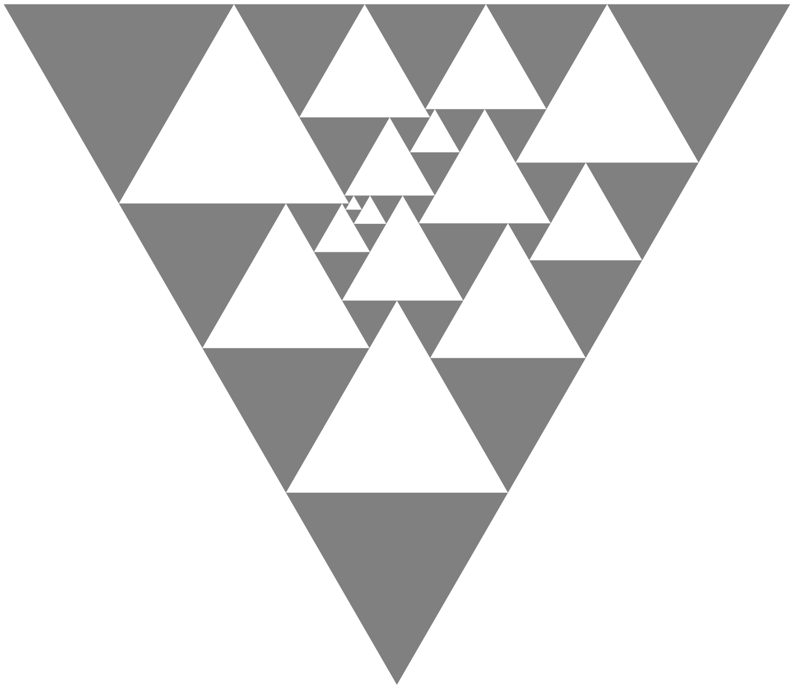
Crash Course:
Schnyder Woods and Applications

Dagstuhl Seminar 10461
– Schematization –
November 15. 2010



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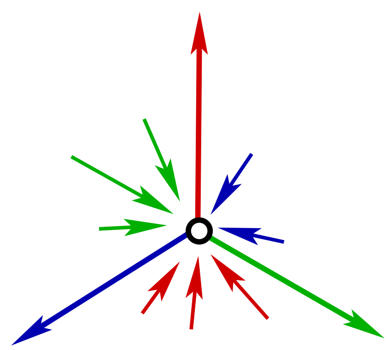
Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors $1, 2, 3$ is a Schnyder wood of G iff

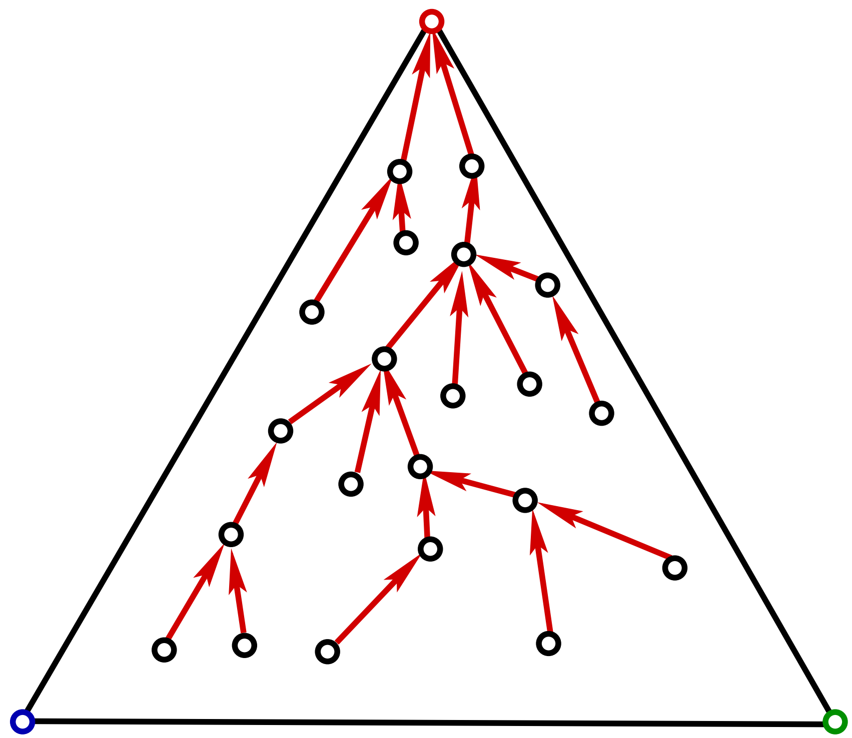
- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Woods - Trees

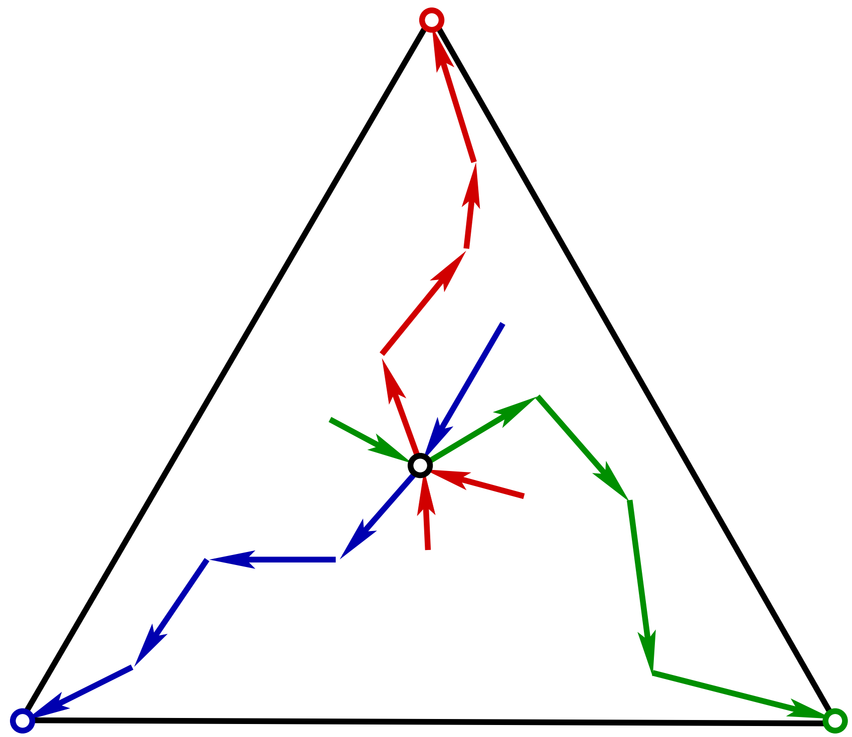
- The set T_i of edges colored i is a tree rooted at a_i.



Proof. Count edges in a cycle — Euler ↯

Schnyder Woods - Paths

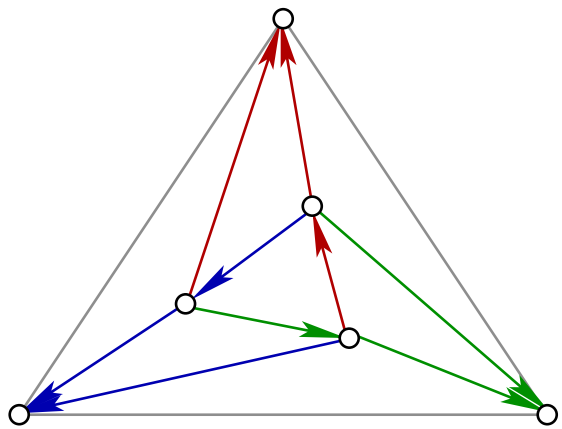
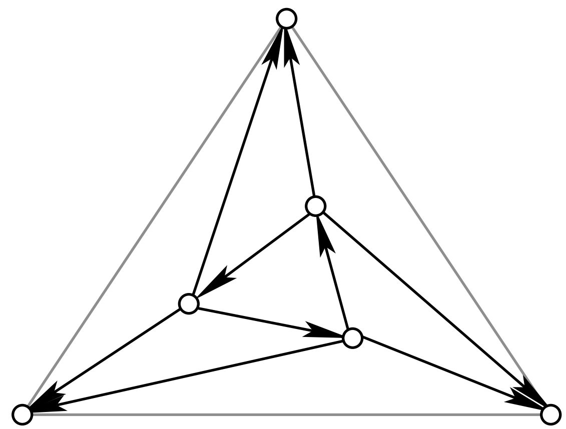
- Paths of different color have at most one vertex in common.



3-orientations

Definition. A 3-orientation of a planar triangulation with a triangle a_1, a_2, a_3 is an orientation of edges such that every vertex v ($v \neq a_i$, $i = 1, 2, 3$) has out-degree 3.

- A Schnyder wood induces a 3-orientation.

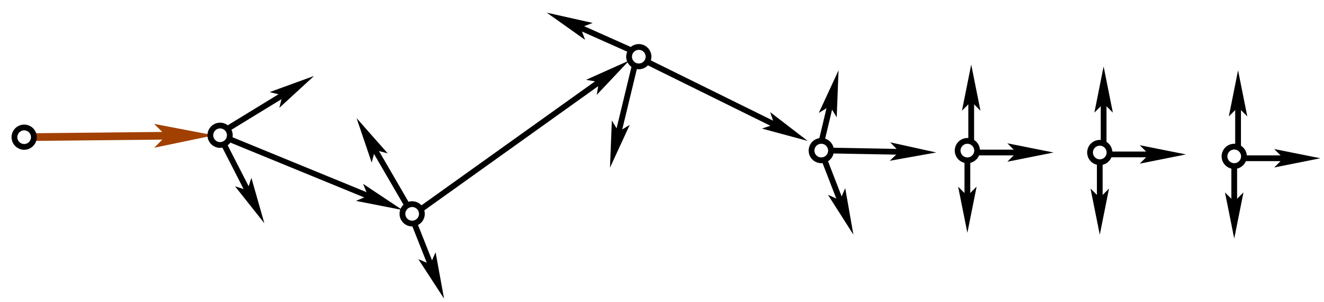



3-orientations

Theorem. Up to a permutations of colors
a 3-orientation induces a unique Schnyder wood.

Proof.

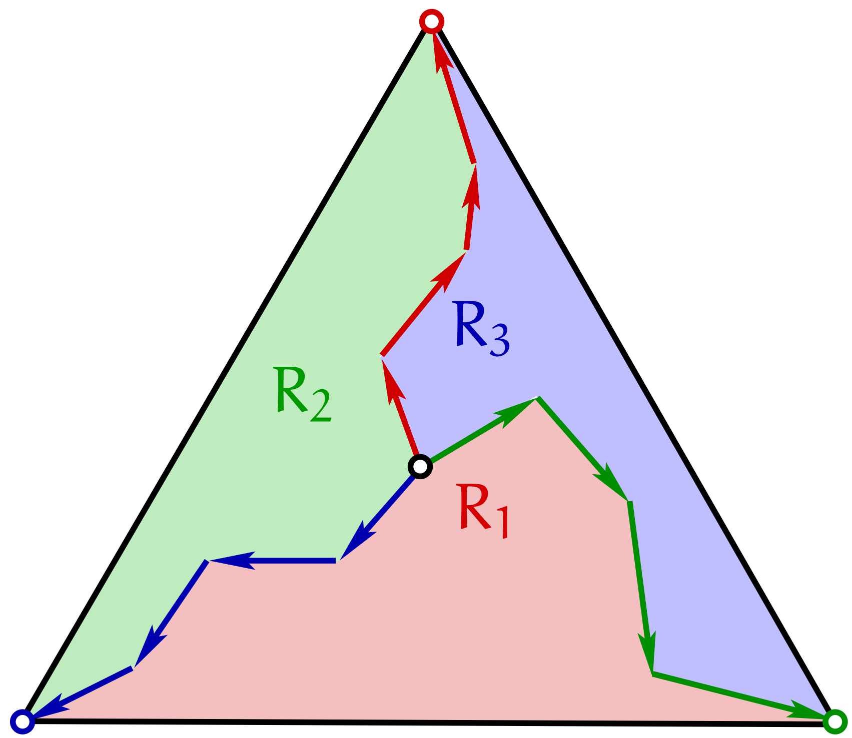
- Claim: All edges incident to a_i are oriented $\rightarrow a_i$.
  G has $3n-9$ interior edges and $n-3$ interior vertices.
- Define the path of an edge:



- The path is simple (Euler), hence, ends at some a_i.

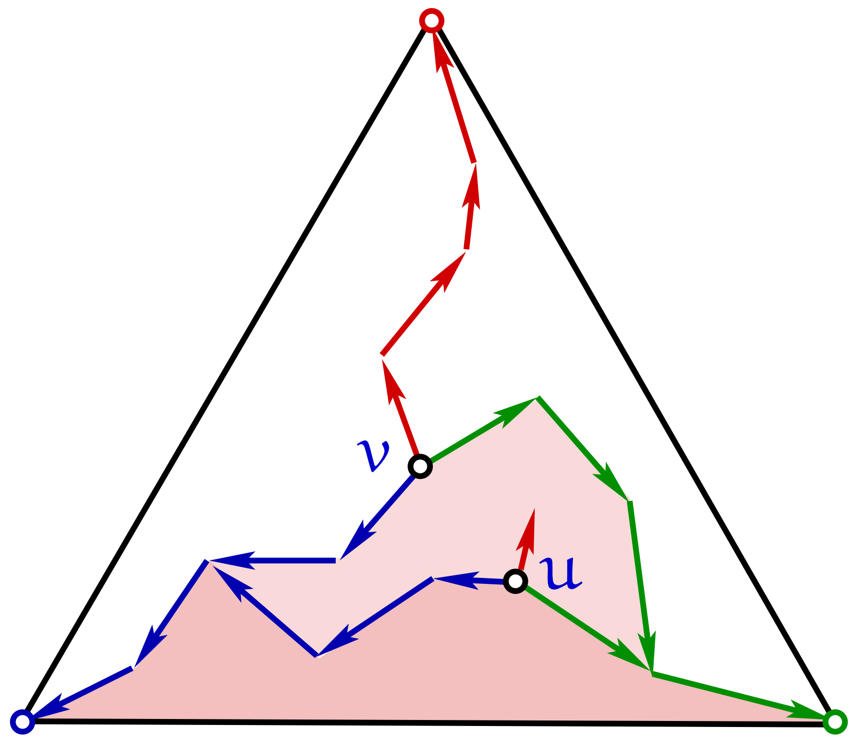
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



Schnyder Woods - Regions

- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.

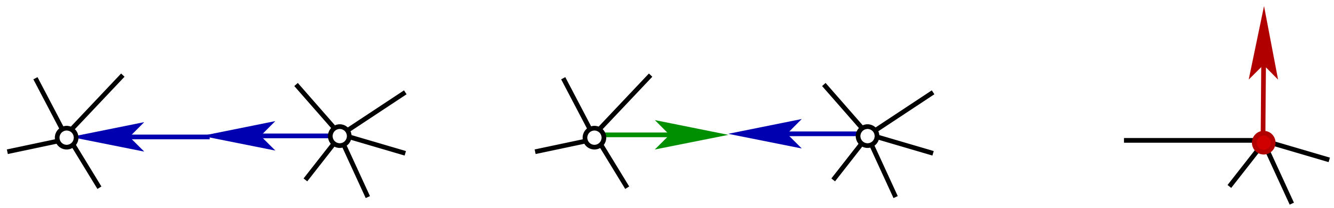


Schnyder Woods – Generalized

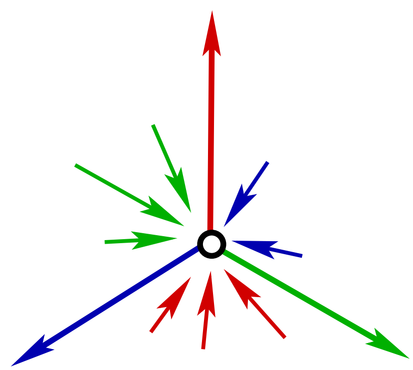
G a 3-connected planar graph with special vertices a_1, a_2, a_3 on the outer face.

Axioms for 3-coloring and orientation of edges:

(W1 - W2) Rule of edges and half-edges:



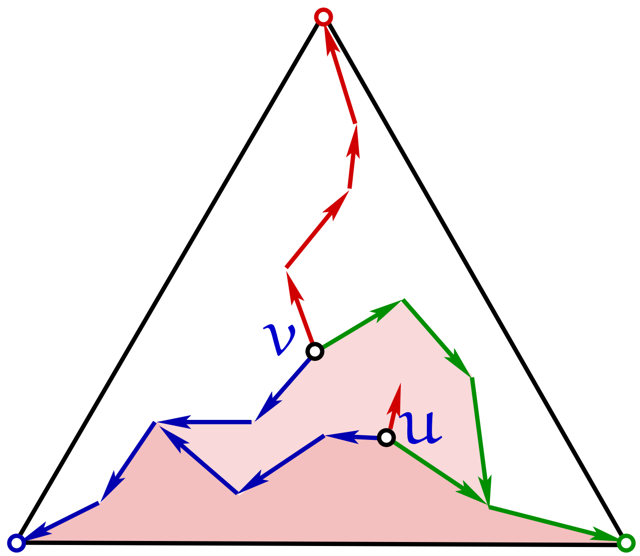
(W3) Rule of vertices:

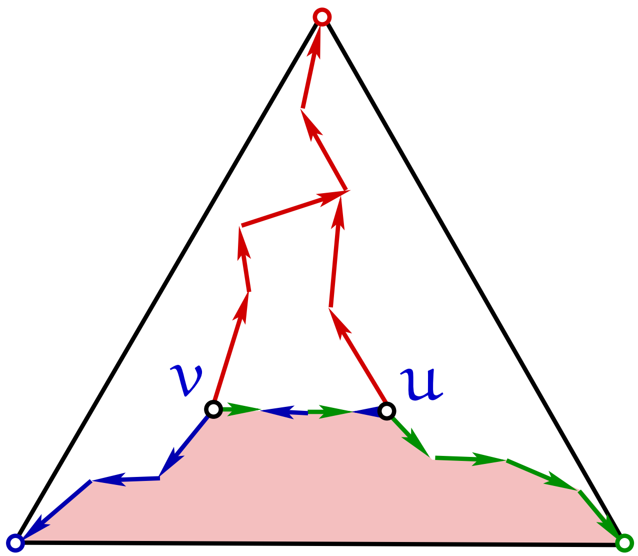


(W4) No face boundary is a directed cycle in one color.

Schnyder Woods - Regions

- If $u \in R_i^o(v)$ then $R_i(u) \subset R_i(v)$.
- If $u \in \partial R_i(v)$ then $R_i(u) \subseteq R_i(v)$
  (equality, iff there is a bi-directed path between u and v.)

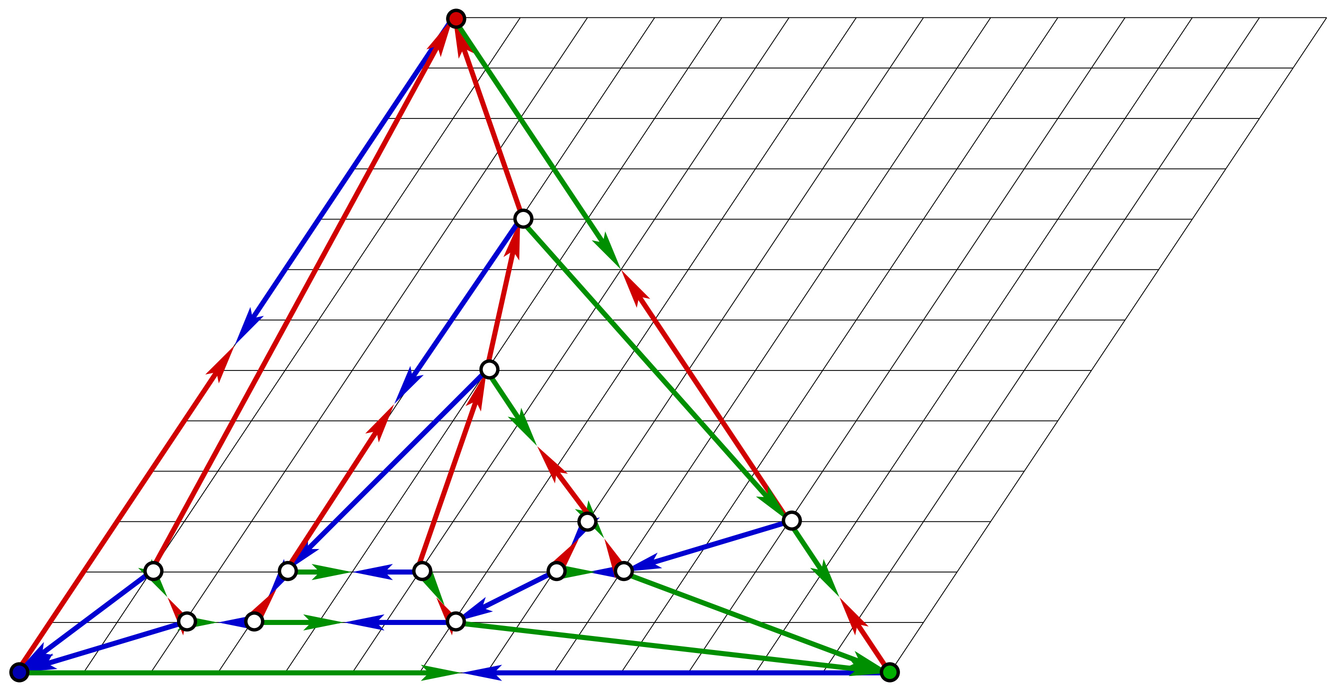




Drawings by Counting Faces

$\phi_i(v) = \#$ faces in $R_i(v)$.

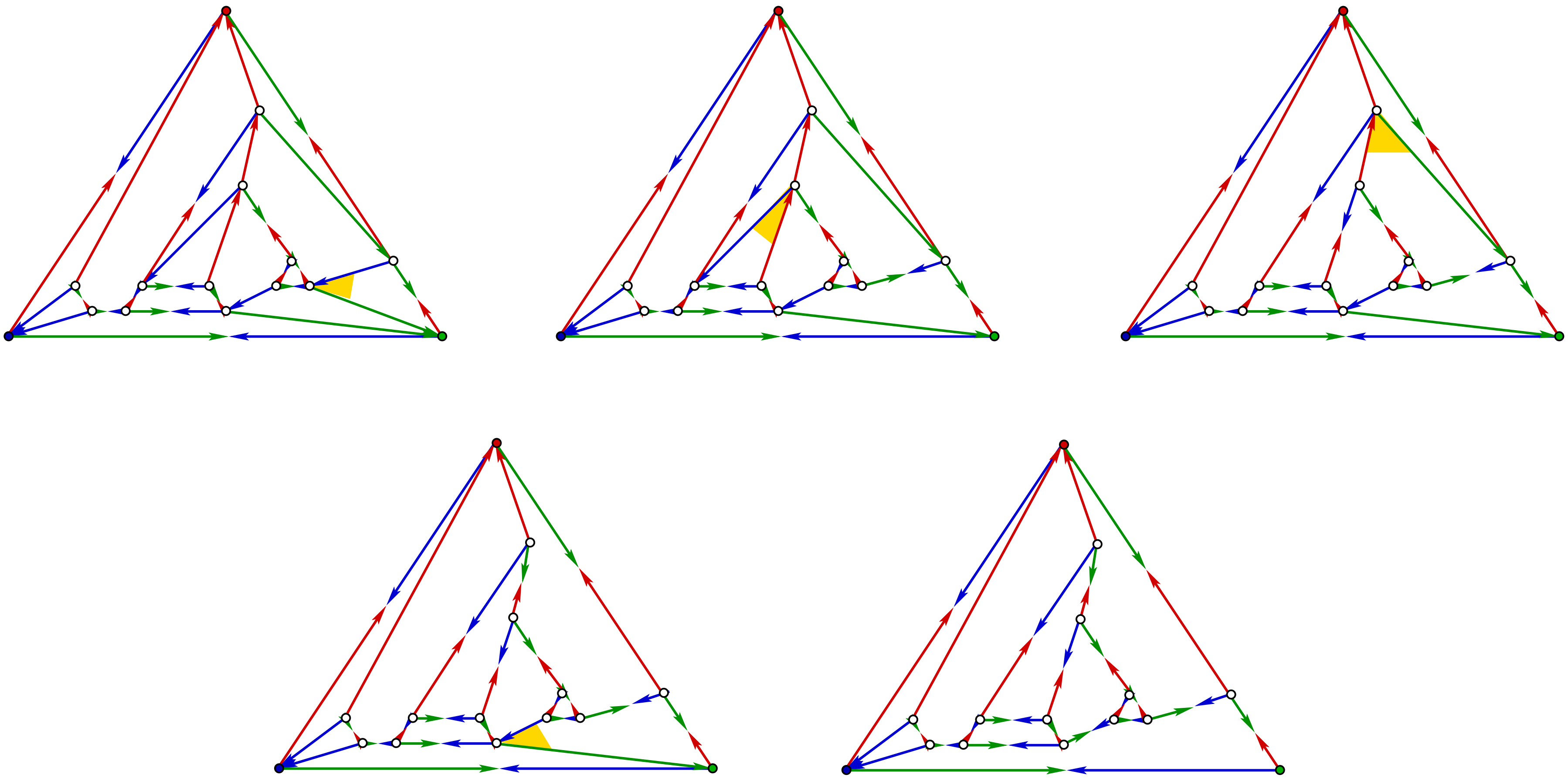
Embed v at $(\phi_1(v), \phi_2(v))$



Theorem. 3-connected planar graphs admit convex drawings on the $(f-1) \times (f-1)$ grid.

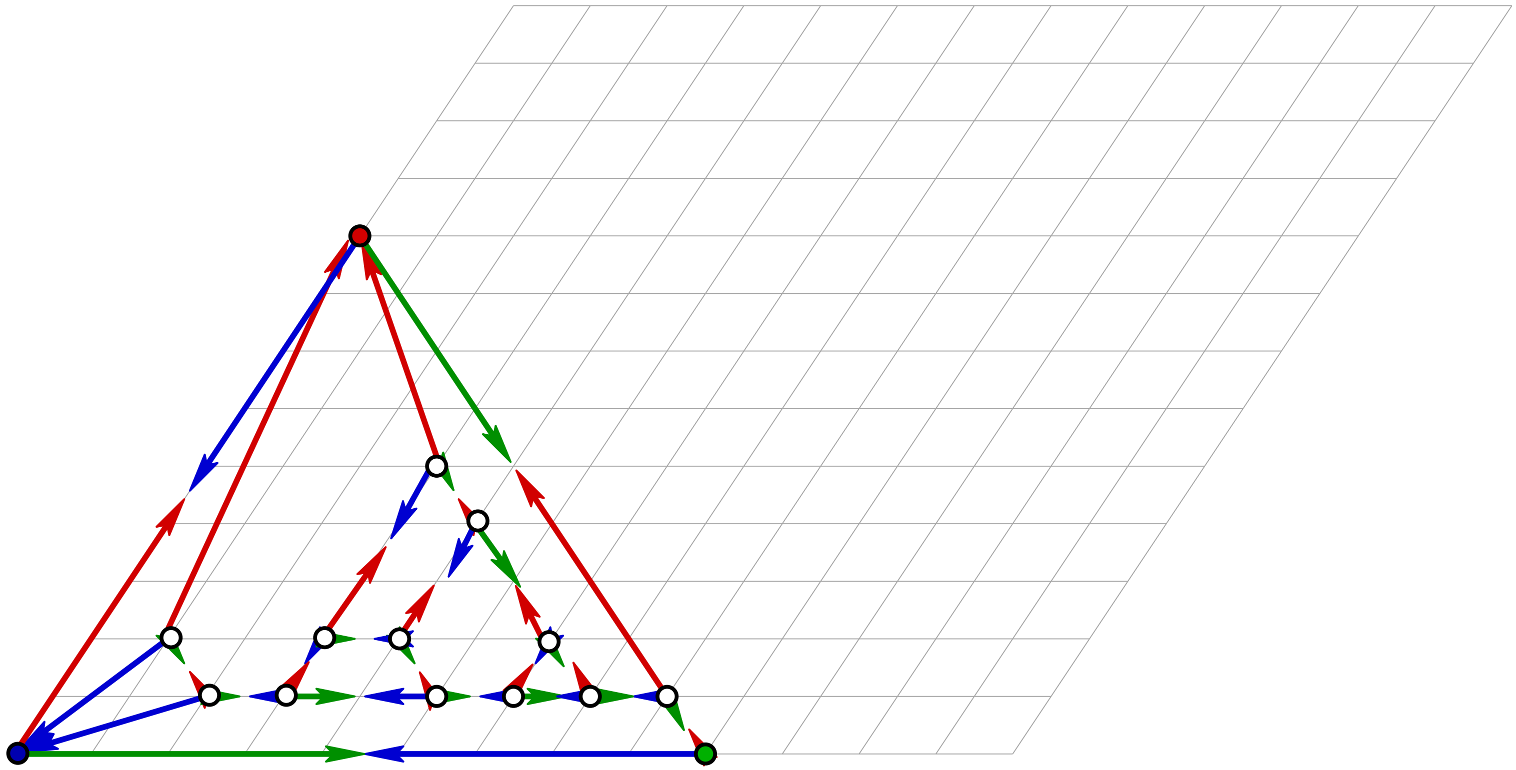
More Compact Drawings – Step I: Reduction

Reduce the face count by merging edges.



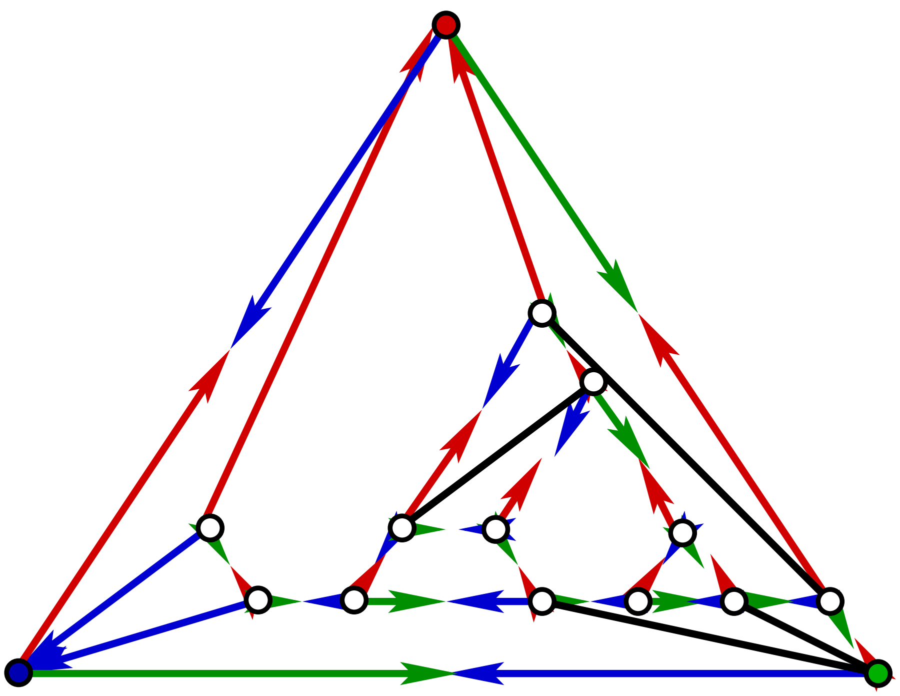
Step II: Drawing

Draw the reduced graph by counting faces on the $(f^{\downarrow} - 1) \times (f^{\downarrow} - 1)$ grid.



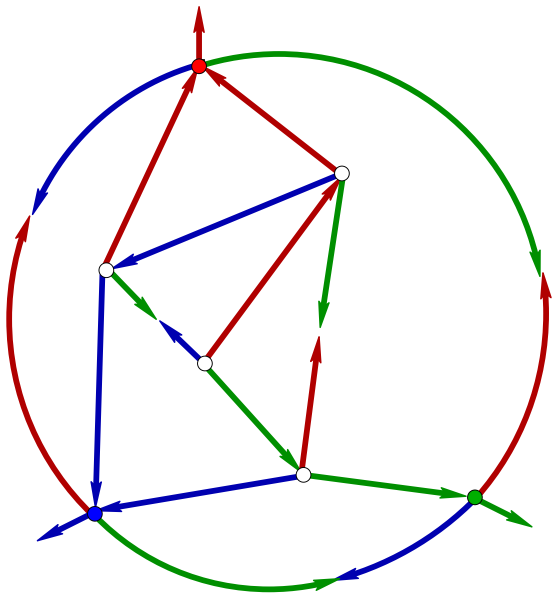
Step III: Drawing More

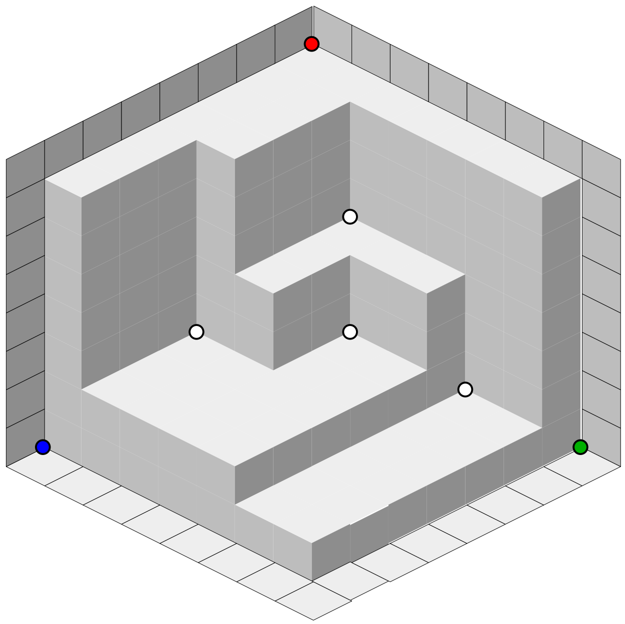
Reinsert the ‘merge edges’.



Counting Faces in Schnyder Regions II

Embed v at $(\phi_1(v), \phi_2(v), \phi_3(v))$

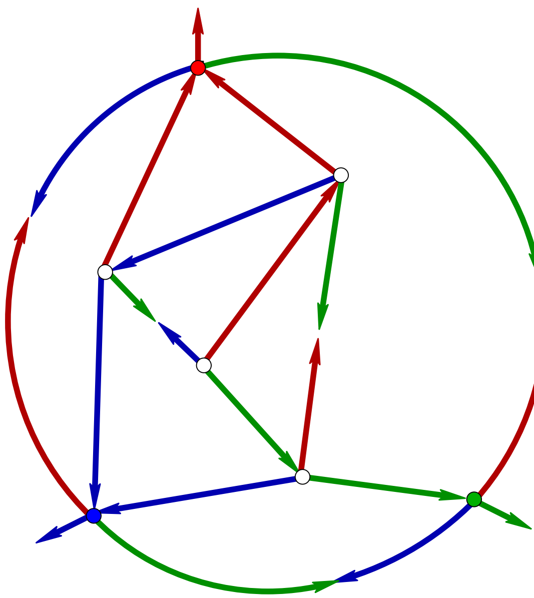


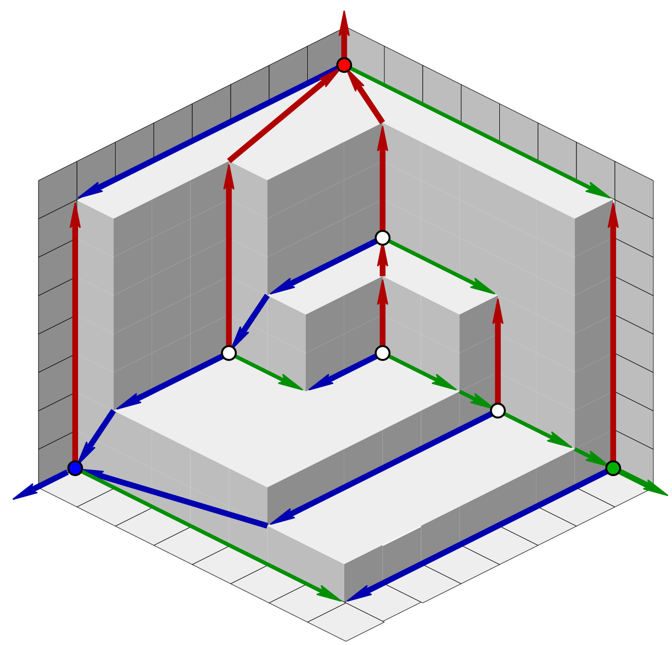


The vertices generate an orthogonal surface.

Counting Faces in Schnyder Regions II

Embed v at $(\phi_1(v), \phi_2(v), \phi_3(v))$

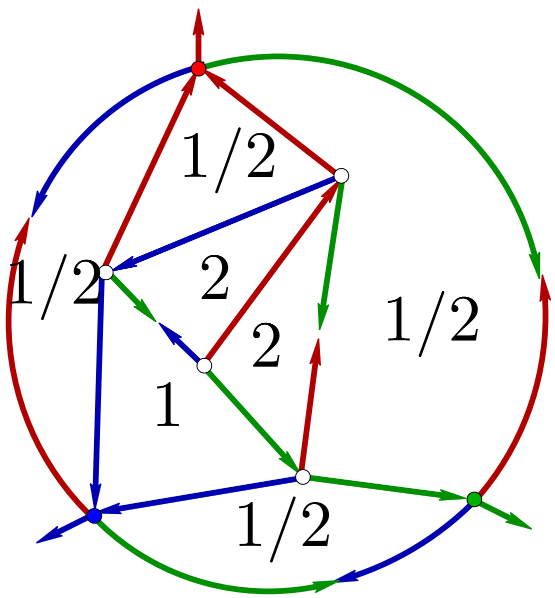


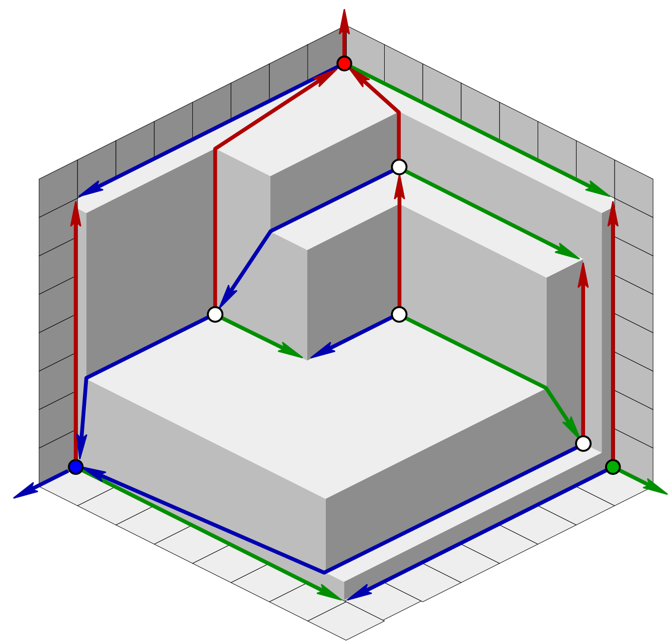


The orthogonal surface supports the Schnyder wood.

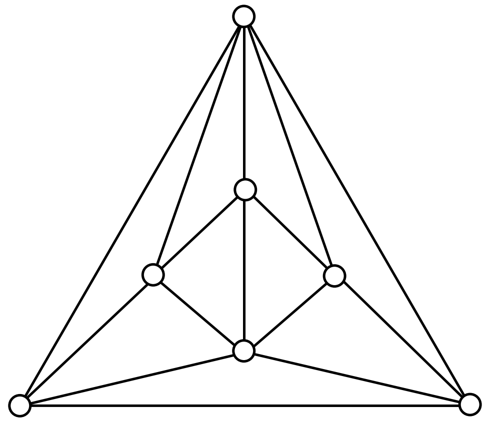
Weighted Count

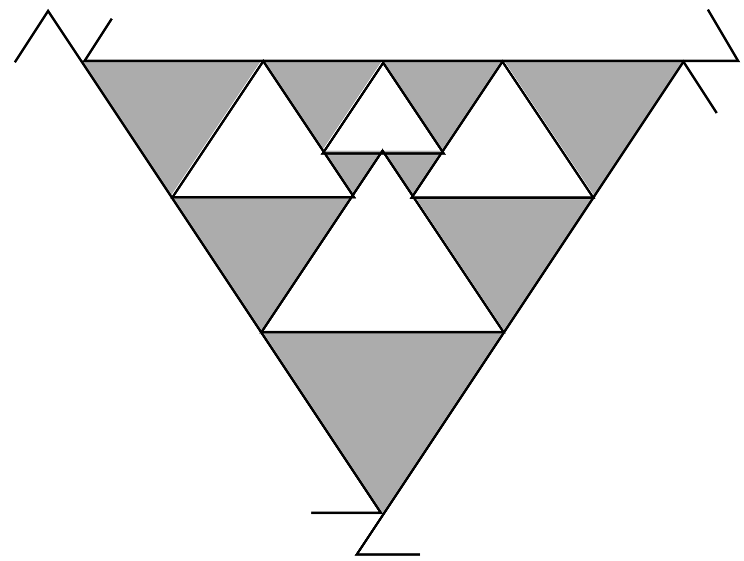
Theorem. Every coplanar orhogonal surface supporting a Schnyder wood S can be obtained from weighted regions.





Triangles and Graphs

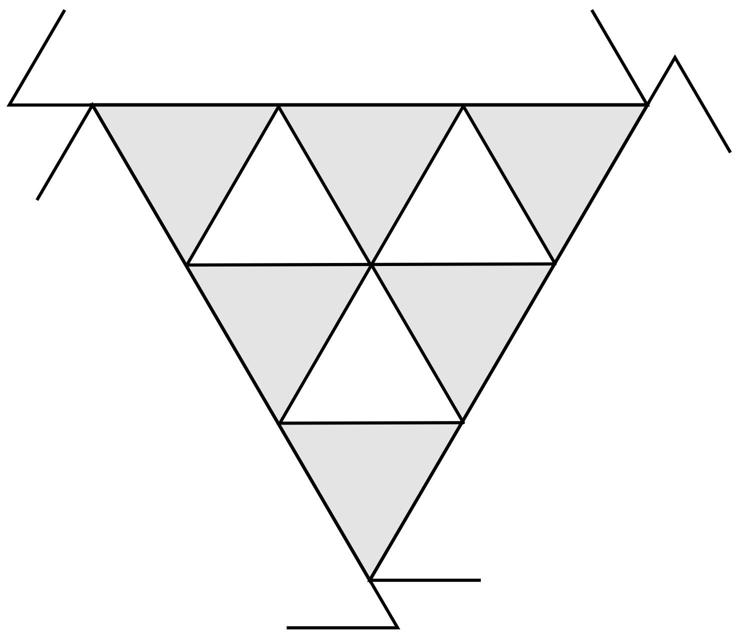


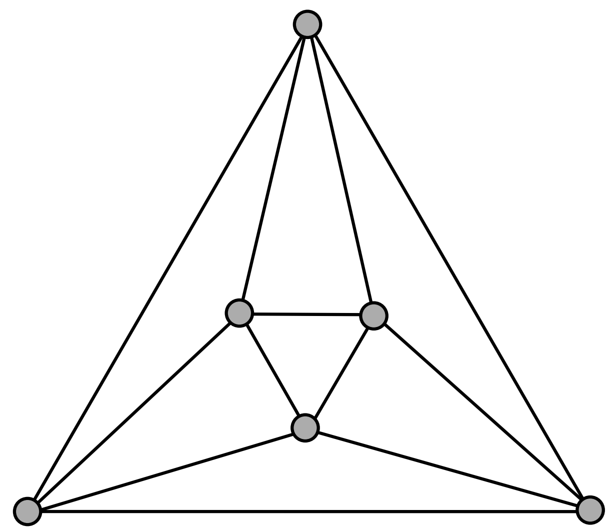


A triangle contact representation
with homothetic triangles.

Triangle Contact Representations

Conjecture. [Bertinoro 2007]
Every 4-connected triangulation has a triangle contact representation with homothetic triangles.





Triangle Contact Representations

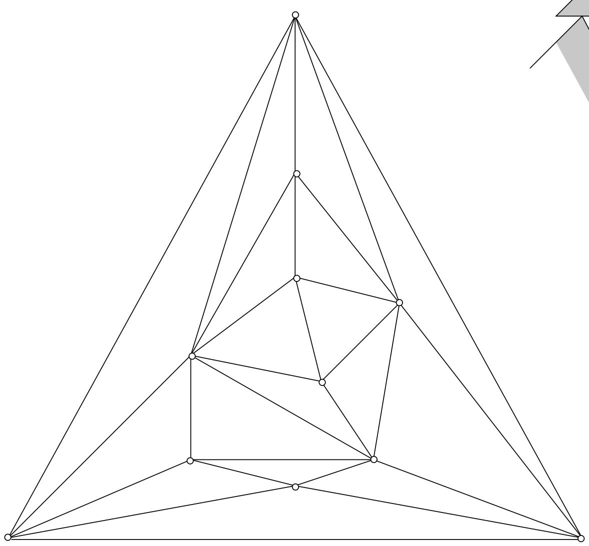
Gonçalves, Lévêque, Pinlou (GD 2010) observe that the conjecture follows from a corollary of Schramm's "Monster Packing Theorem" from *Combinatorially Prescribed Packings and applications to Conformal and Quasiconformal Maps*.

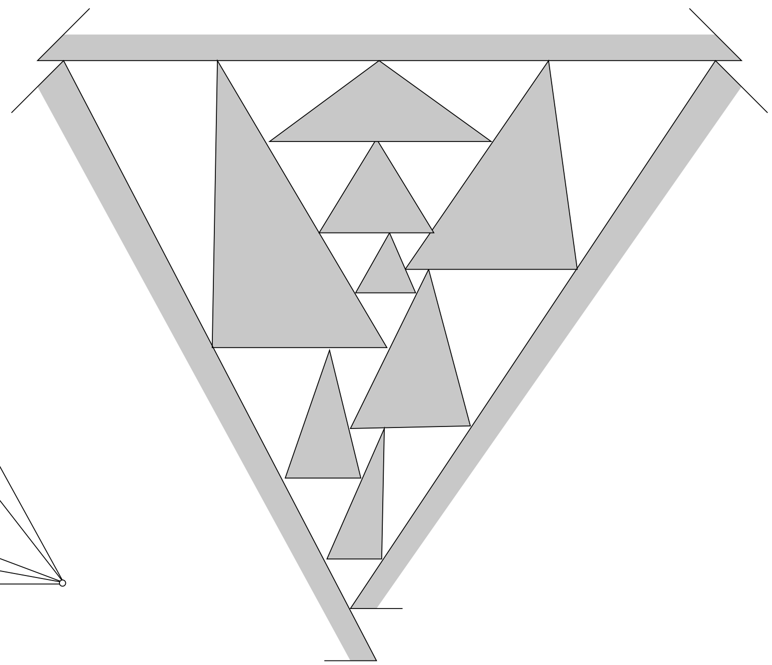
Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v containing more than one point. Then there is a contact representation of T with homothetic copies.

Remark. In general homothetic copies of the Q_v can degenerate to a point. Gonçalves et al. show that this is impossible if T is 4-connected.

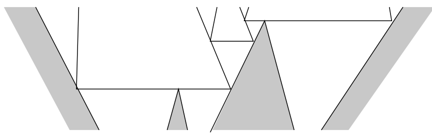
Combinatorial Methods

de Fraysseix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.

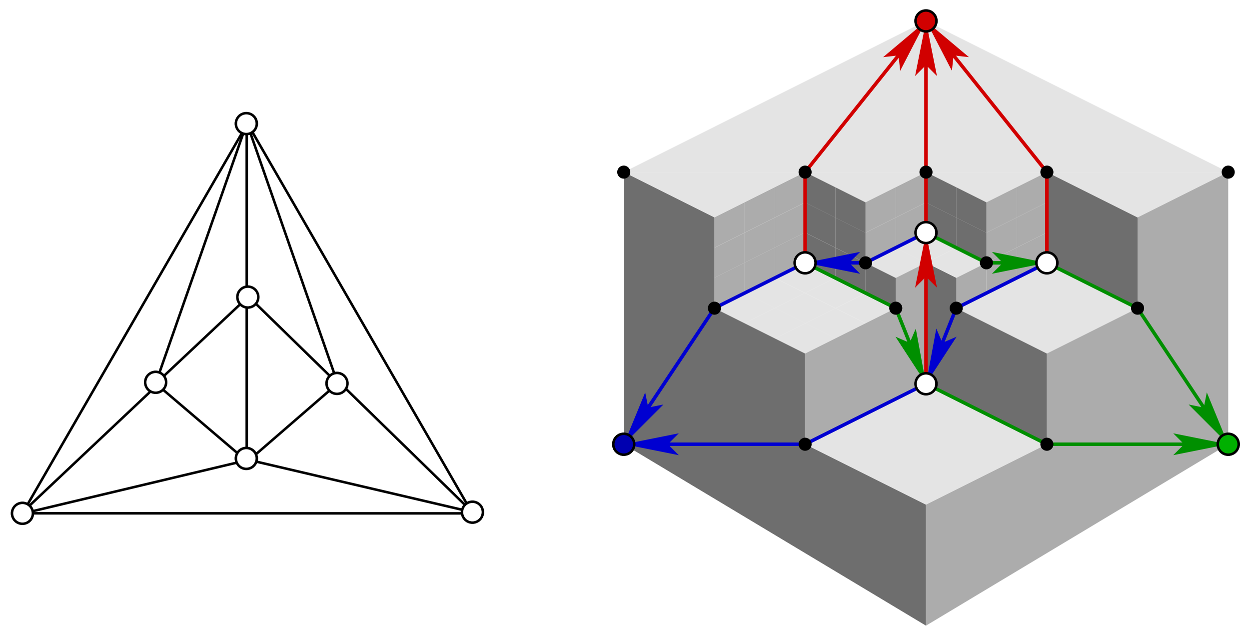




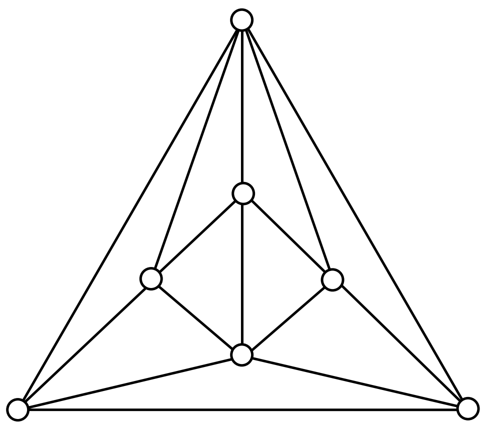
Take vertices in order of increasing red region:

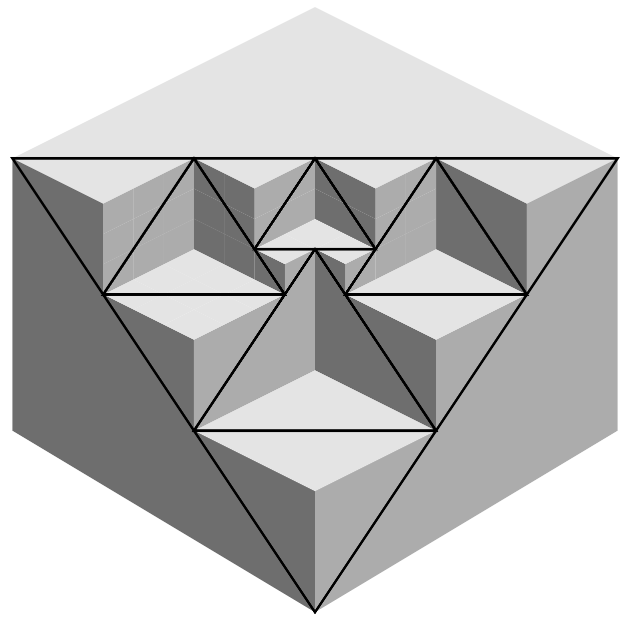


Edge-Coplanar Orthogonal Surfaces

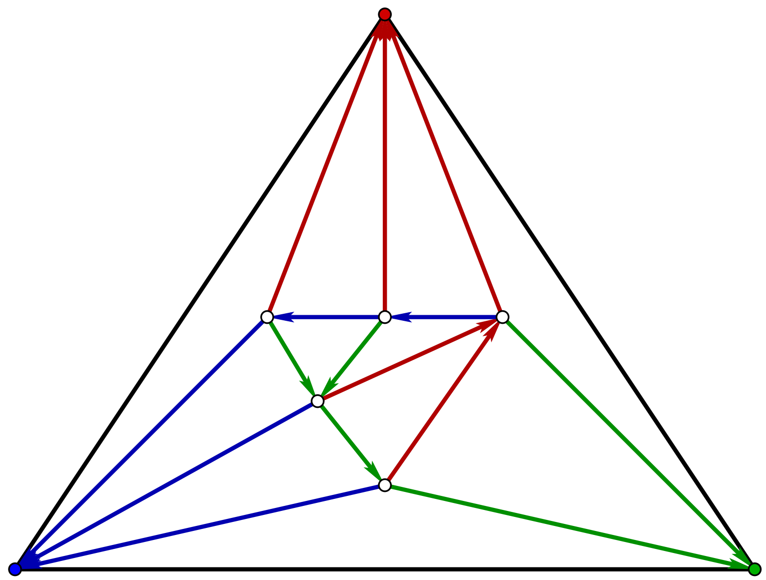


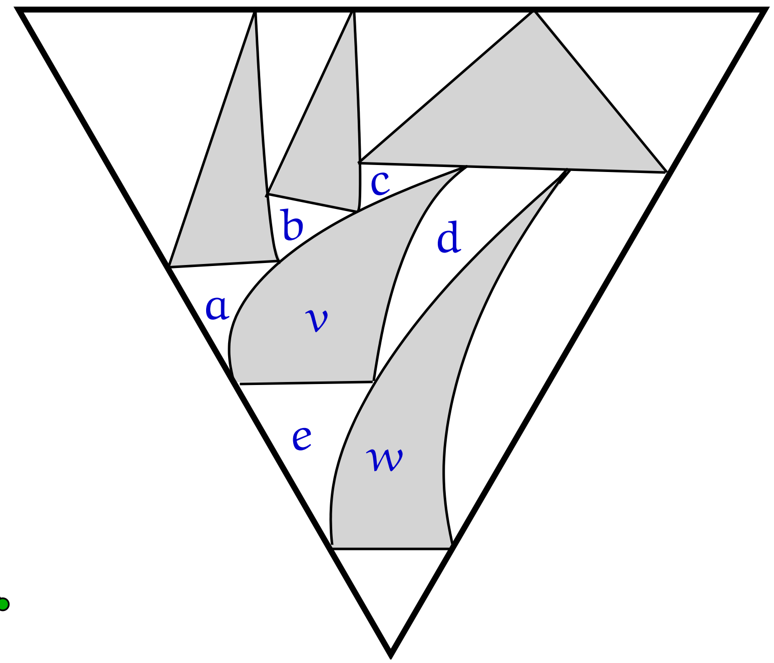
Edge-Coplanar Orthogonal Surfaces





Triangle Contacts and Equations





A Schnyder wood induces an *abstract triangle contact representation*. Equations for the sidelength:

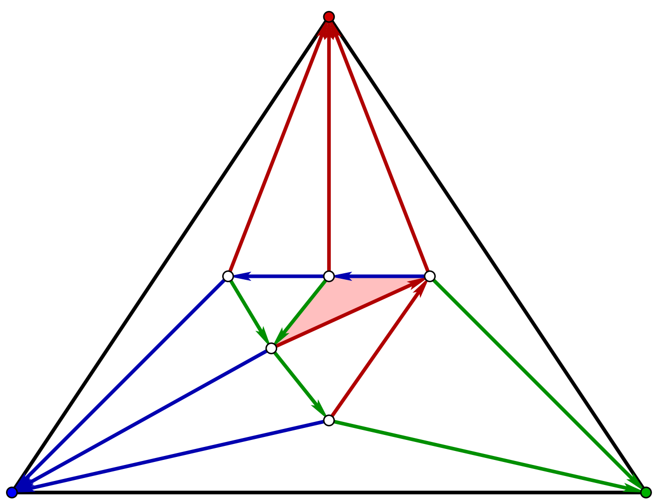
$x_a + x_b + x_c = x_v$ and $x_d = x_v$ and $x_e = x_v$ and $x_d + x_e = x_w$ and ...

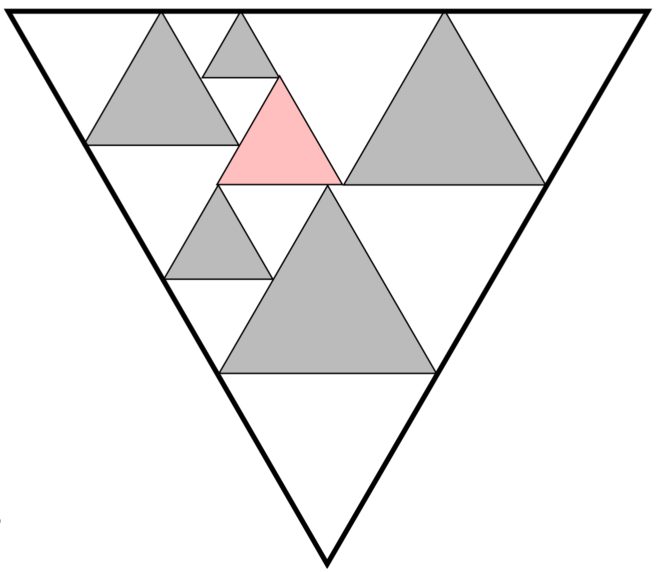
Solving the Equations

Theorem. The system of equations has a uniqe solution.

The proof is based on counting matchings.

- In the solution some variables may be negative.
- Still the solution yields a triangle contact representation.





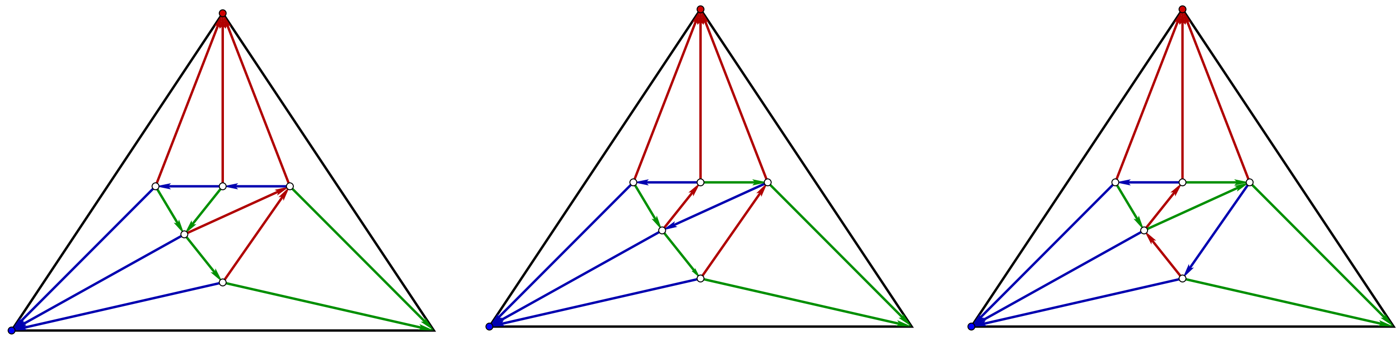
Flipping Cycles

Proposition. The boundary of a negative area is a directed cycle in the underlying Schnyder wood.

From the bijection

$$\text{Schnyder woods} \iff \text{3-orientations}$$

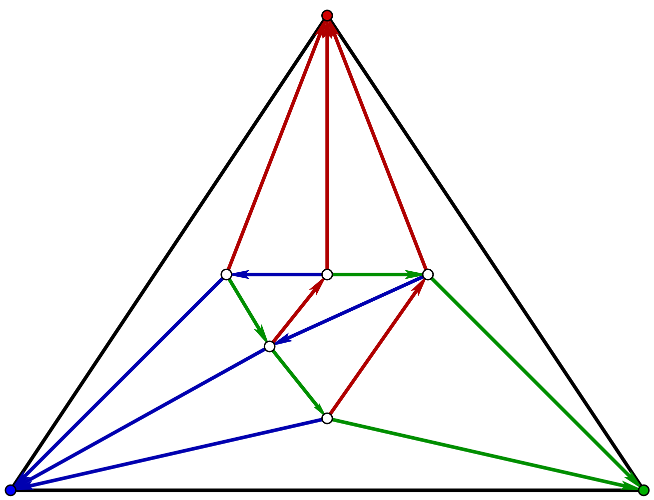
we see that cycles can be reverted (flipped).

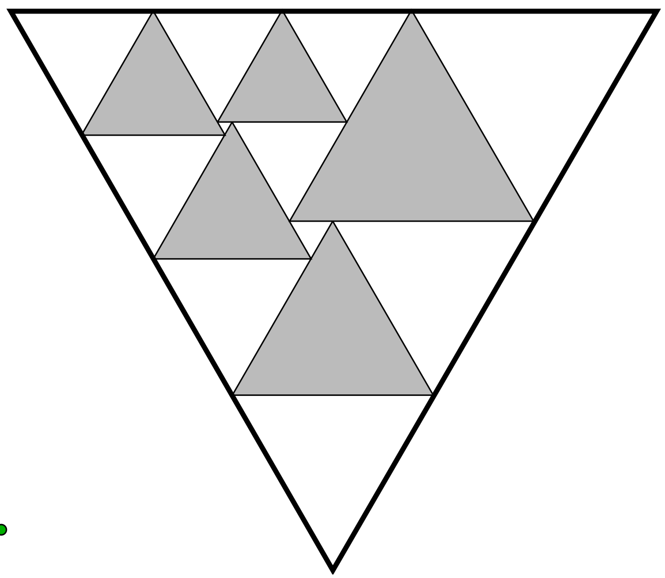


Resolving

A new Schnyder wood yields new equations and a new solution.

Theorem. A negative triangle becomes positive by flipping.





Additional Applications of Schnyder Woods

- Dimension Theory of Posets.

  (W. Schnyder, G. Brightwell, W.T. Trotter, S. Felsner)

- Visibility Representations.

  (C.C. Lin, H. Lu, I-F. Sun, H. Zhang)

- Counting:

  (E. Fusy, O. Bernardi, G. Schaeffer)

- Greedy Routing.

  (R. Dhandapani, X. He)

The End

Thank you.