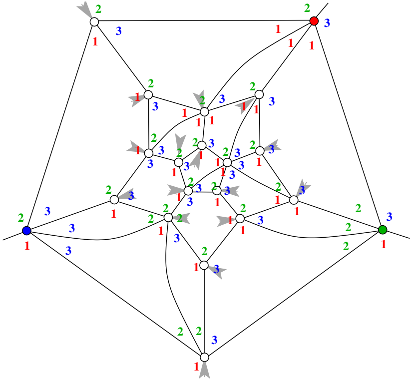
(A3) Rule of faces:



Corner Compatibility

Definition. A Schnyder labeling σ and an FAA ψ with the same suspensions are a corner compatible pair if

- Every face has corners in ψ that are labeled 1, 2, and 3 in σ.

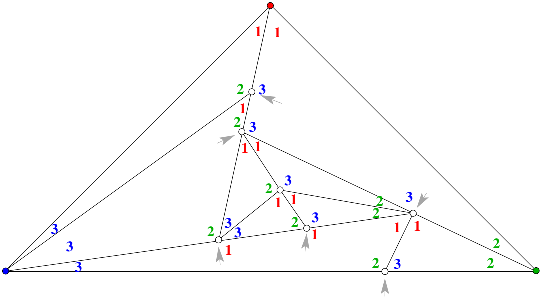


Corner Compatibility

Theorem. G a suspended, internally 3-connected graph. G has an SLTR if and only if it has a corner compatible pair.

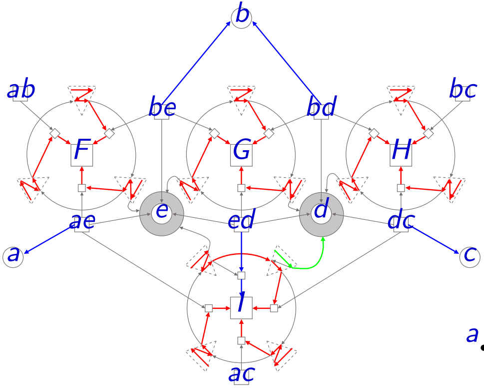
($\Leftarrow$) Use the convex drawing induced by the Schnyder labeling to show that the FAA is good.

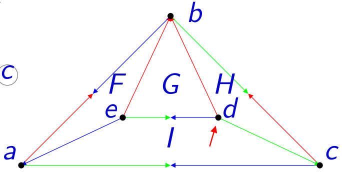
($\Rightarrow$) Inductive construction of a Schnyder labeling from an SLTR (15 pages).



Constructing Corner Compatible Pairs

- Schnyder labelings and flat angle assignments (FAA) can be modeled via flow.
- The compatibility condition can be added in a two-commodity problem.

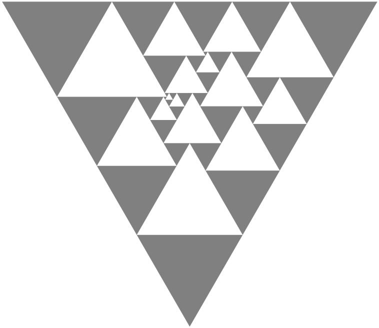




Logic and GFAA

Theorem [Yi-Jun Chang and Hsu-Chun Yen 2015].
The existence of a GFAA can be encoded by a Monadic Second Order Formula.

This implies (via Courcelle's Theorem) that the question can be answered in polynomial time if the corresponding auxiliary graph has bounded treewidth.



Thank You