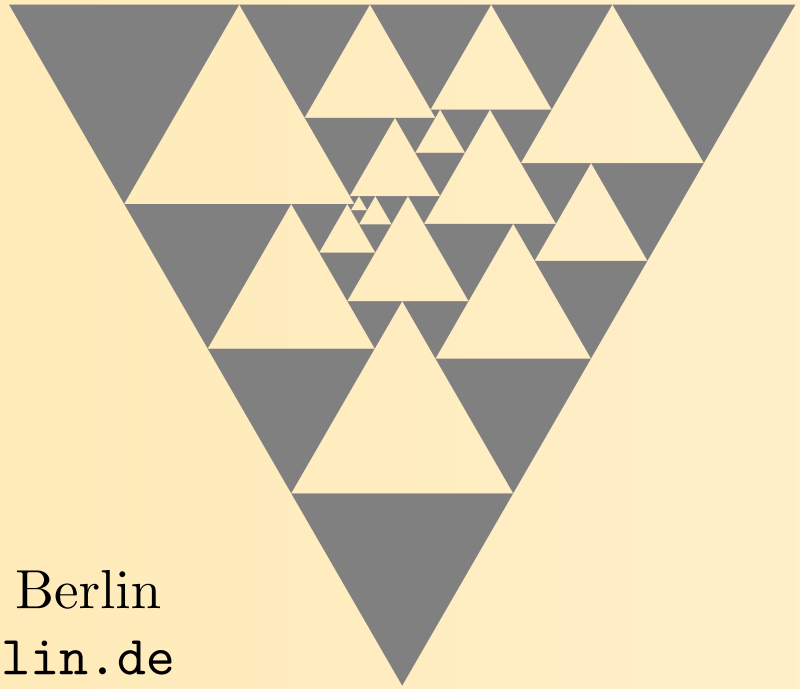
Representing Planar Graphs with Rectangles and Triangles



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Lausanne
Oktober 14. 2010

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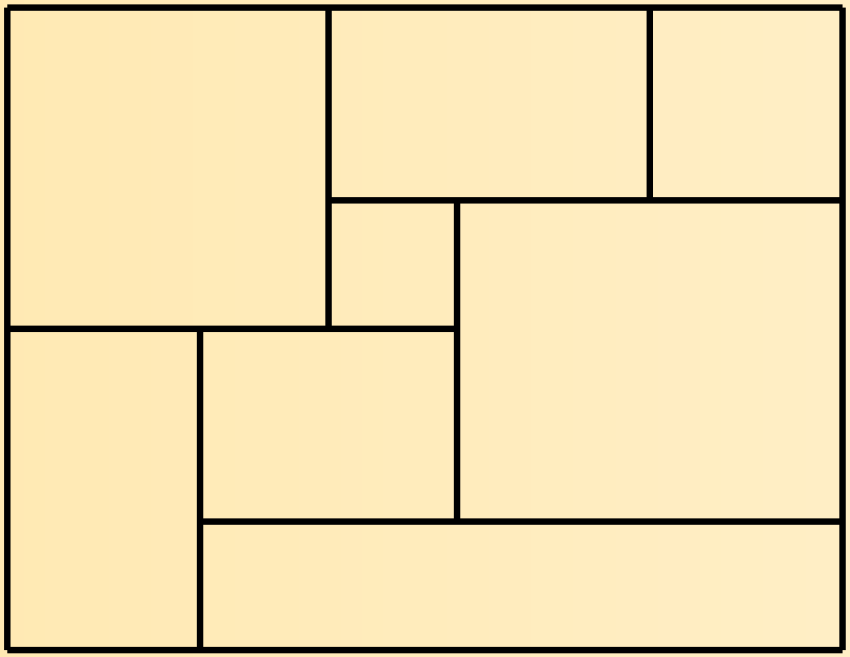
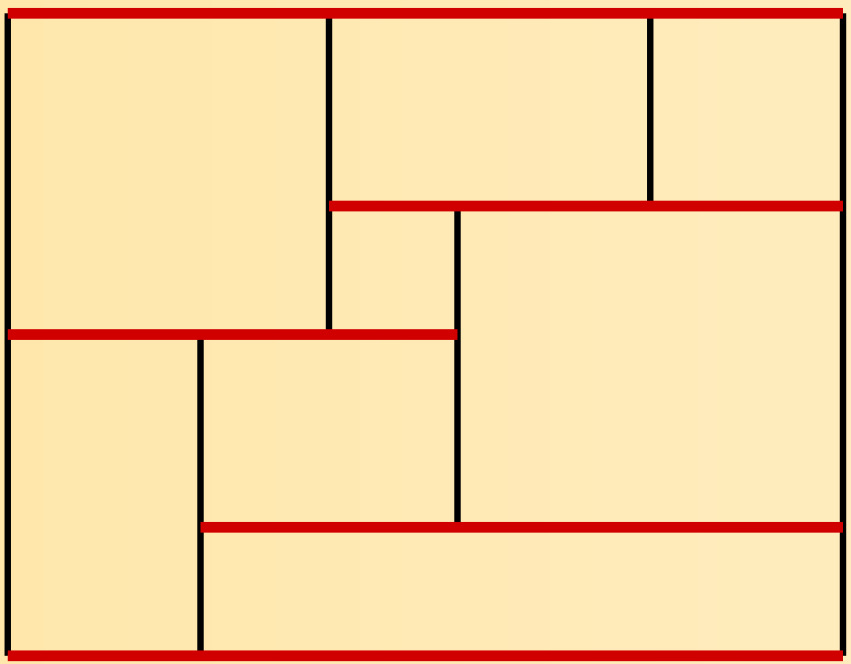
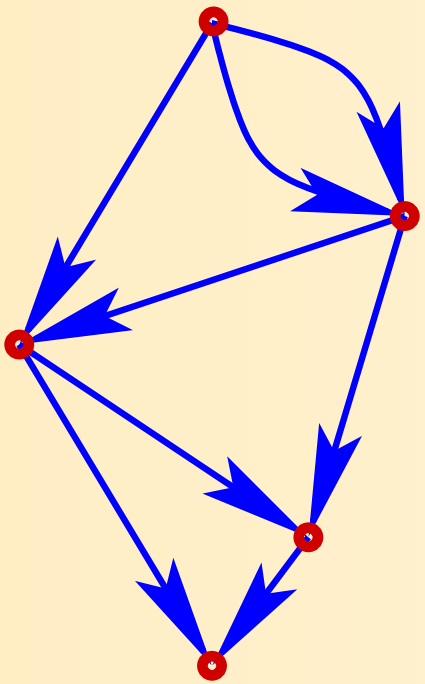
A Rectangular Dissection



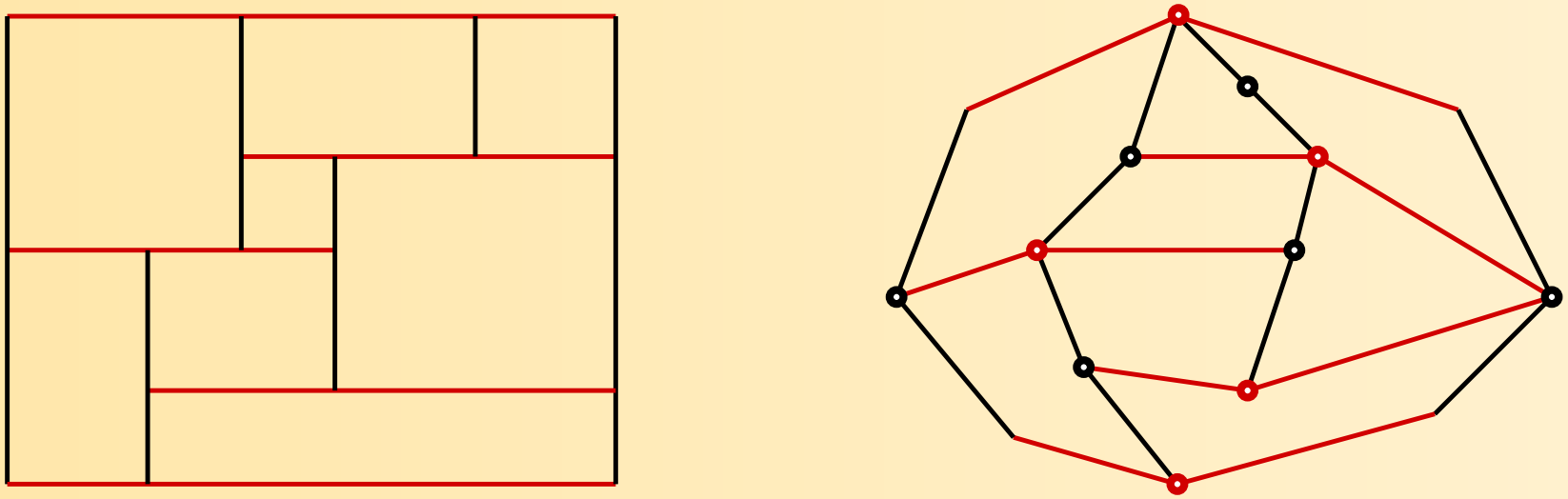
Rectangular Dissections Induce Graphs





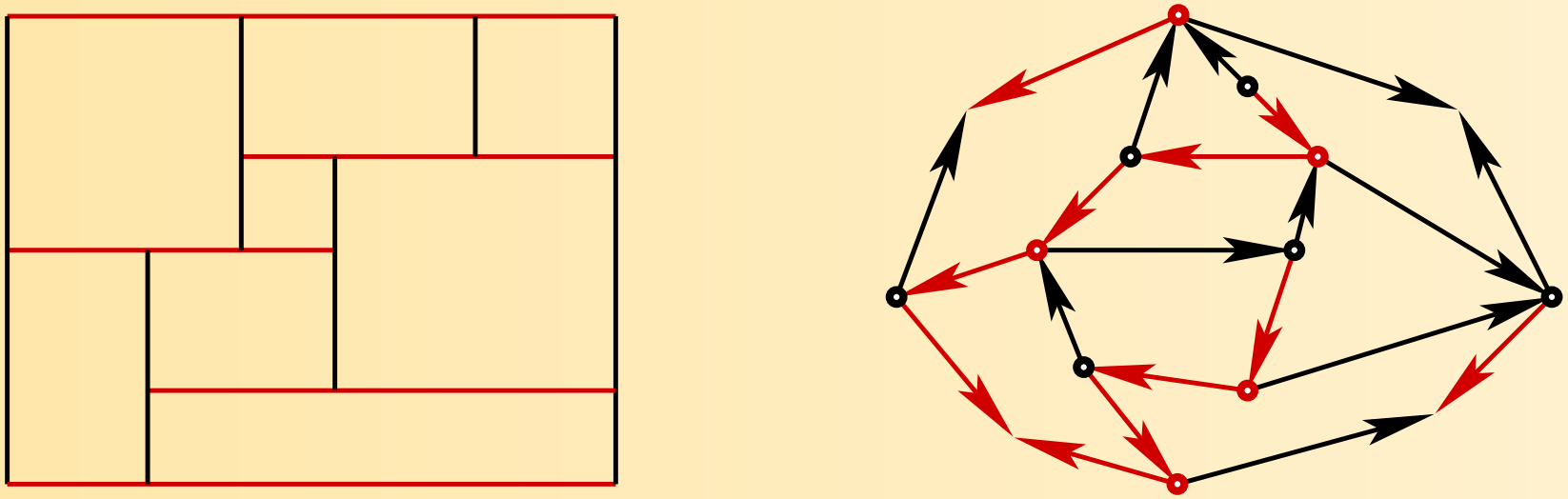
A bipolar graph induced by R.

Rectangular Dissections Induce Graphs



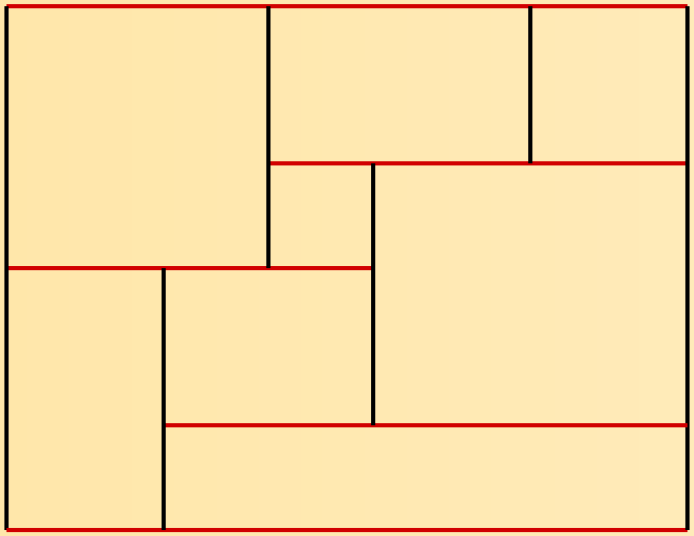
A quadrangulation based on segment contacts.

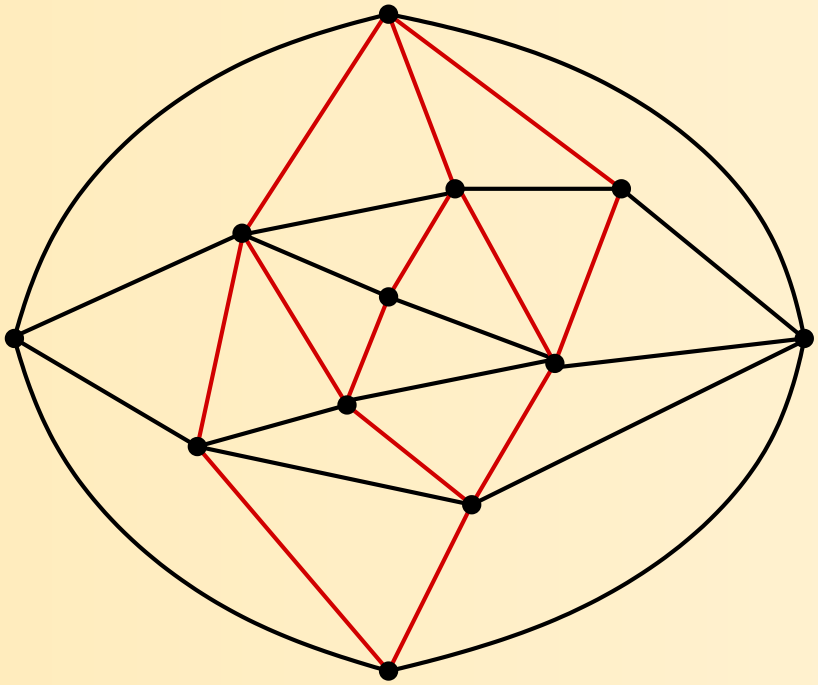
Rectangular Dissections Induce Graphs



A separating decomposition of the quadrangulation.

Rectangular Dissections Induce Graphs





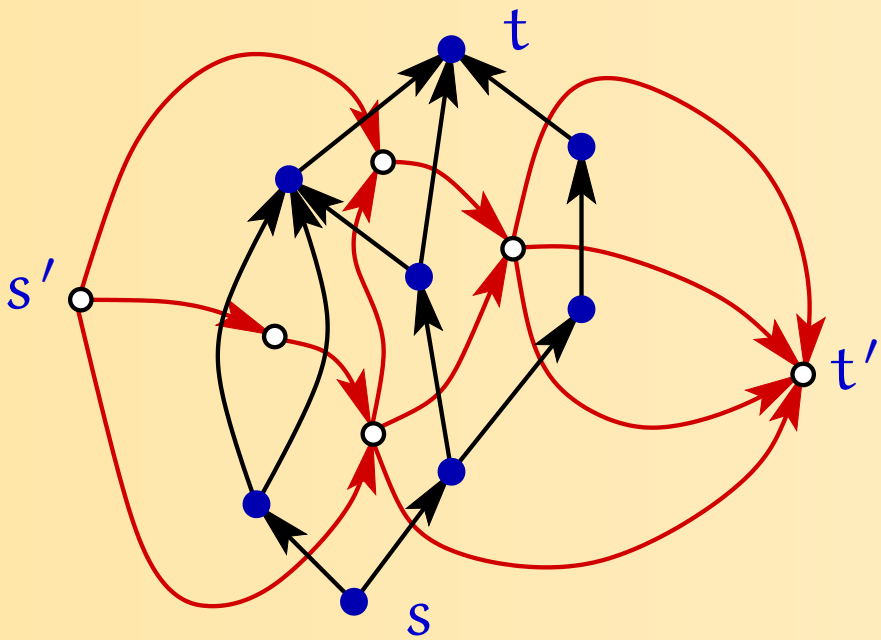
An inner triangualation of a quadrangle.
Vertices correspond to rectangles.

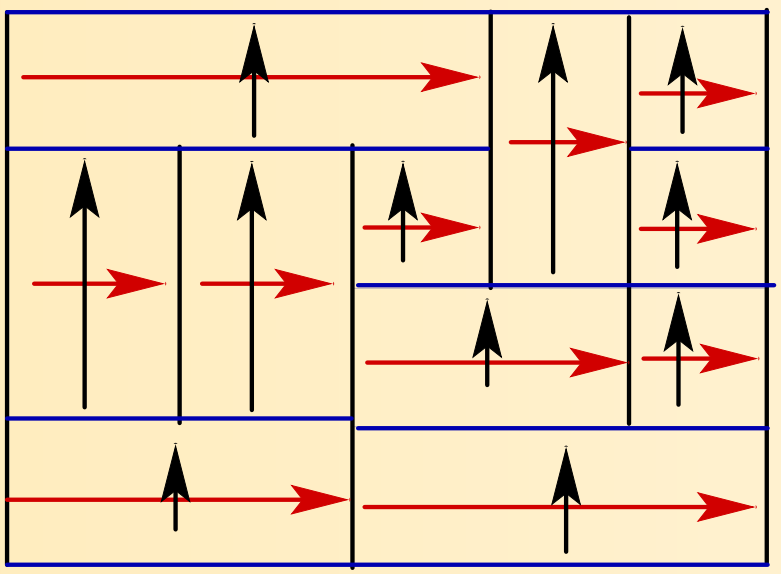
Representation Problems

- Given a bipolar graph G_B find a rectangulation R representing G_B.
- Given a planar quadrangulation Q find some R representing Q.
- Given a triangualation of a quadrangle G find some R representing Q (in this case R is called rectangular dual, resp. floorplan of G).

Sketch: Bipolar Orientation

From the bipolar orientation compute its dual orientation.
Together they yield a rectangular dissection.

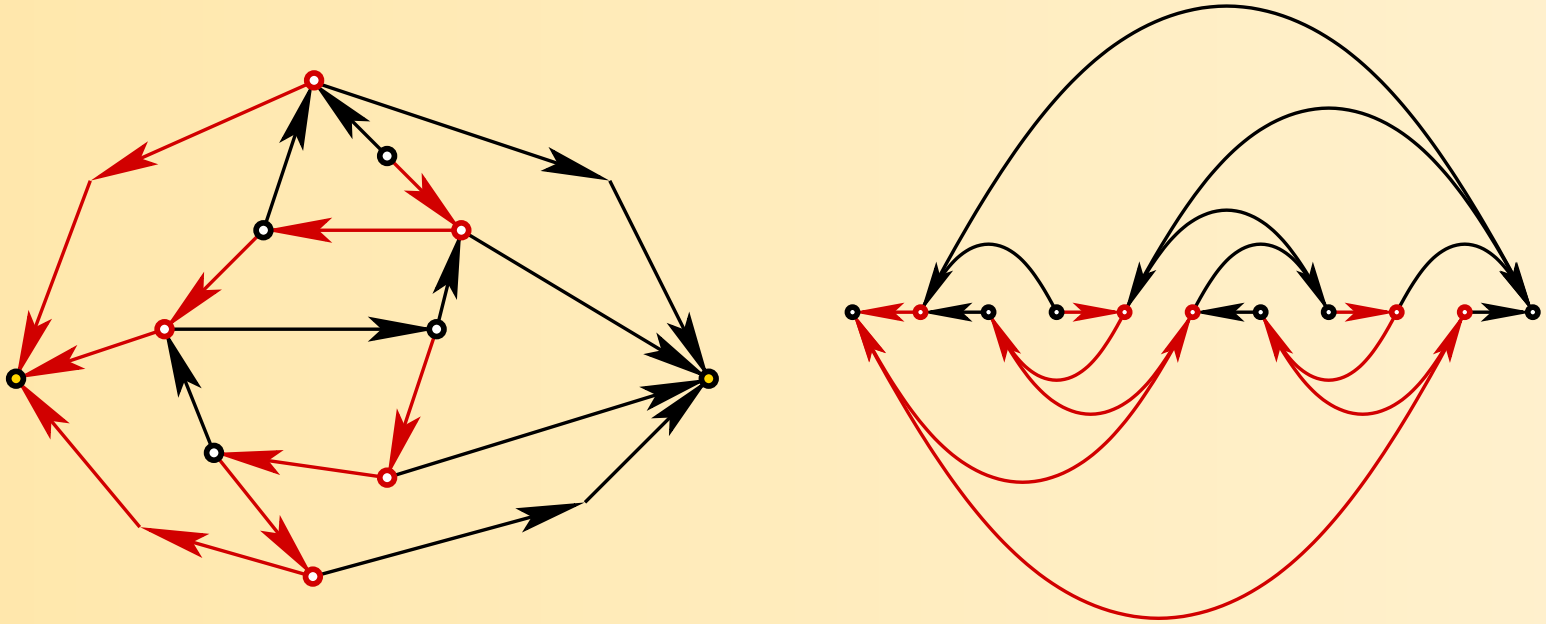




coordinates from longest paths

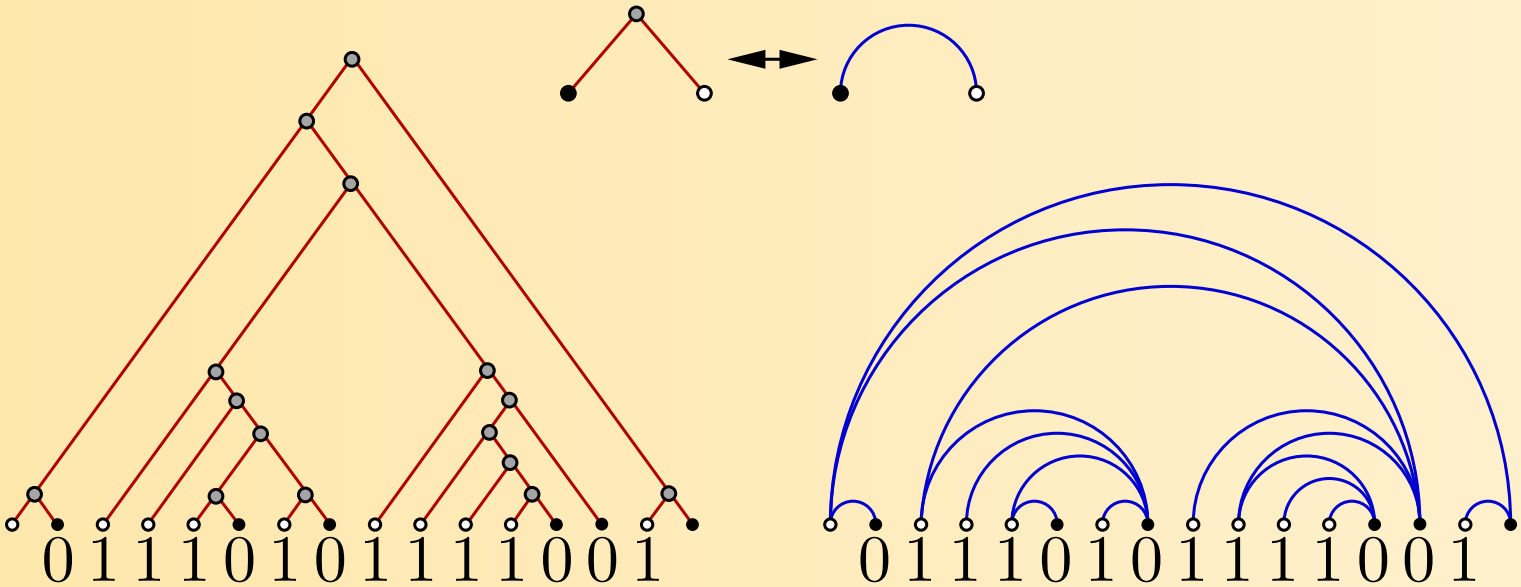
Sketch: Quadrangulation

- Compute a separating decomposition.
- Separate the two alternating trees.



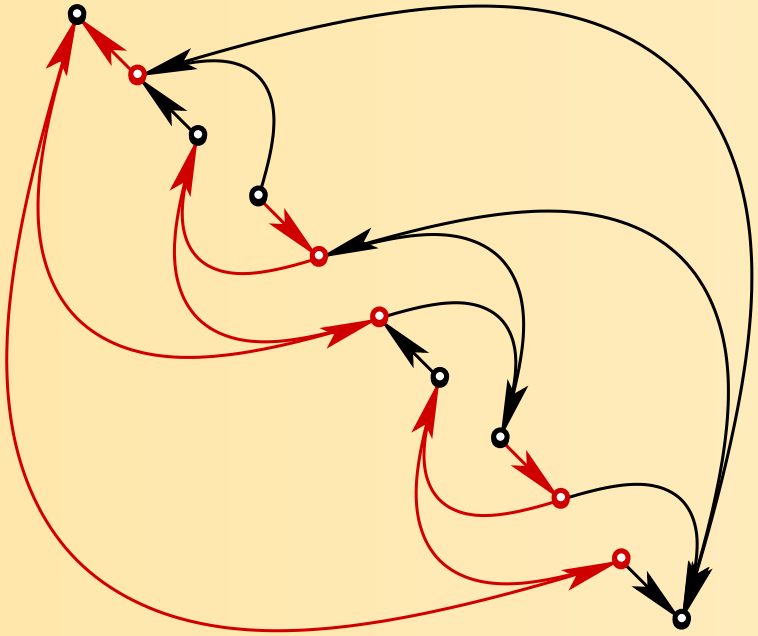
Alternating and Full Binary Trees

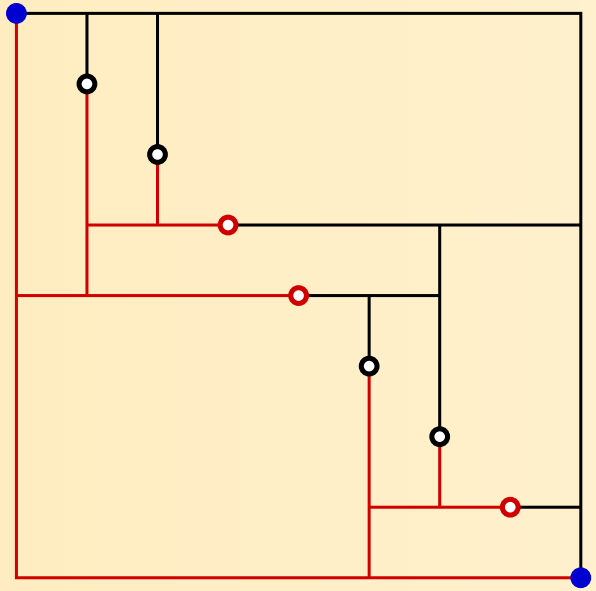
Proposition. There is bijection between alternating and binary trees that preserves fingerprints.



Sketch: Quadrangulation

- The two binary trees obtained from the separating decomposition fit together.





More Representation Problems

Add some conditions.

- Find a representing rectangulation R in a square.
  (Trivial scaling)

- Find a representing rectangulation R in a square such that all rectangles intersect the diagonal.
  (We just saw a solution)

- Find a representing rectangulation R such that all inner rectangles are squares.

  The dissection of rectangles into squares
  Brooks, Smith, Stone and Tutte 1940.

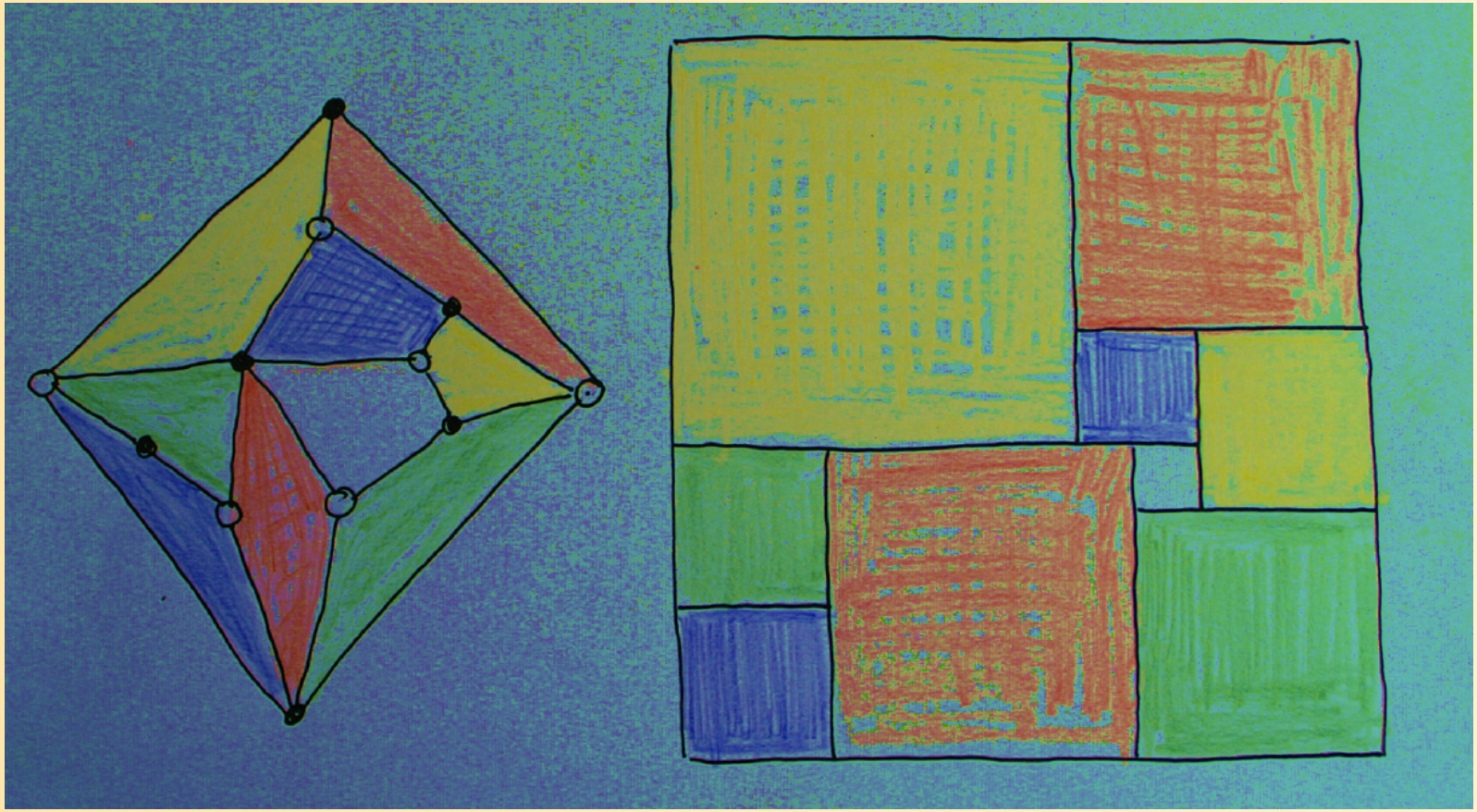
Squarings a la BSST

They interpret the bipolar graph as electrical network with edges of resistance 1 Ohm. Consider an $\mathtt{s} \to \mathtt{t}$ flow in this network. The distribution of current in edges corresponds to a squaring.

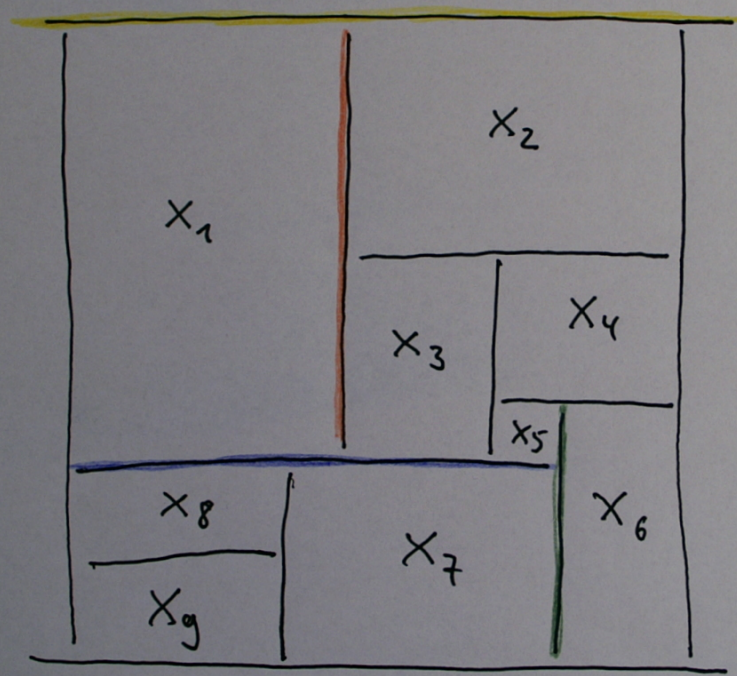
Based on this theory they can give explicit solutions:

$$\begin{aligned} \mathrm{size}(\mathtt{i},\mathtt{j}) = & \\ & \#\text{ spanning trees } \mathsf{T} \text{ with } (\mathtt{i},\mathtt{j}) \text{ on the } \mathtt{s} \to \mathtt{t} \text{ path in } \mathsf{T} \\ - & \#\text{ spanning trees } \mathsf{T} \text{ with } (\mathtt{j},\mathtt{i}) \text{ on the } \mathtt{s} \to \mathtt{t} \text{ path in } \mathsf{T}. \end{aligned}$$

Squarings with Segment Contacts



How to square a segment contact representation



- $x_1 = x_2 + x_3$
- $x_1 + x_3 + x_5 = x_7 + x_8$
- $x_5 + x_7 = x_6$
- $x_1 + x_2 = 1$

\# variables
= # inner faces of Q

\# equations
= # inner vertices of Q
$+ 1$

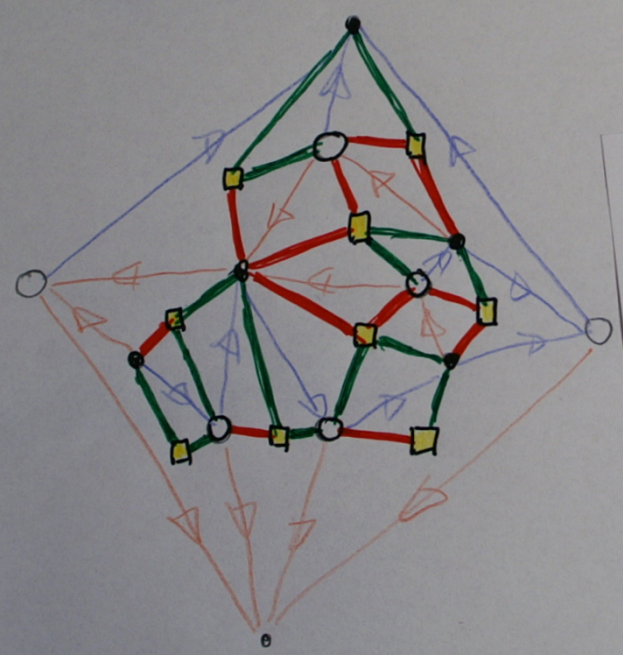
$$A_s x = e_1$$

A_s
$(n-3) \times (n-3)$

The matrix A_S is non-degenerate: $\det(A_S) \neq 0$

- A_S as incidence matrix of a bip. graph. H

$$\det(A_S) = \sum_{\sigma \in S_{n-3}} \operatorname{sgn}(\sigma) \prod a_{i\sigma(i)}$$



a non-zero contribution corresponds to a perfect matching M_σ of H

The matrix A_S is non-degenerate: $\det(A_S) \neq 0$

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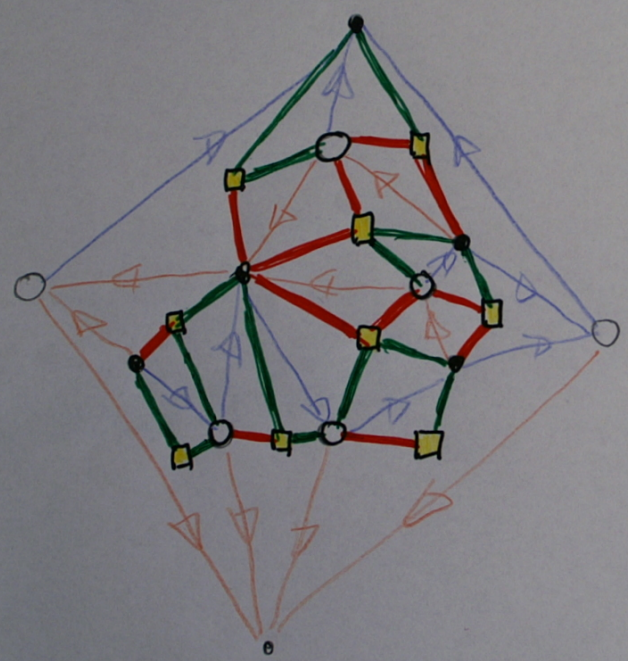


$$\det(A_S) = \sum_{\sigma \in S_{n-3}} \operatorname{sgn}(\sigma) \prod a_{i\sigma(i)}$$

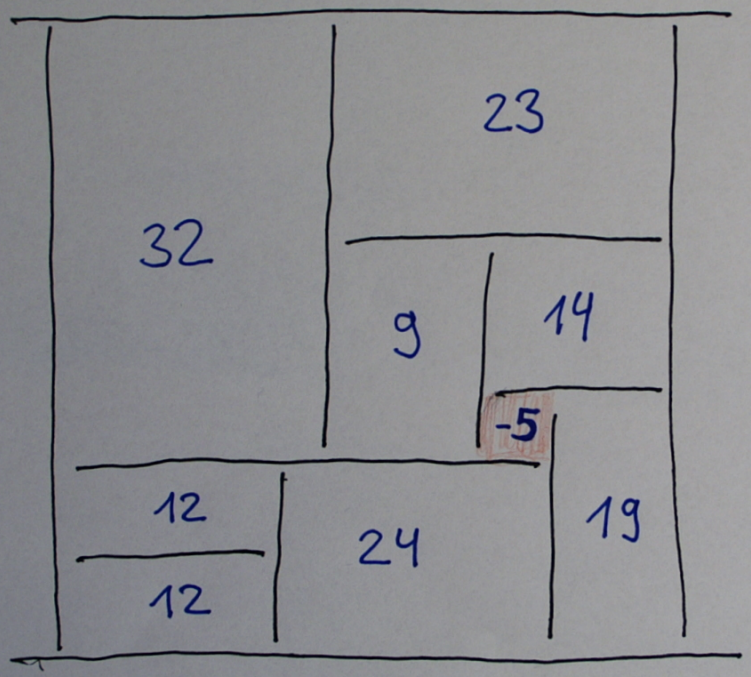
$$= \sum_{M \in \mathcal{M}(H)} \operatorname{sgn}(M)$$

Claim A: $\exists$ perf. matching

Claim B: $M, M' \in \mathcal{M}(H)$

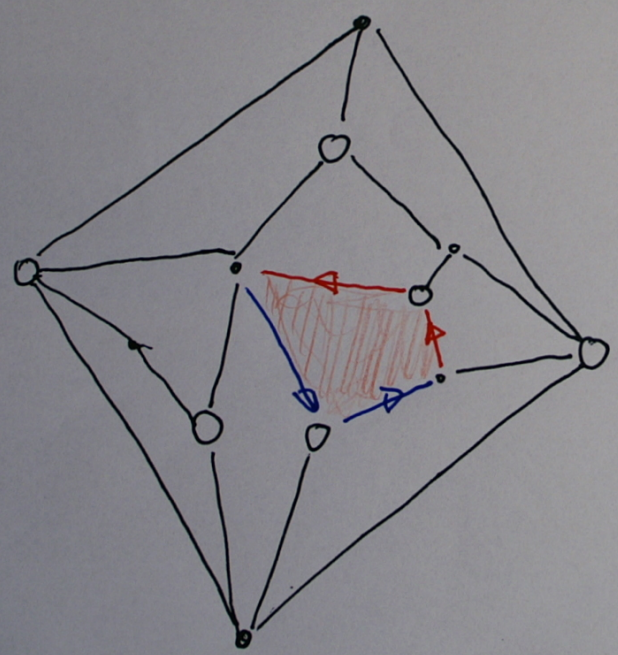
$\Rightarrow \operatorname{sgn}(M) = \operatorname{sgn}(M')$

The solution (scaled)



Proposition

The boundary of the set of negative faces is a (union) of directed cycles in the separating decomposition



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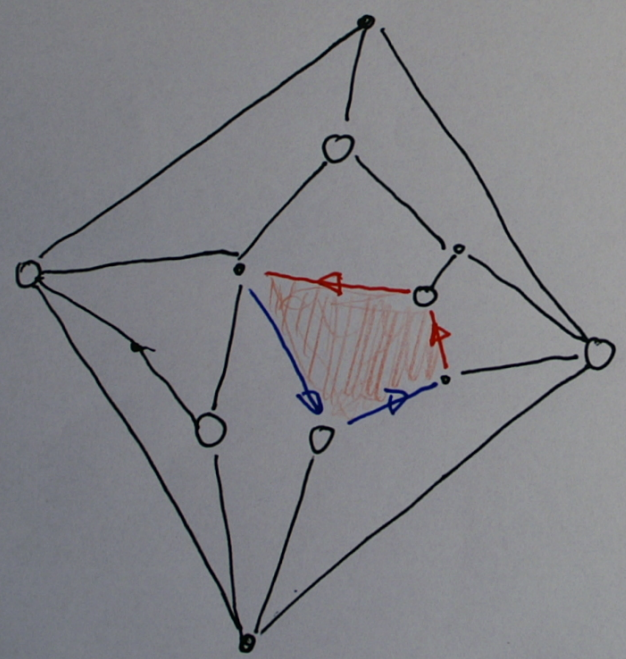
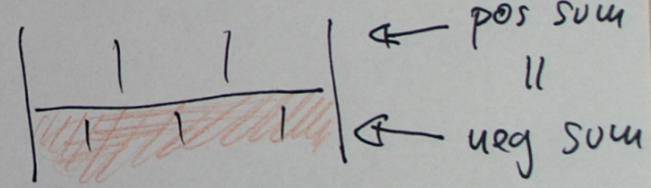


Lemma: The boundary contains no full segment in the segment cont repr.



Proposition

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Tracing a boundary

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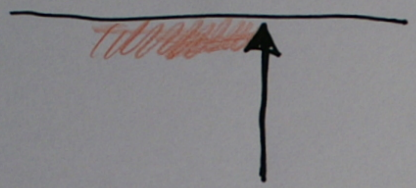
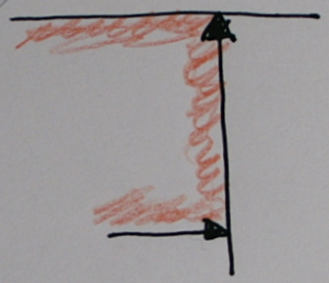
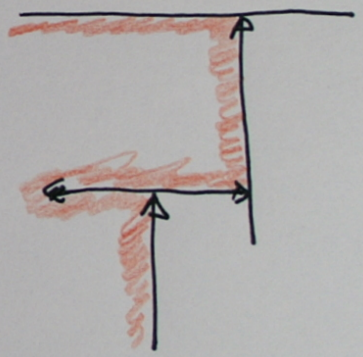
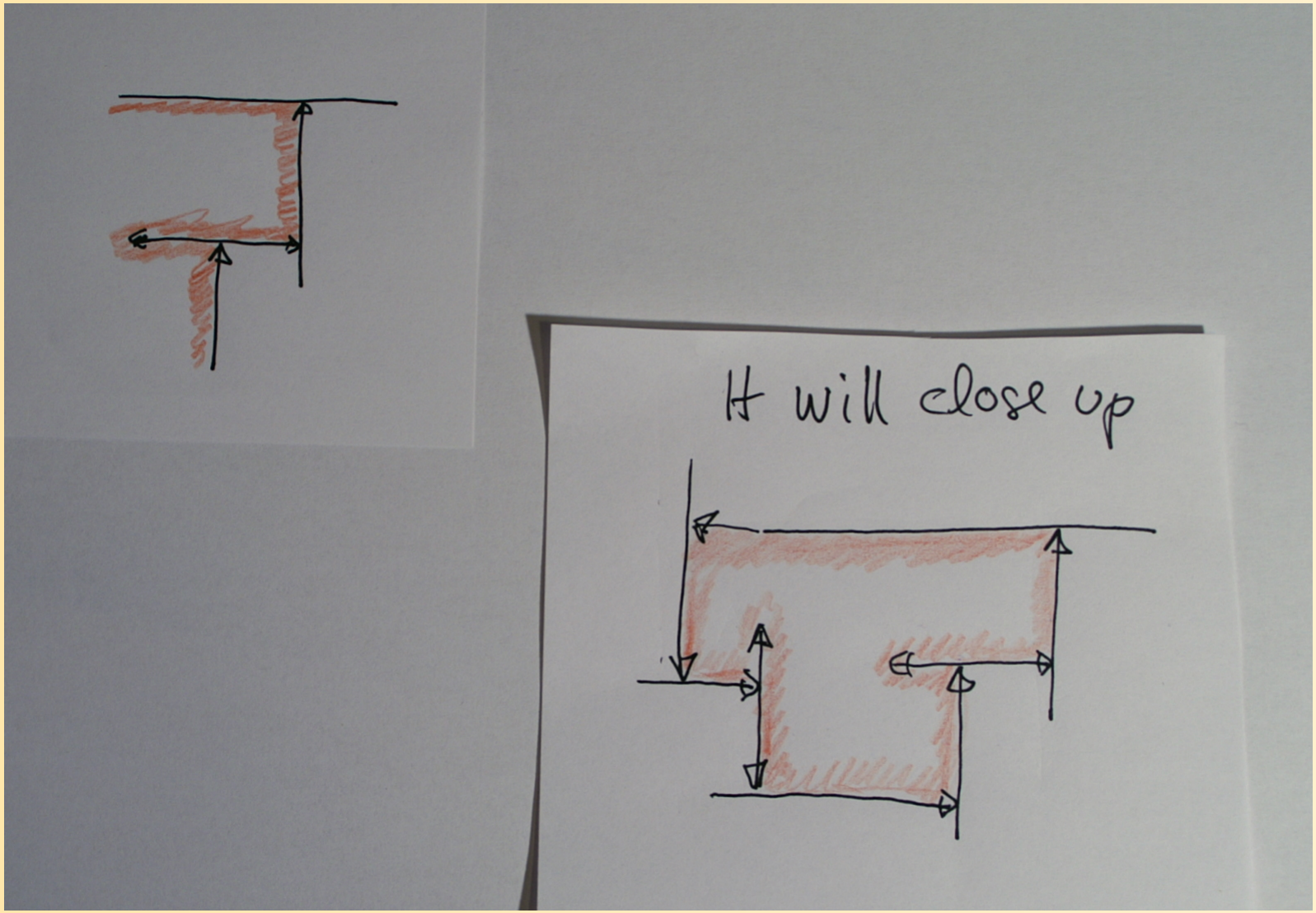
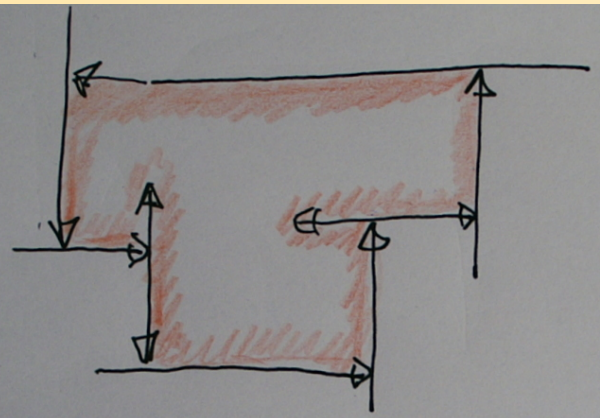


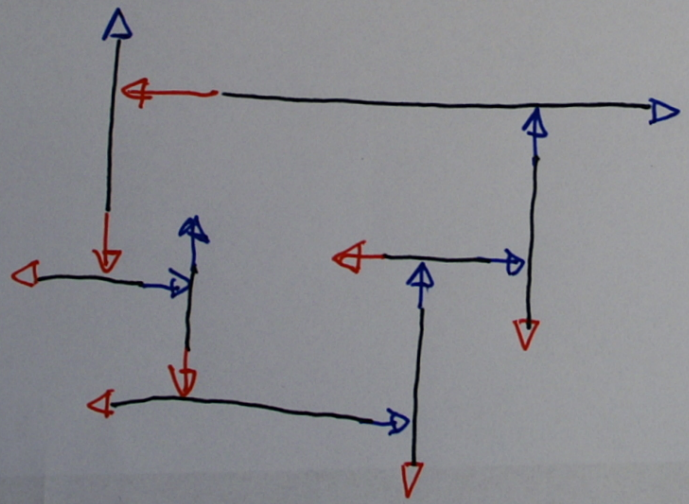


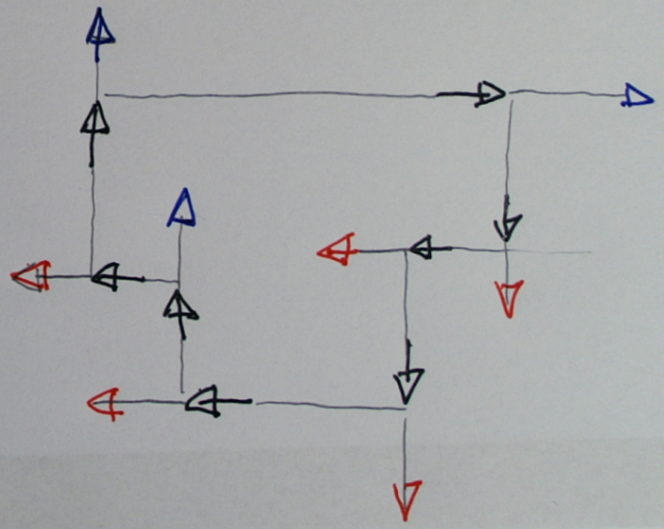



It will close up



Changing the orientation





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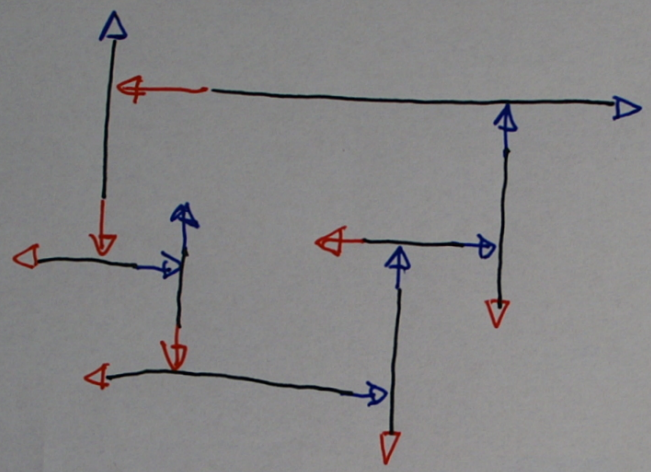
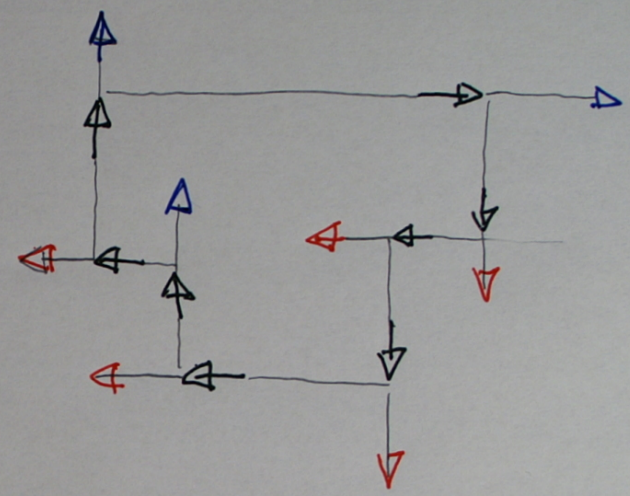


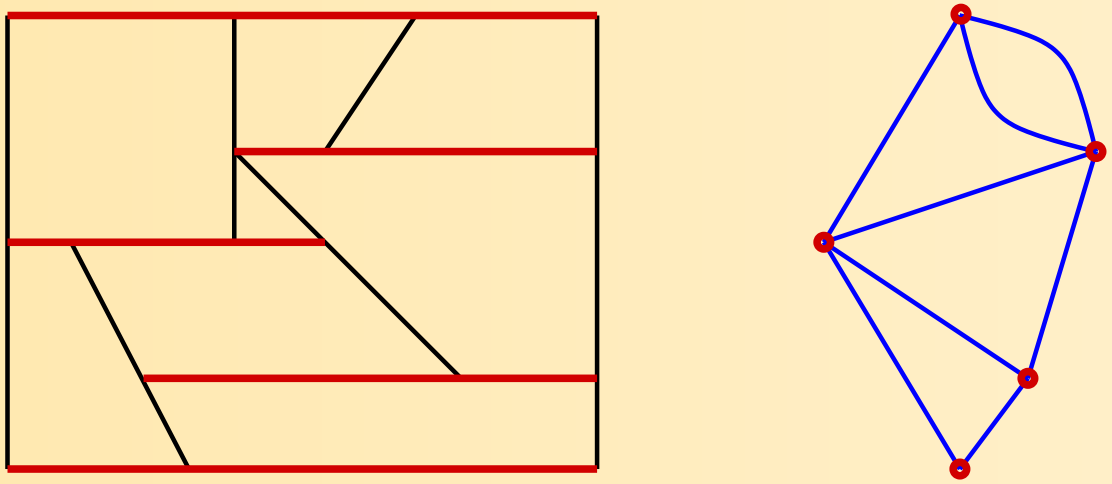


This moves the faces with negative values to the other side of the equations

$\Rightarrow$ same values with pos. sign yields a solution

Trapezoidal Dissections and Markov Chains

Based on R. Kenyon *Tilings and Discrete Dirichlet Problems* 1998.



At (horizontal) segment i the transition probabilities are

$$p(i,j) \propto m(i,j) = \frac{\mathrm{width}_i(T_{ij})}{\mathrm{height}(T_{ij})}.$$

Trapezoidal Dissections and Markov Chains

Proposition. $f(i) = y_i$ is harmonic with respect to p for all $i \notin \{s, t\}$, i.e.,

$$f(i) = \sum_j f(j)p(i,j).$$

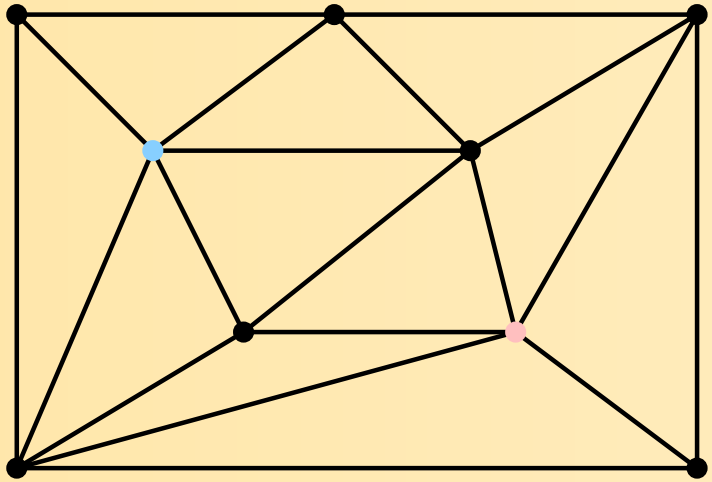
Trapezoidal Dissections and Markov Chains

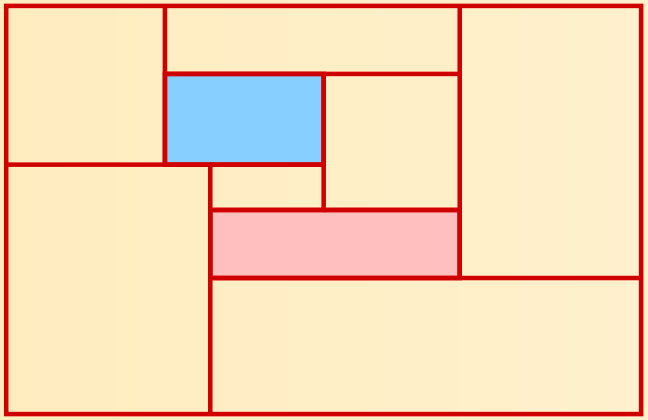
Theorem. G planar, p transition probabilities, s, t on the outer face $\implies$ the stationary distribution π together with the unique p-harmonic function f on $V \setminus \{s, t\}$ yield a trapezoidal dissection of a rectangle.

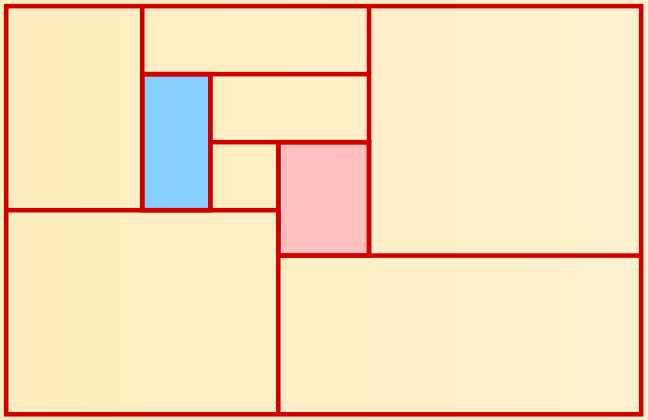
If in addition $\pi(i)p(i,j) = \pi(j)p(j,i)$ for all edges, then we get a rectangulation.

If $p(i,j) = \frac{1}{\deg(i)}$ we get a squaring.

Squarings for Inner Triangulations

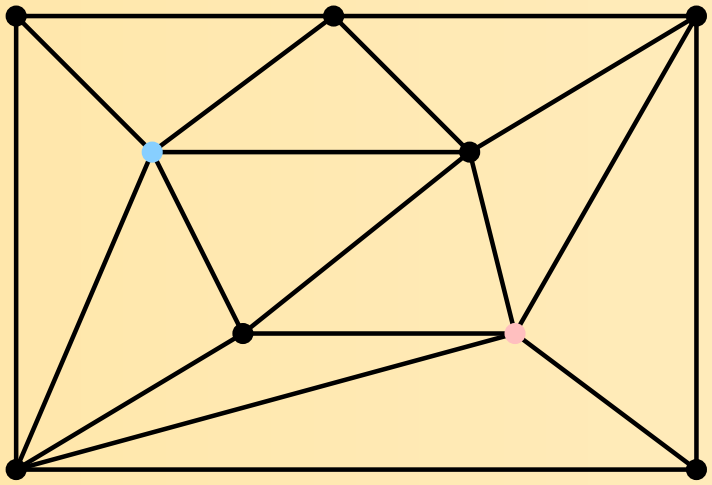


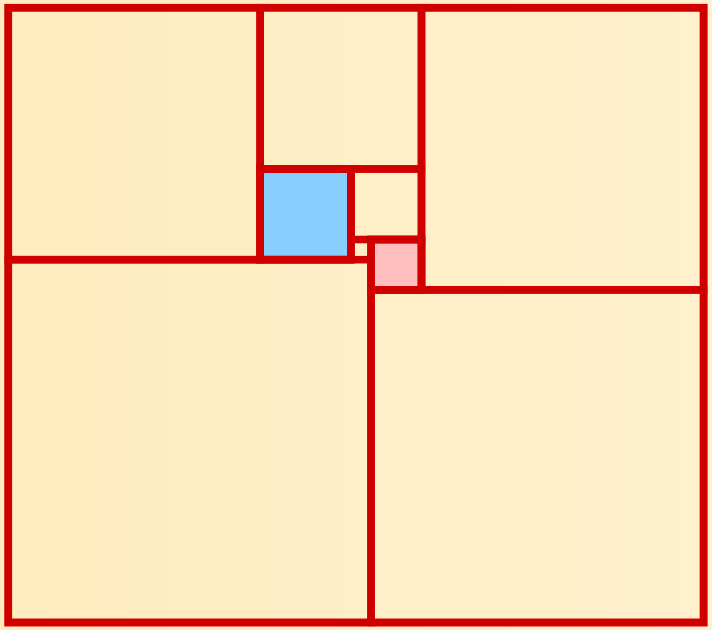




Squarings for Inner Triangulations

The squaring of G is unique.





Extremal Length

Based on O. Schramm *Square Tilings with prescribed Combinatorics.*

- $m : V \to \mathbb{R}^+$ discrete metric on G
- Length of a path: $\ell_m(\gamma)$
- Distance between sets: $\ell_m(A, B) = \min_{\gamma \in \Gamma(A,B)} \ell_m(\gamma)$
- $\text{area}(m) = \sum_v m(v)^2 = \|m\|^2$

Extremal Length

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- Length of a path: $\ell_m(\gamma)$
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- $\text{area}(m) = \sum_v m(v)^2 = \|m\|^2$
- Normalized distance $\ell^*_m(A, B) = \frac{\ell_m(A,B)^2}{\|m\|^2}$
- Extremal length $L(A, B) = \sup_m \ell^*_m(A, B)$

Extremal Length and Squarings

Theorem. For G, A, B there is a (up to scaling) unique extremal metric.

Proof. Normalized distance is invariant under scaling. Hence, we only have to look at metrics with $\ell_m(A, B) = \min_{\gamma \in \Gamma(A,B)} \ell_m(\gamma) = 1$.

These m form a polyhedral set P (ineq. $\ell_m(\gamma) \geq 1$).

Extremal metric is the unique m with minimal norm in P.

Extremal Length and Squarings

Theorem. A squaring of G, with A and B at top and bottom induces an extremal metric.

Proof. Let $h = \text{height}(R)$ and $w = \text{width}(R)$ we may assume $h \cdot w = 1$.

For the side length $s(v)$:
$\|s\|^2 = \sum s(v)^2 = h \cdot w = 1$, hence $\|s\| = 1$.

For $t \in [0, w]$ the squaring induces a path γ_t. By definition $\ell_m(A, B) \leq \sum_{v \in \gamma_t} m(v)$.

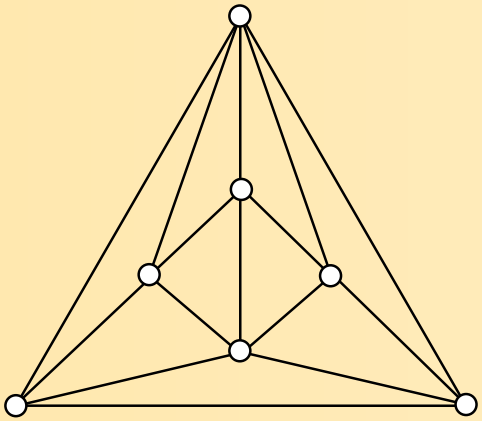
Extremal Length and Squarings

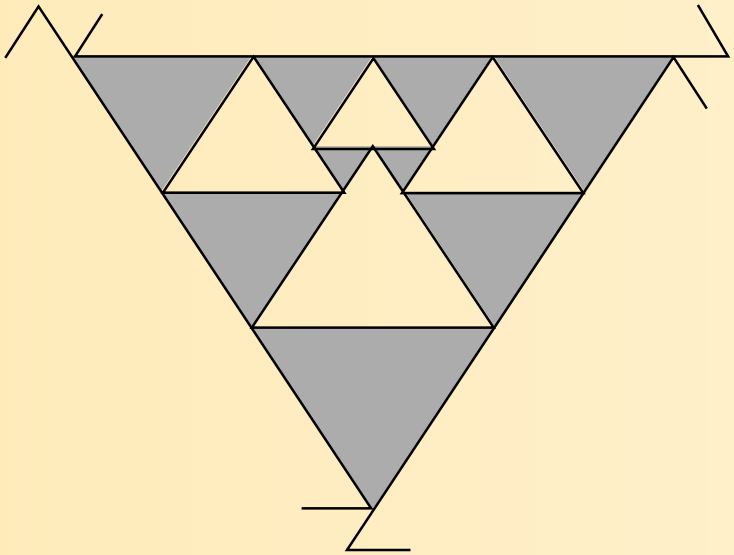
$$
\begin{aligned}
w \cdot \ell_m(A, B) &\leq \int_0^w \sum_{v \in \gamma_t} m(v) dt \\
&= \int_0^w \sum_{v \in V} m(v) \delta_{[v \in \gamma_t]} dt \\
&= \sum_{v \in V} m(v) \int_0^w \delta_{[v \in \gamma_t]} dt \\
&= \sum_{v \in V} m(v) s(v) \\
&\leq \langle m, s \rangle \leq ||m|| \cdot ||s|| = ||m||
\end{aligned}
$$

Hence:

$$
\ell_m^*(A, B) = \frac{\ell_m(A, B)^2}{||m||^2} \leq \frac{1}{w^2} = h^2 = \frac{h^2}{||s||^2} = \ell_s^*(A, B)
$$

Triangles and Graphs

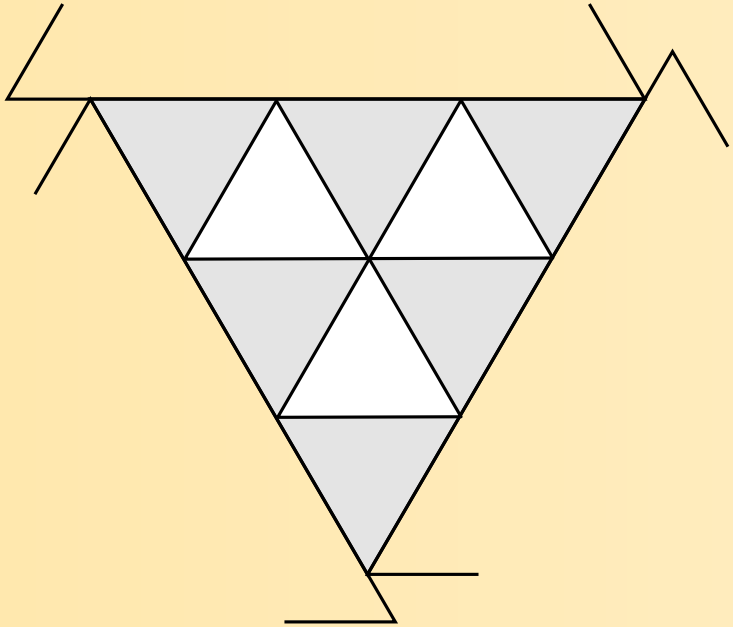


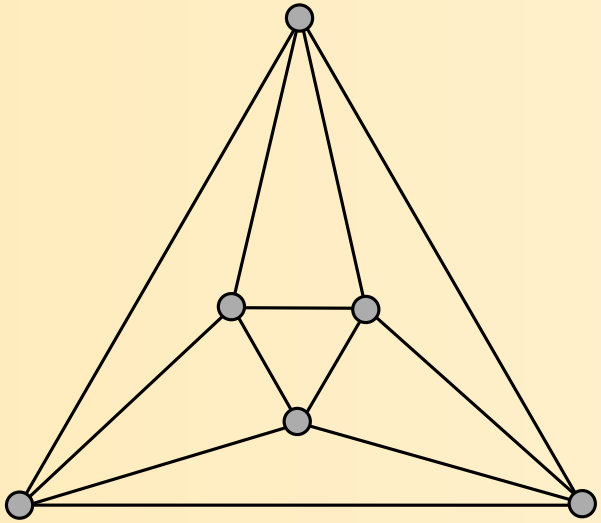


A triangle contact representation
with homothetic triangles.

Triangle Contact Representations

Conjecture. Every 4-connected triangulation has a triangle contact representation with homothetic triangles.





Triangle Contact Representations

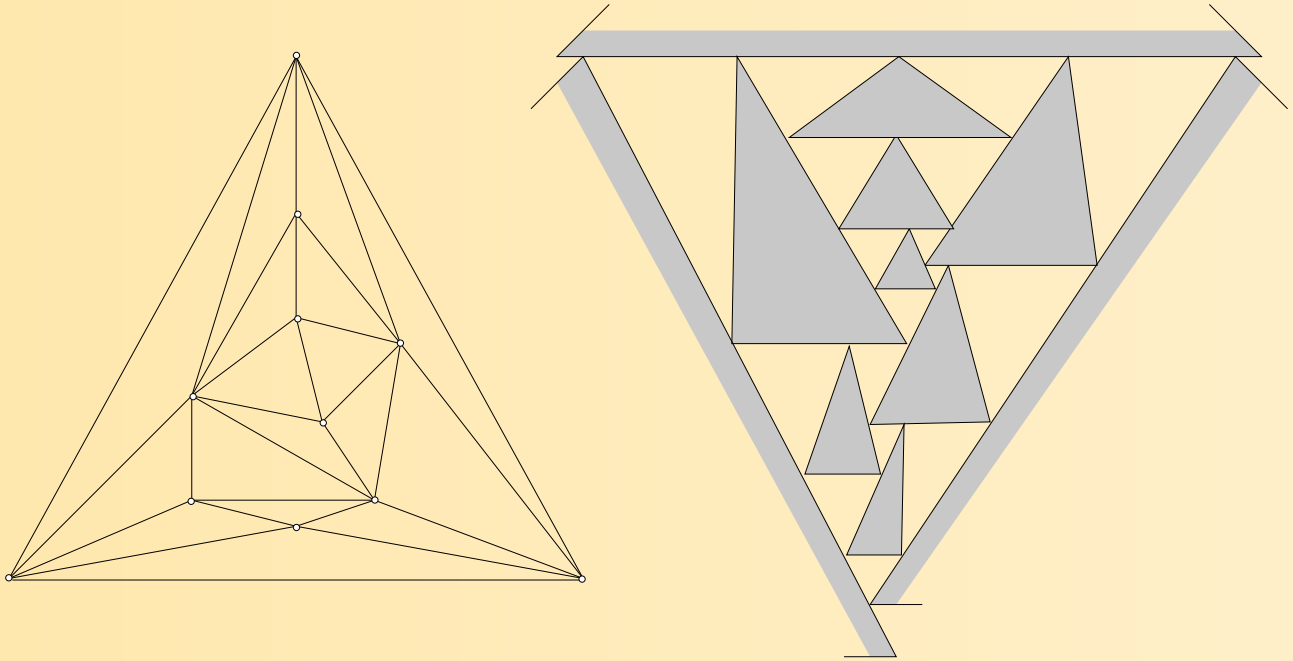
Gonçalves, Lévêque, Pinlou (GD 2010) observe that the conjecture follows from a corollary of Schramm's "Monster Packing Theorem" from *Combinatorially Prescribed Packings and applications to Conformal and Quasiconformal Maps*.

Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{P_a, P_b, P_c\}$. For each interior vertex v of T prescribe a convex set Q_v containing more than one point. Then there is a contact representation of T with homothetic copies.

Remark. In general homothetic copies of the Q_v can degenerate to a point. Gonçalves et al. show that this is impossible if T is 4-connected.

Combinatorial Methods

de Fraysseix, de Mendez and Rosenstiehl construct triangle contact representations of triangulations.



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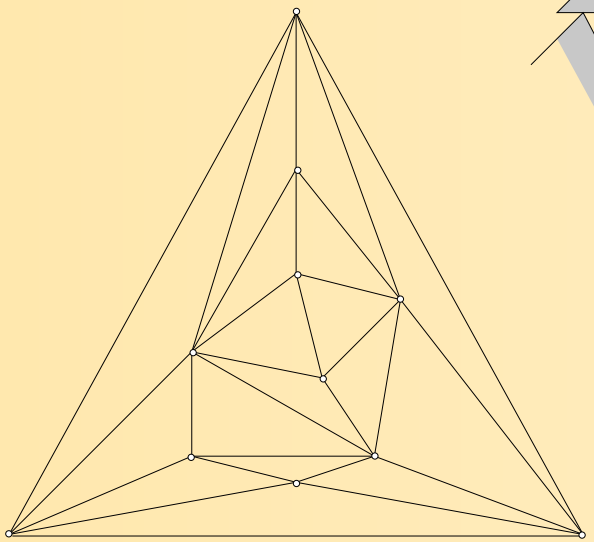
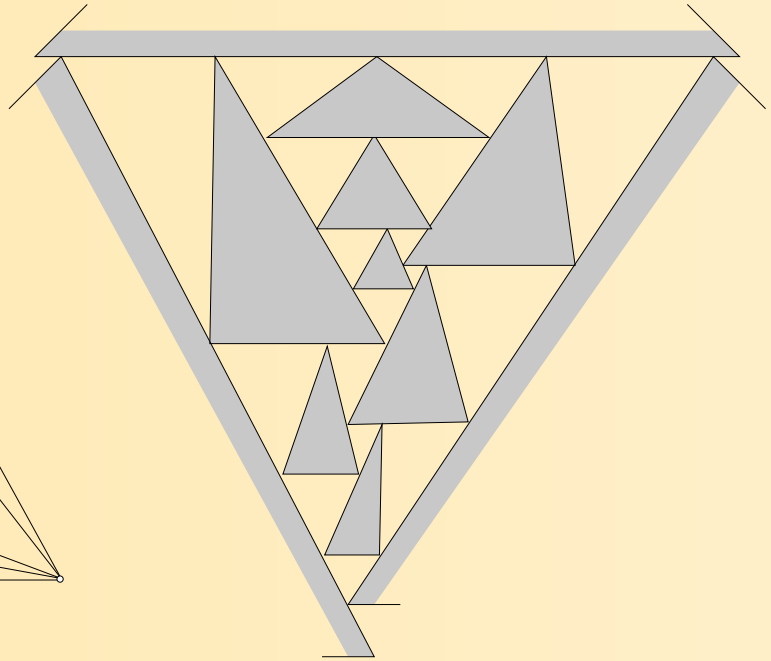
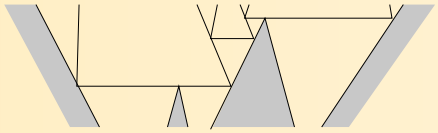




Construct along a good ordering of vertices:

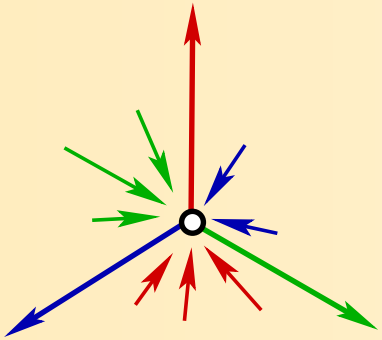


Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.

A coloring and orientation of the interior edges of G with colors $1, 2, 3$ is a Schnyder wood of G iff

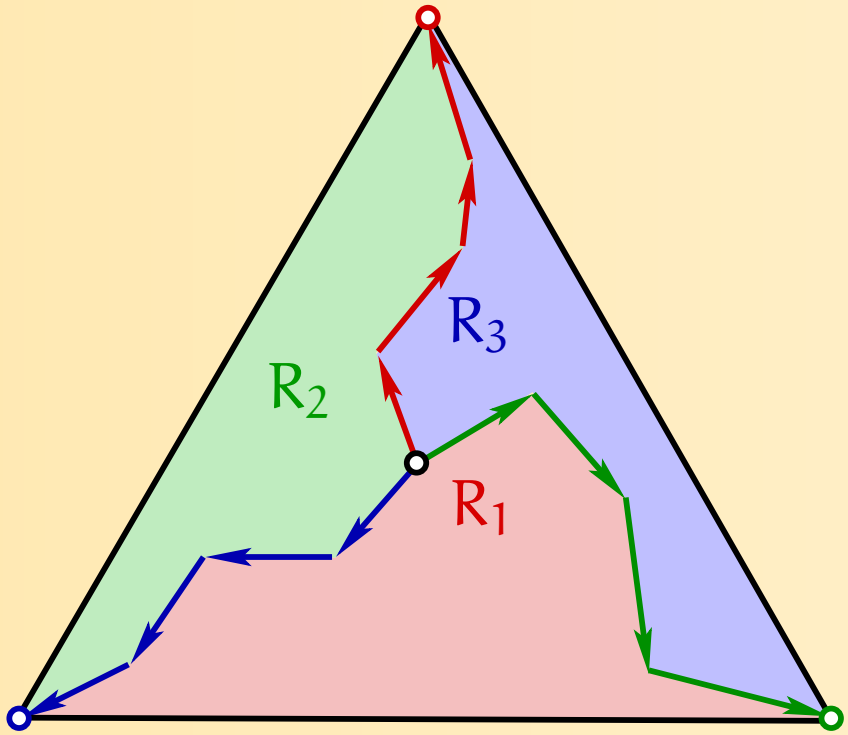
- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

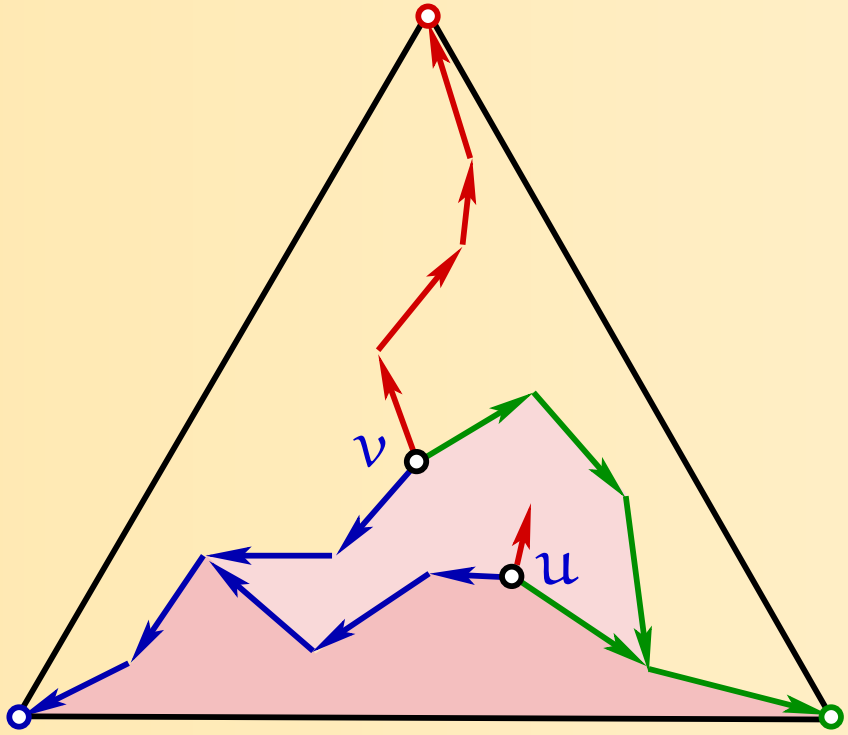
Schnyder Woods - Regions

- Every vertex has three distinguished regions.



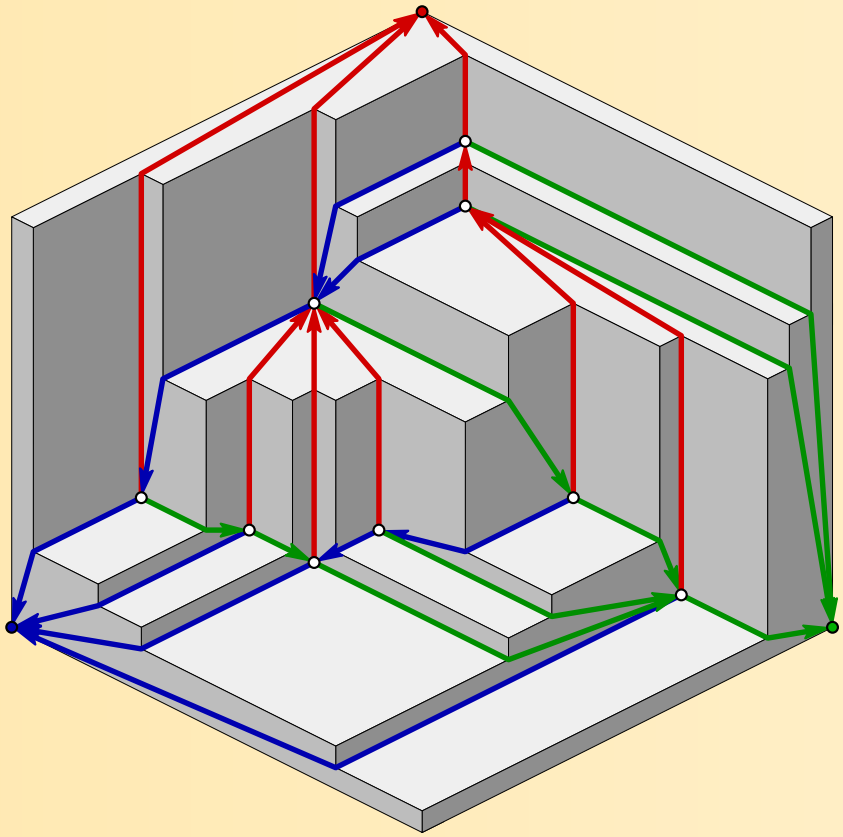
Schnyder Woods - Regions

- If $u \in R_i(v)$ then $R_i(u) \subset R_i(v)$.

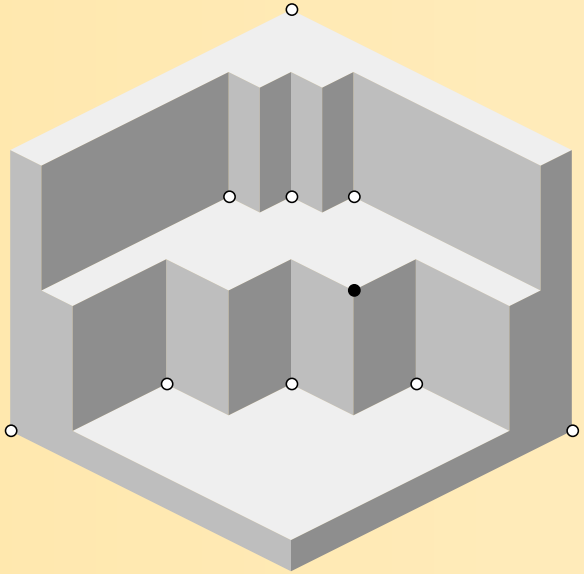


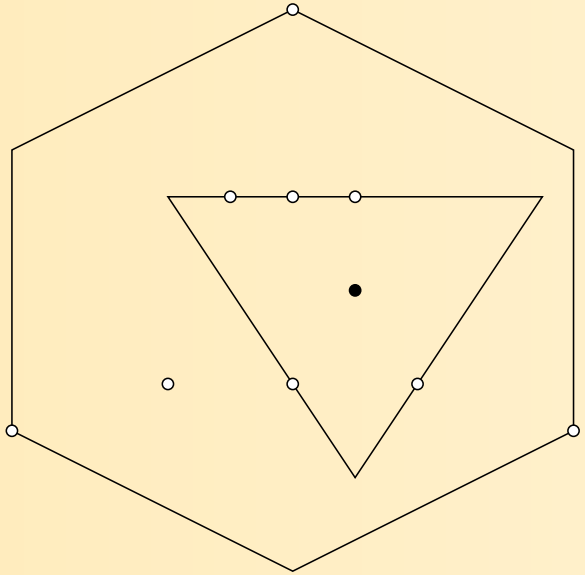
Face Count Coordinates

Using all three face count coordinates we obtain an embedding of T on an orthogonal surface.



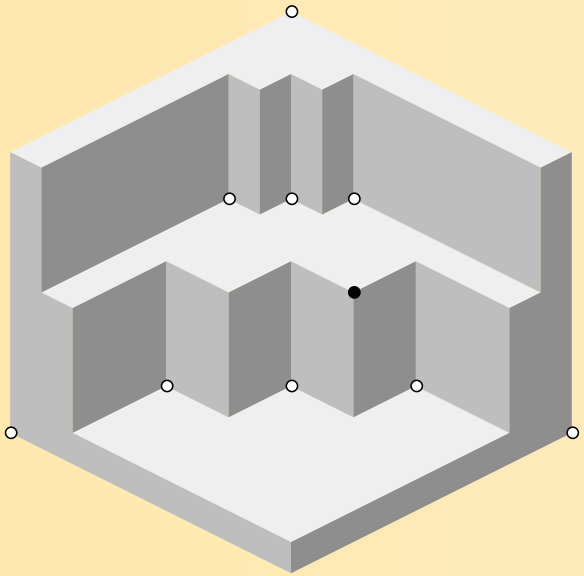
Cutting Orthogonal Surfaces

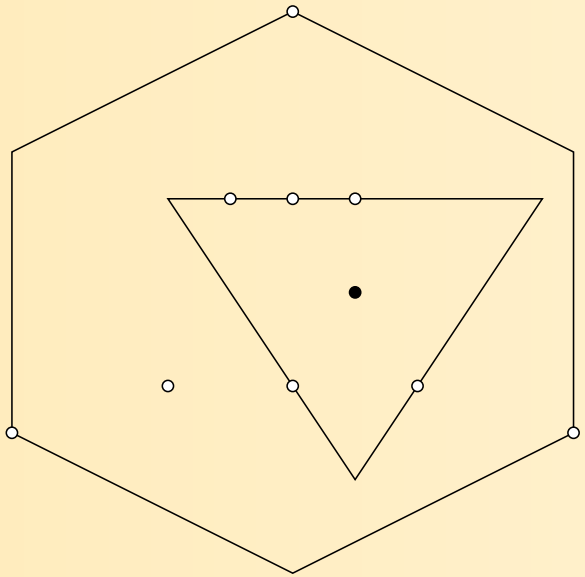




Region-count yields vertex-coplanar orthogonal surfaces.

Cutting Orthogonal Surfaces

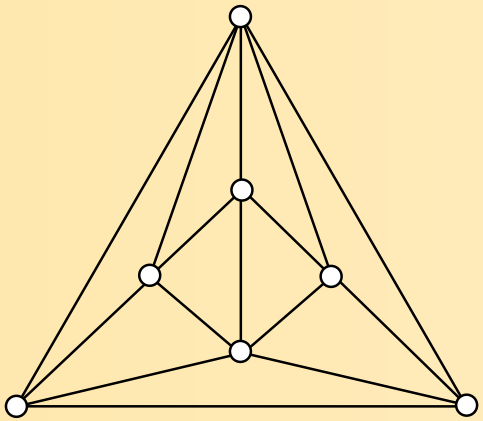


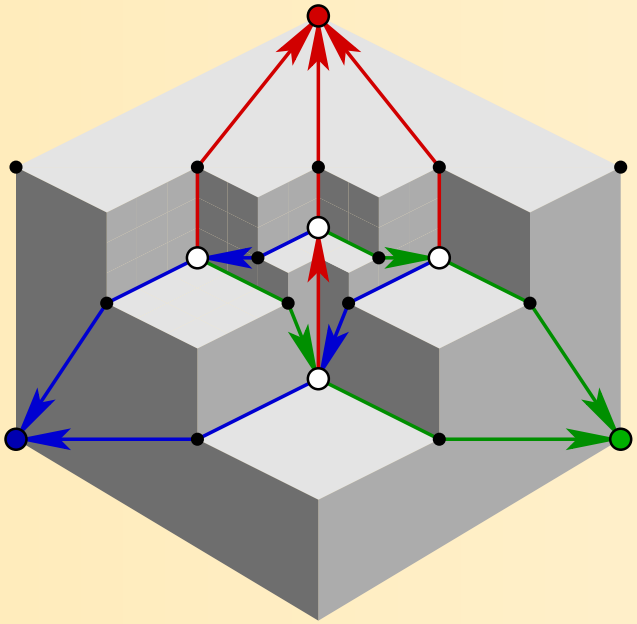


Region-count yields vertex-coplanar orthogonal surfaces.

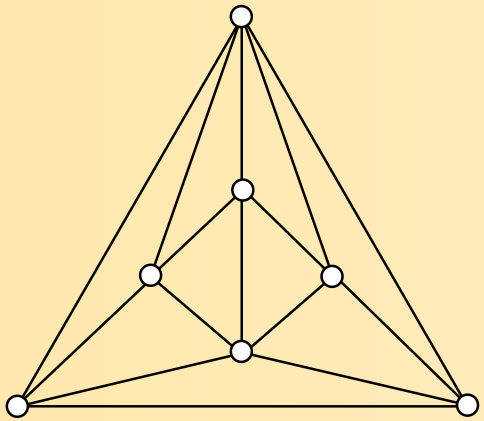
Theorem. Every coplanar orthogonal surface can be obtained via weighted region-count.

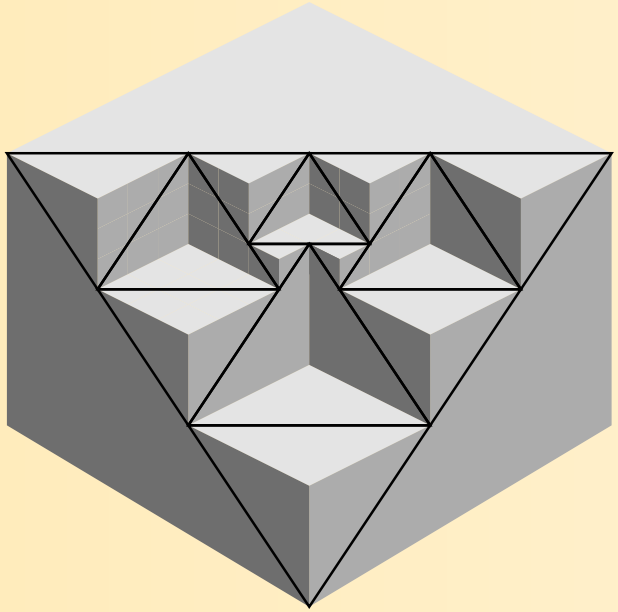
Edge-Coplanar Orthogonal Surfaces



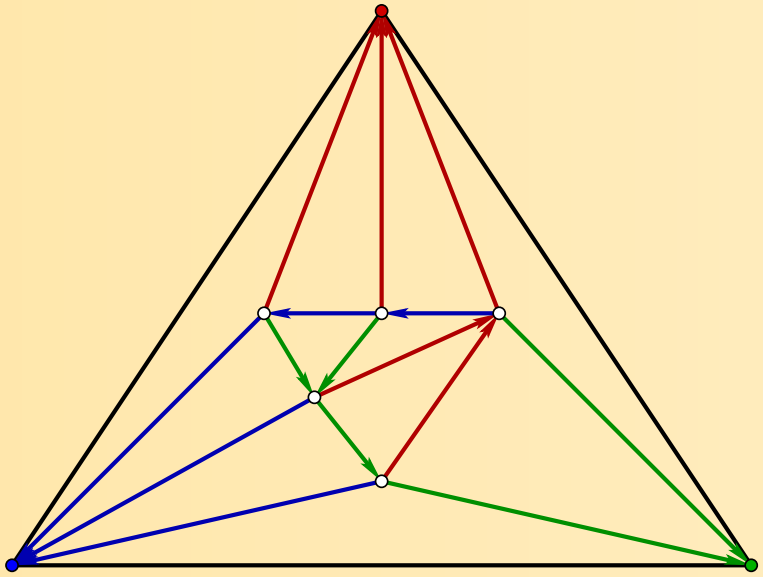


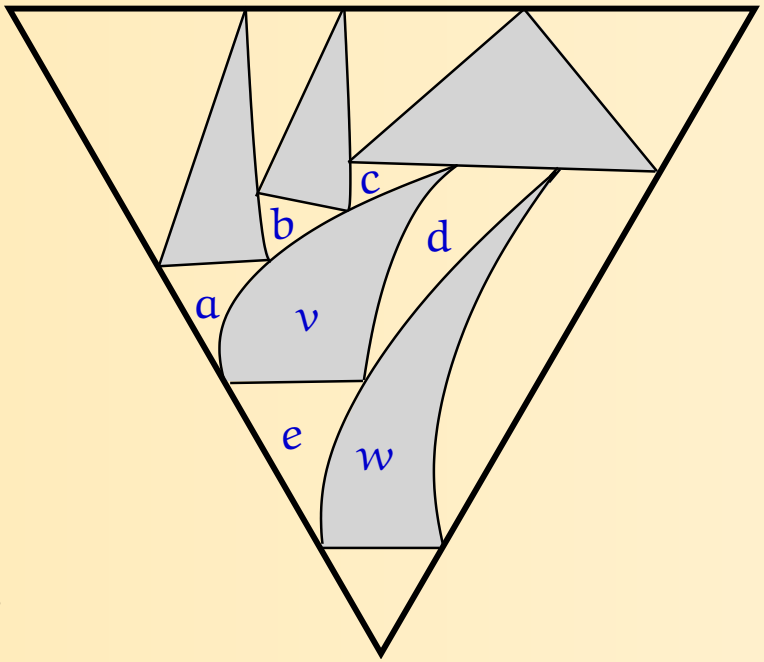
Edge-Coplanar Orthogonal Surfaces





Triangle Contacts and Equations





A Schnyder wood induces an *abstract triangle contact representation*. Equations for the sidelength:

$x_a + x_b + x_c = x_v$ and $x_d = x_v$ and $x_e = x_v$ and $x_d + x_e = x_w$ and $\ldots$

Solving the Equations

Theorem. The system of equations has a uniqe solution.

The proof is based on counting matchings.

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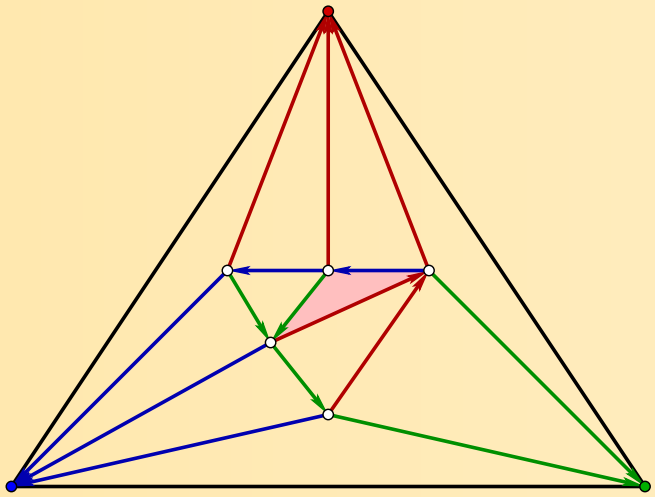
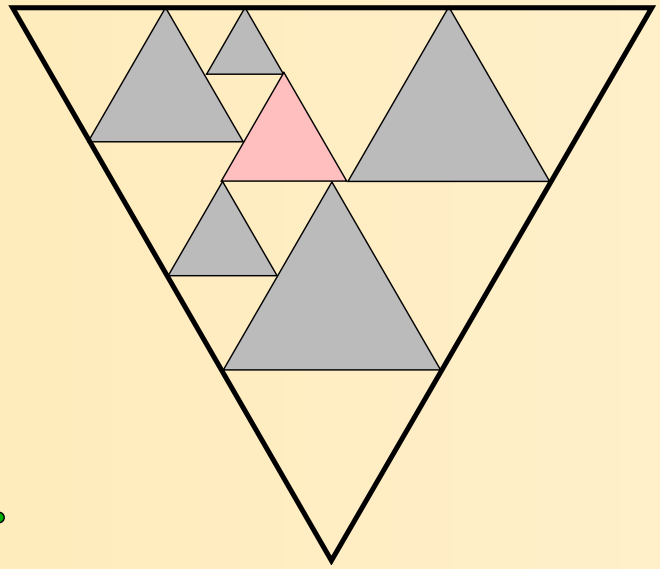
The proof is based on counting matchings.
In the solution some variables may be negative.
Still the solution yields a triangle contact representation.





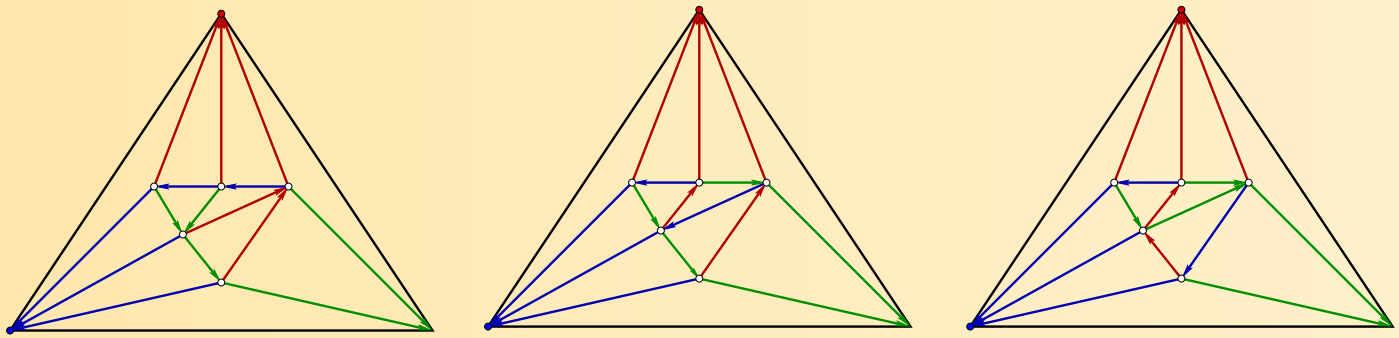
Flipping Cycles

Proposition. The boundary of a negative area is a directed cycle in the underlying Schnyder wood.

From the bijection

$$\text{Schnyder woods} \iff \text{3-orientations}$$

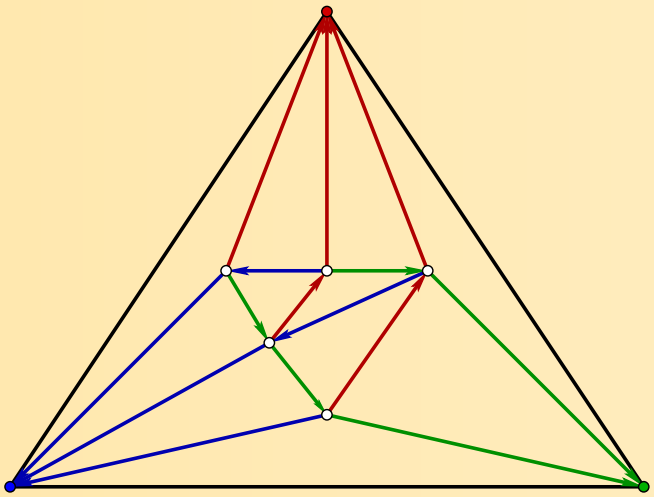
we see that cycles can be reverted (flipped).

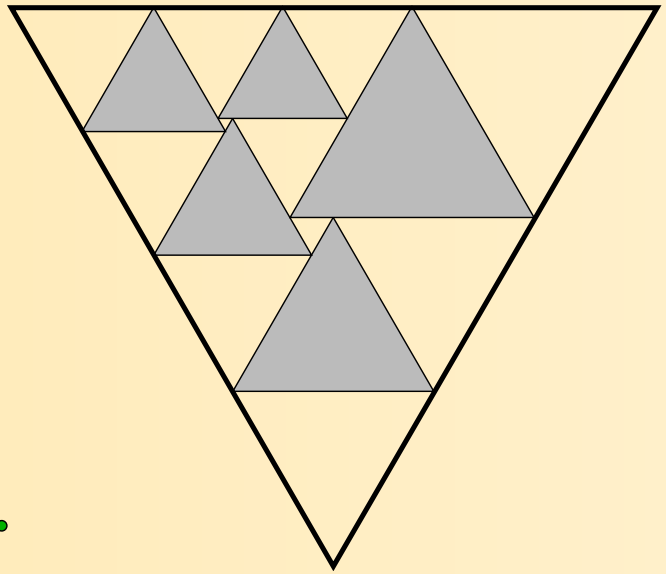


Resolving

A new Schnyder wood yields new equations and a new solution.

Theorem. A negative triangle becomes positive by flipping.





The Status

- We have no proof that the process always ends with a homothetic triangle representation.
- From a program written by Julia Rucker we have strong experimental evidence that it does.

The End

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Thank you.