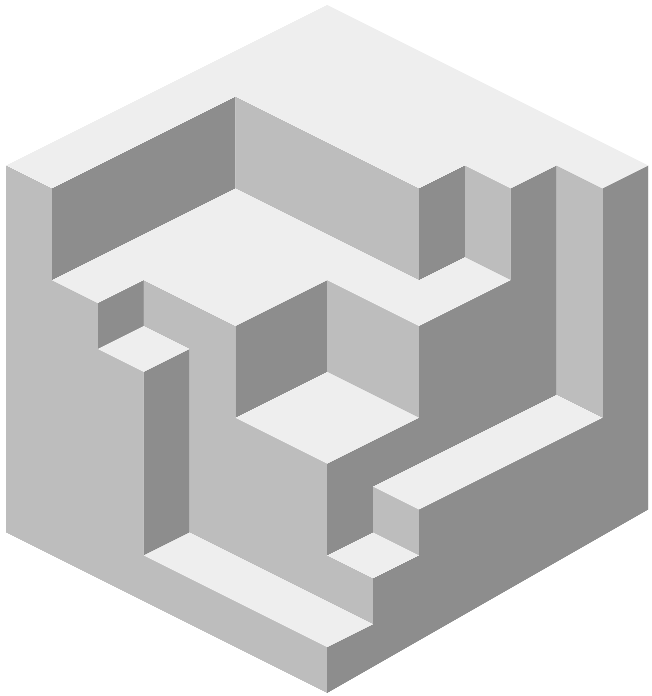
Geometry of Orthogonal Surfaces

CanaDAM 2007
Banff
May 28, 2007



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contains joint work with
Sarah Kappes and
Florian Zickfeld

Orthogonal Surfaces

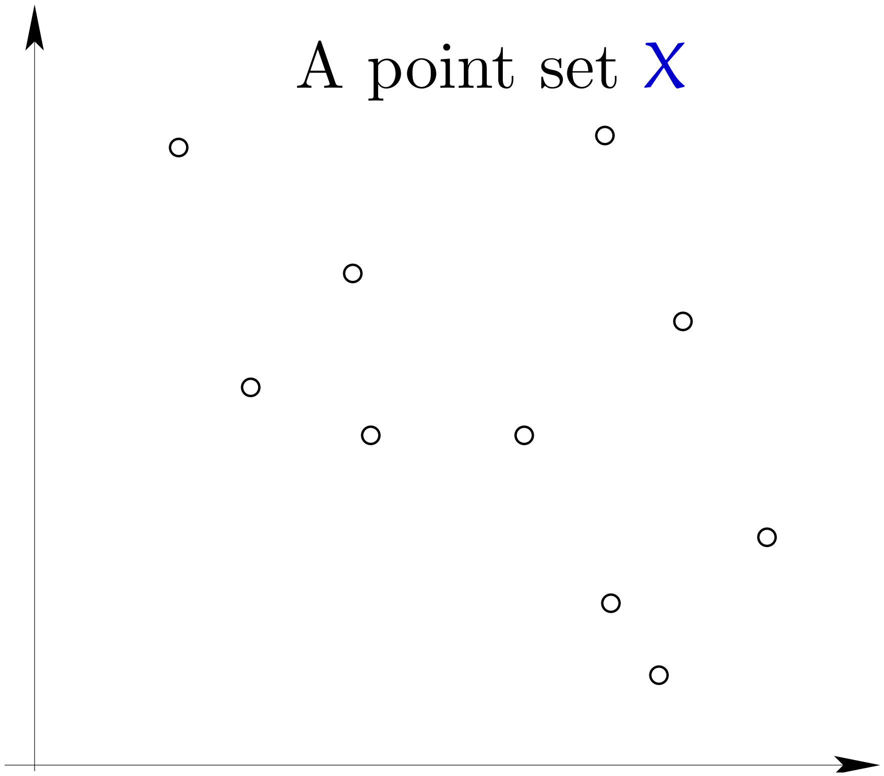
The dominance order on $\mathbb{R}^d$:

$$x \le y \iff x_i \le y_i \quad \text{for} \quad i = 1, .., d$$

The orthogonal surface $\mathbf{S}_X$ generated by a finite $X \subset \mathbb{R}^d$ is the boundary of the filter

$$\langle X \rangle = \{ y \in \mathbb{R}^d : \exists x \in X \text{ with } y \ge x \}.$$

An Example in 2-D



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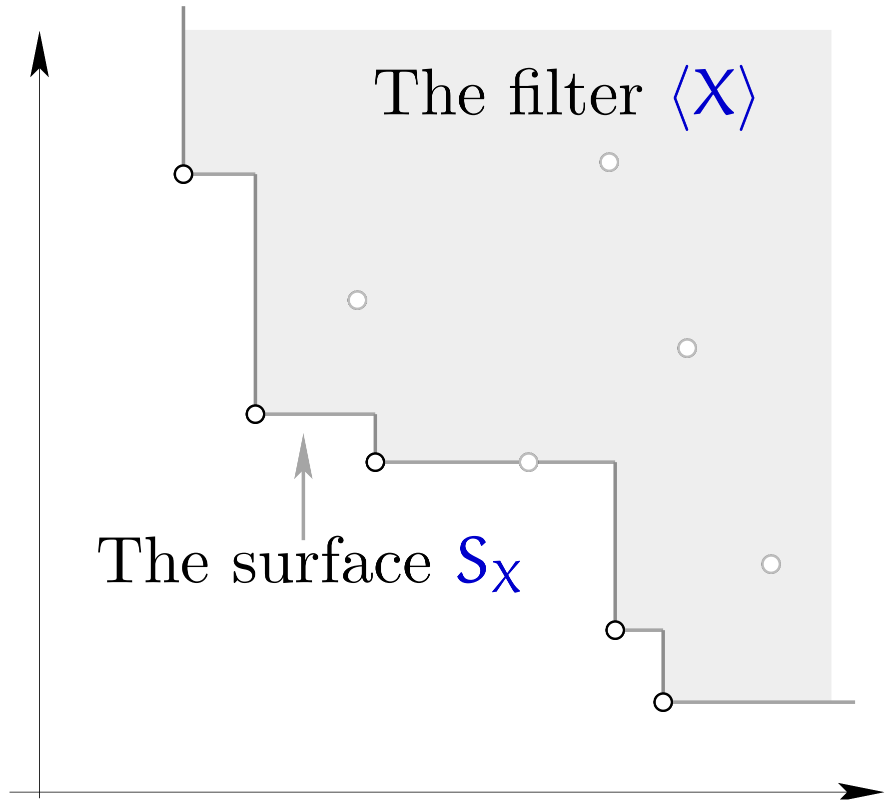
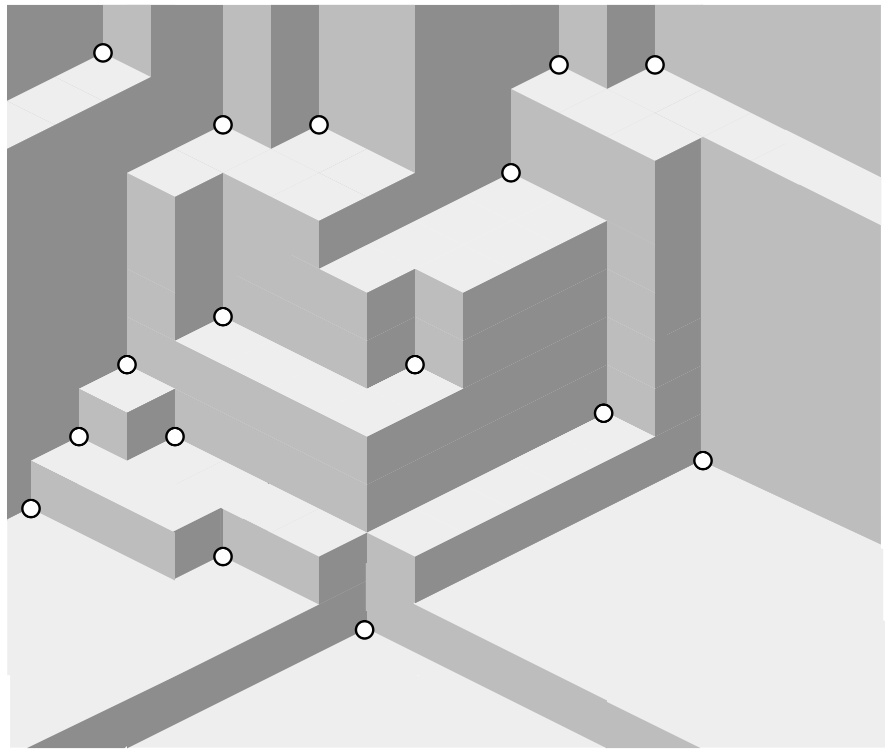
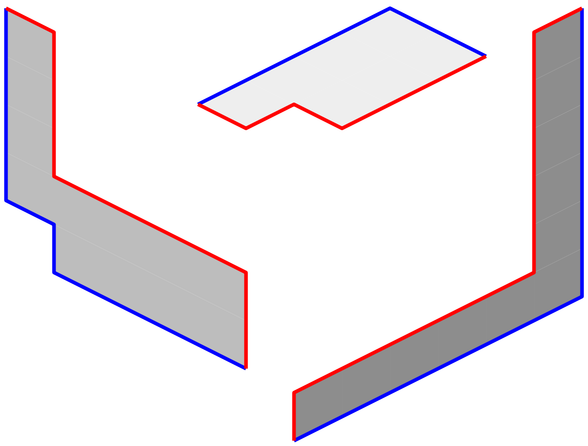


An Example in 3-D



Flats and their Features

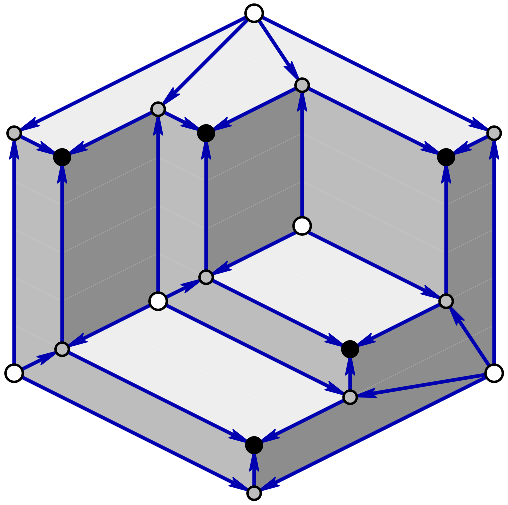


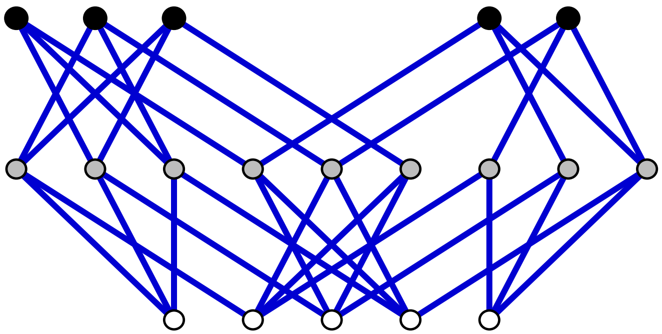
A Flat is a connected piece of the intersection with an orthogonal hyperplane.

Upper and lower boundary are pieces of orthogonal surfaces of dimension one less.

Characteristic points

Characteristic points are points incident to flats of all colors.





The CP-order is the dominance order on characteristic points.

More Terminology

Surface S_X is generic if every flat has a single minimum.

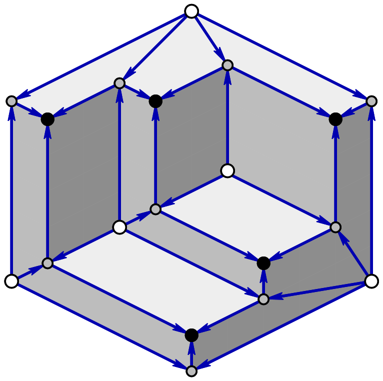
Surface S_X is suspended if it has exactly d unbounded flats.

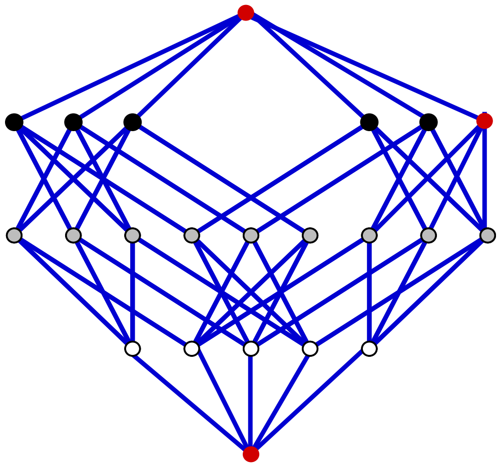
Surface S_X is rigid if the CP-order is ranked

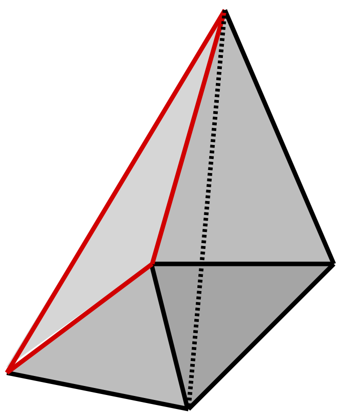
Fact. A generic surface is rigid.

Connections with Polytopes I

Theorem [Scarf 1979]**.**
The CP-order of a generic suspended orthogonal surface in $\mathbb{R}^{d}$ is isomorphic to the face lattice of a simplicial d-polytope (minus **0**, **1** and one facet).







Connections with Polytopes II

Theorem [Schnyder 1989].
The face lattice of every simplicial 3-polytope (minus $\mathbf{0}$, $\mathbf{1}$ and one facet) is the CP-order of a generic suspended orthogonal surfaces in $\mathbb{R}^3$.

Theorem [Felsner 2003].
The face lattice of every 3-polytope (minus $\mathbf{0}$, $\mathbf{1}$ and one facet) is the CP-order of a rigid suspended orthogonal surfaces in $\mathbb{R}^3$. (Implies the Brightwell-Trotter Theorem.)

Realizability Problems

- Which orthogonal surfaces in $\mathbb{R}^d$ have a corresponding d-polytope? (Scarf: generic; YES).
- Which d-polytopes have a corresponding orthogonal surface in $\mathbb{R}^d$? (Schnyder/F: $d = 3$; YES).

Realizability Problems

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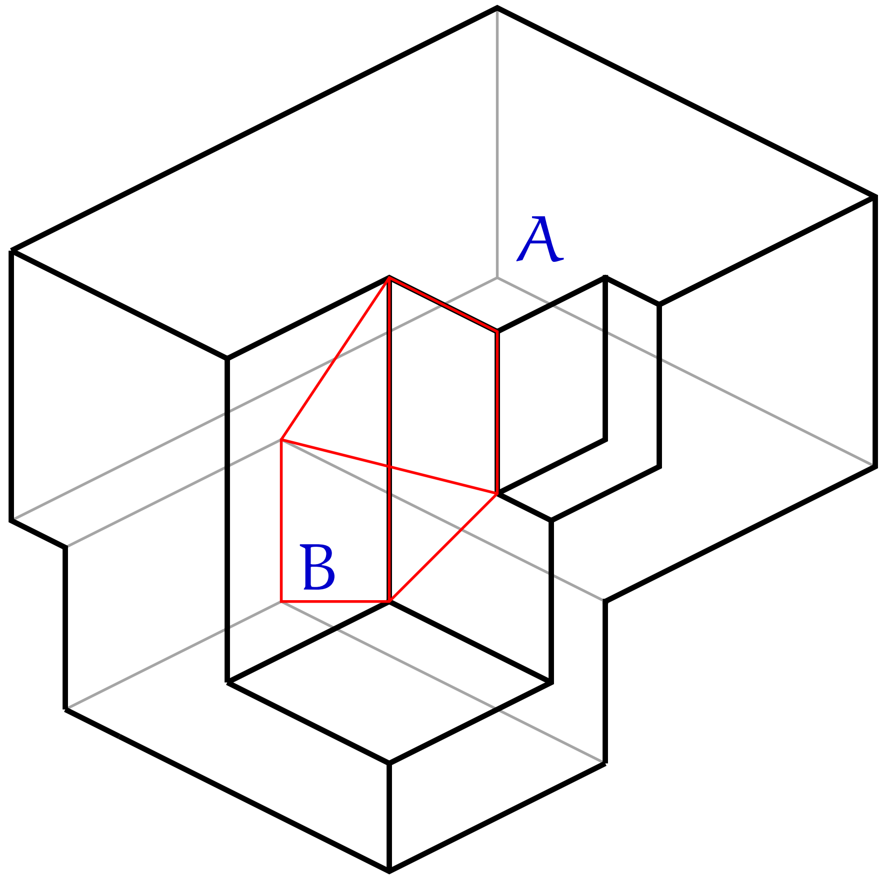
- Which d-polytopes have a corresponding orthogonal surface in $\mathbb{R}^d$? (Schnyder/F: $d = 3$; YES).

- generic suspended orthogonal surface in $\mathbb{R}^d$ $\not\longleftrightarrow$ simplicial d-polytope.

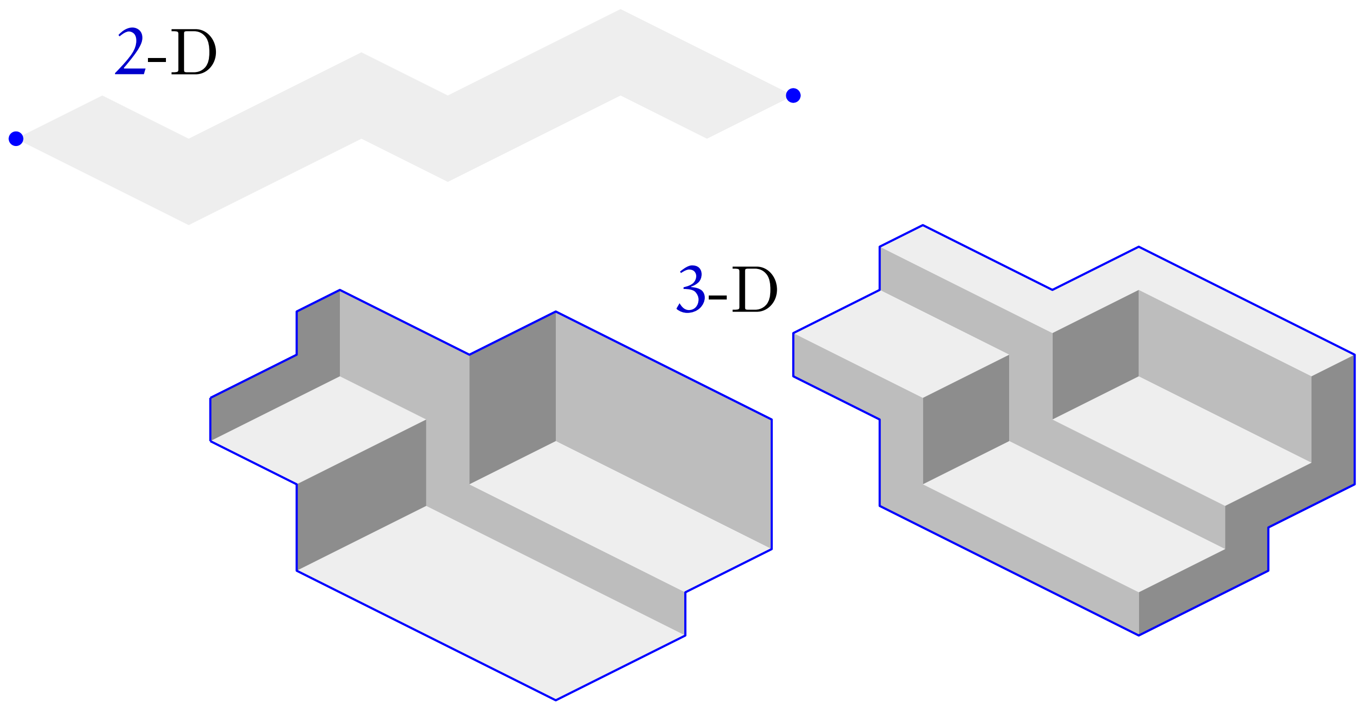
  Proof. Neighbourly 4-polytopes have complete graphs as 2-skeletons, but $\dim(K_{13}) = 5$.

Bad Surfaces



$A = 1, 3, 3, 1$
$B = 3, 1, 3, 2$
$C = 4, 2, 1, 3$
$D = 2, 4, 2, 4$

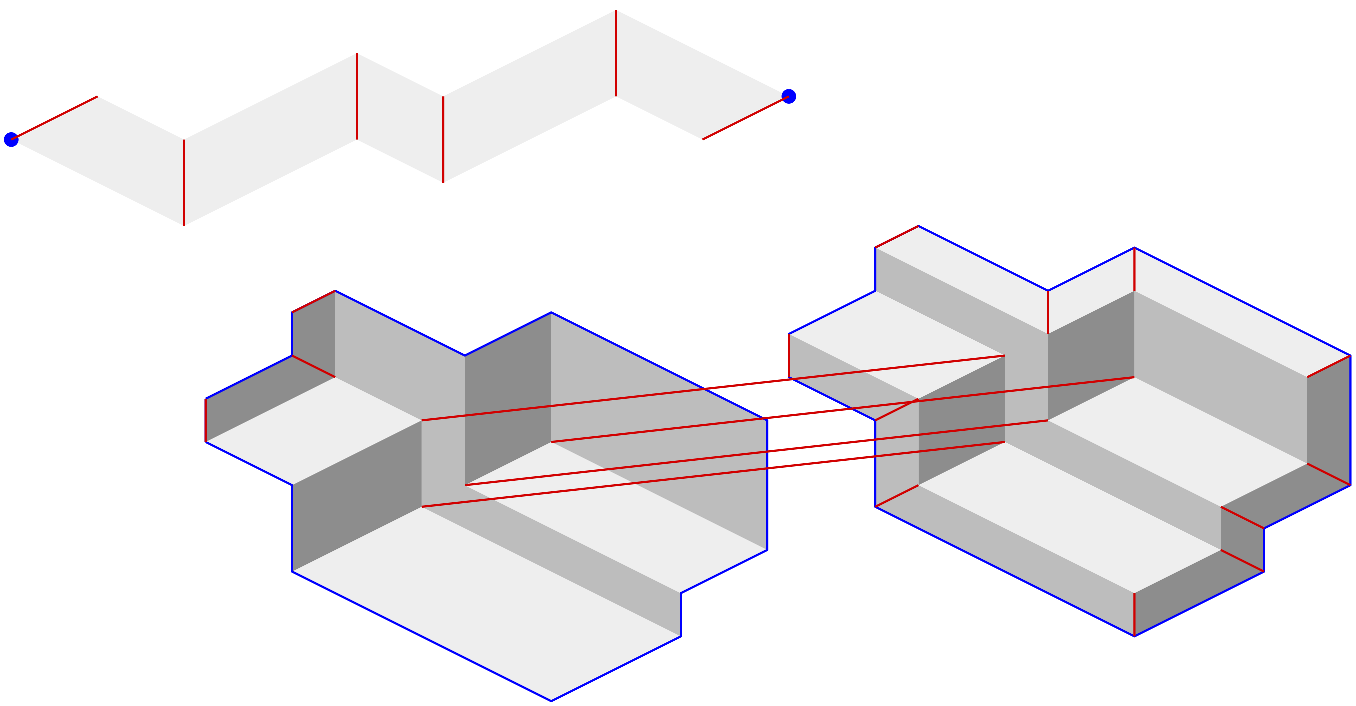
Good Surfaces



Theorem [Kappes 06]. If all flats of a surface are generic, cogeneric or parallel, then the extended CP-order is a CW-poset.

Conjecture. In this situation the CP-order is polytopal.

Good Surfaces



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Good Polytopes

Let a d-polytope P be realizable by an orthogonal surface in $\mathbb{R}^d$

- If F is a simplicial face and P^s is obtained by stacking a new vertex above F, then P^s is realizable.
- If x is a simple vertex and P^c is obtained by cutting x, then P^c is realizable.

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- If P has a suspended realization, then the pyramid over P is realizable.

- If P has a suspended realization, then the product of P with a path is realizable.

Part II

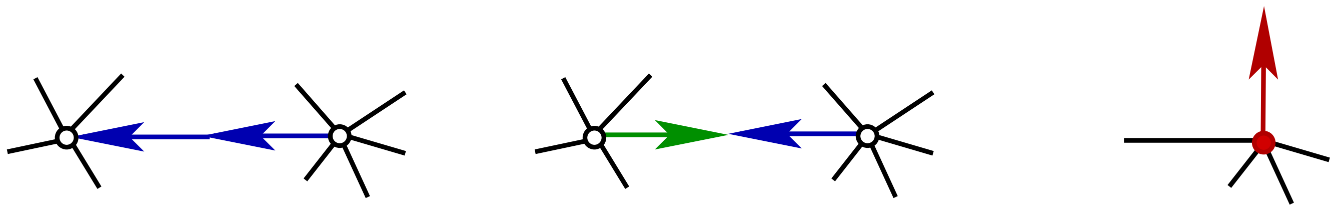
Planar Graphs and Orthogonal Surfaces in 3-D

Schnyder Woods

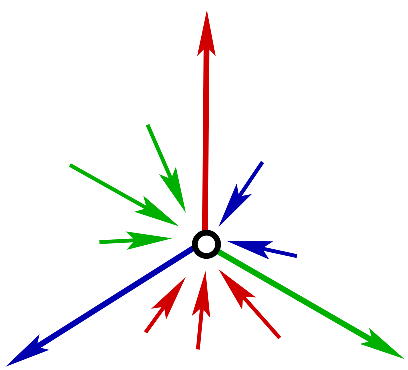
G a 3-connected planar graph with special vertices a_1, a_2, a_3 on the outer face.

Axioms for 3-coloring and orientation of edges:

(W1 - W2) Rule of edges and half-edges:

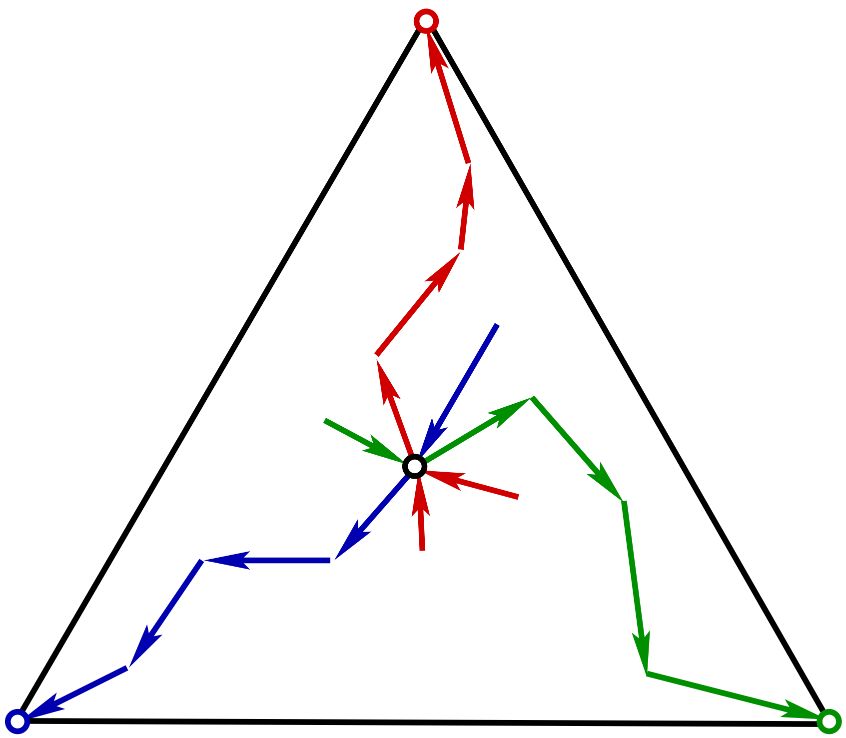


(W3) Rule of vertices:



(W4) No face boundary is a directed cycle in one color.

Schnyder Woods - Paths and Regions



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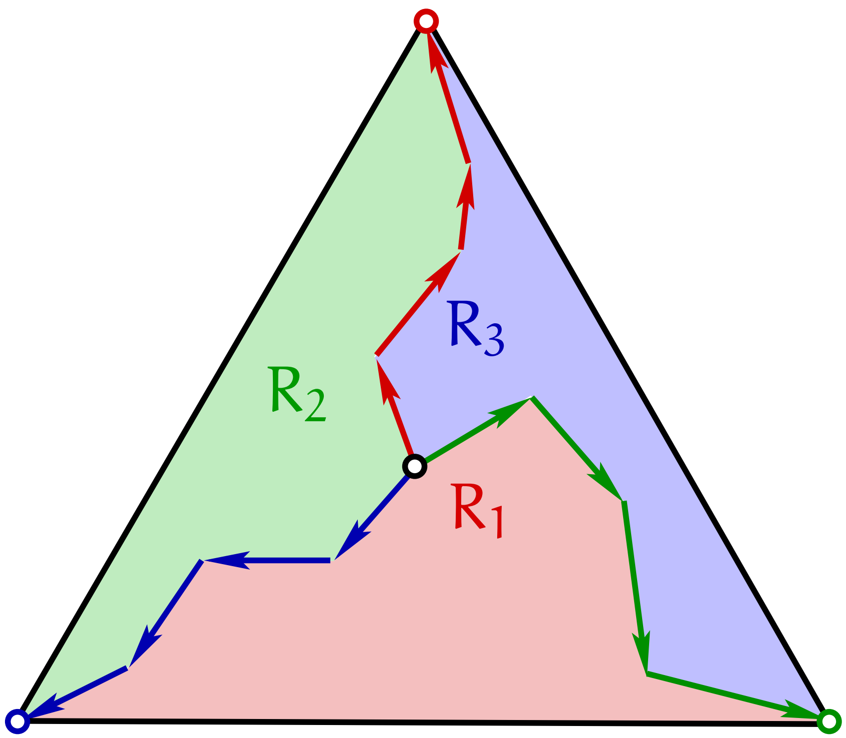
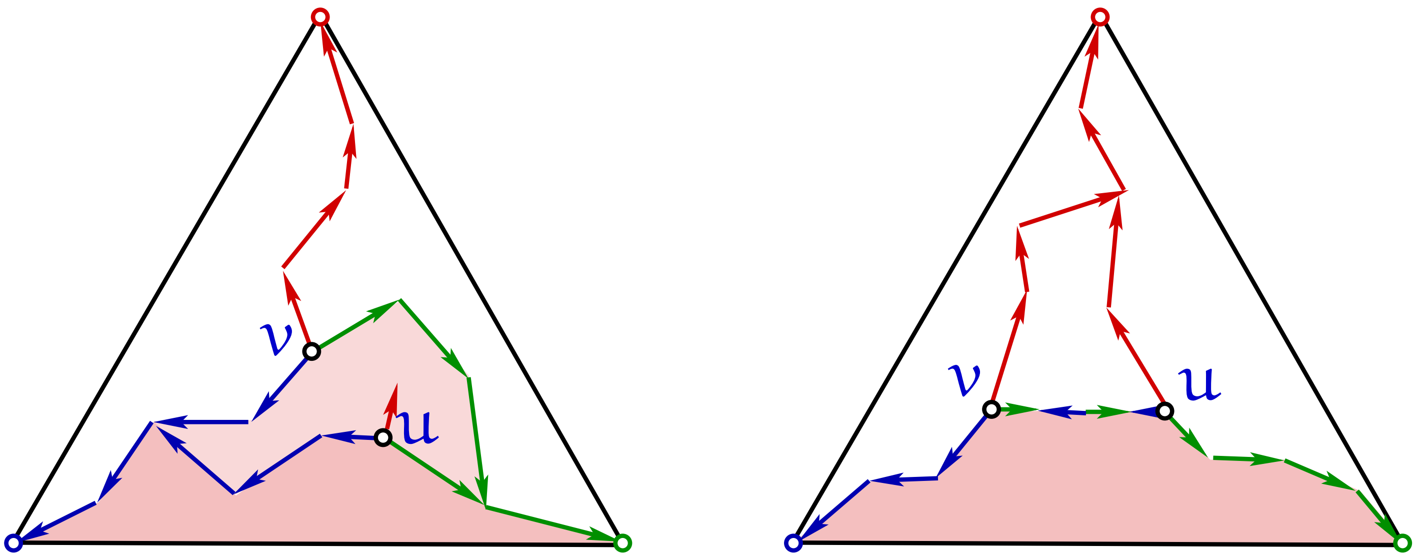


Schnyder Woods - Regions

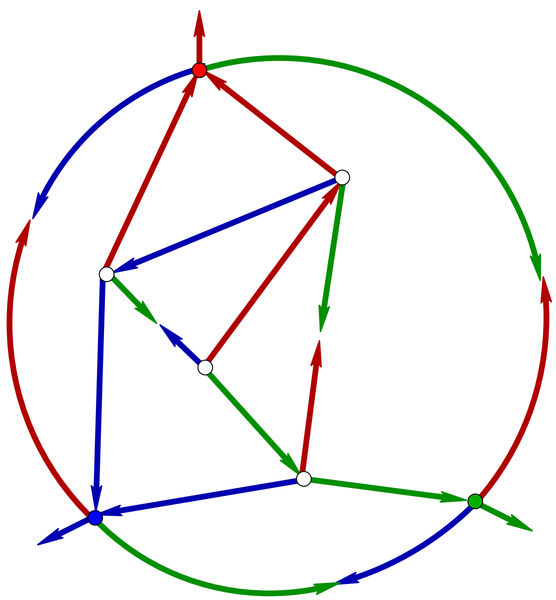
- If $u \in R_i^o(v)$ then $R_i(u) \subset R_i(v)$.
- If $u \in \partial R_i(v)$ then $R_i(u) \subseteq R_i(v)$
  (equality, iff there is a bi-directed path between u and v.)

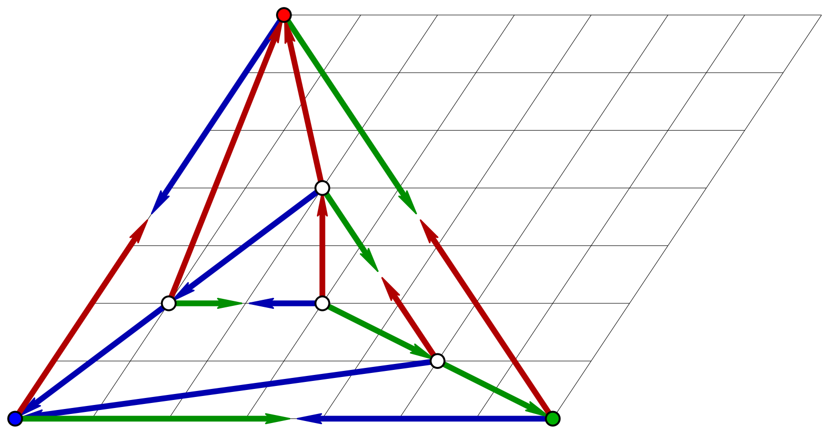


Counting Faces in Schnyder Regions I

$\phi_i(v) = \#$ faces in $R_i(v)$.

Embed v at $(\phi_1(v), \phi_2(v))$

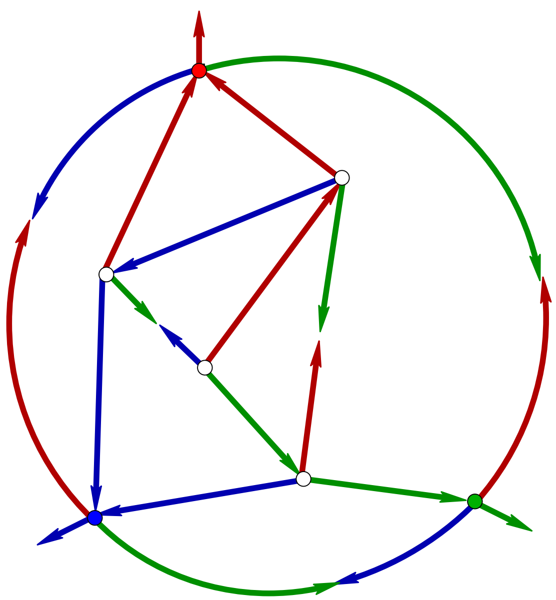


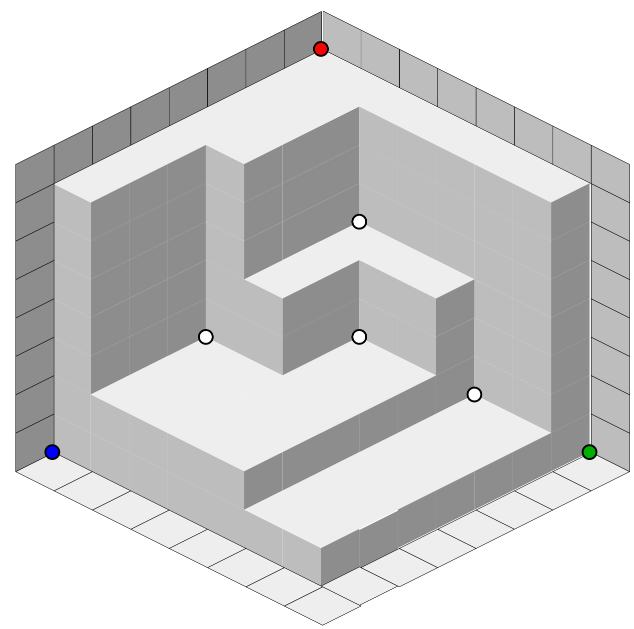


Theorem. 3-connected planar graphs admit convex drawings on the $(f-1) \times (f-1)$ grid.

Counting Faces in Schnyder Regions II

Embed v at $(\phi_1(v), \phi_2(v), \phi_3(v))$





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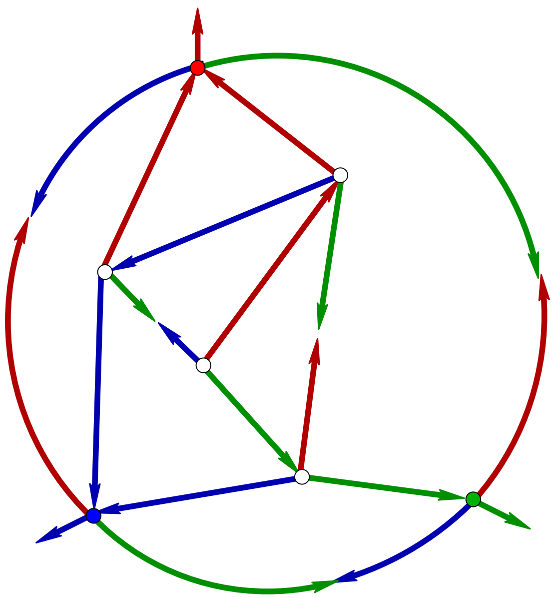
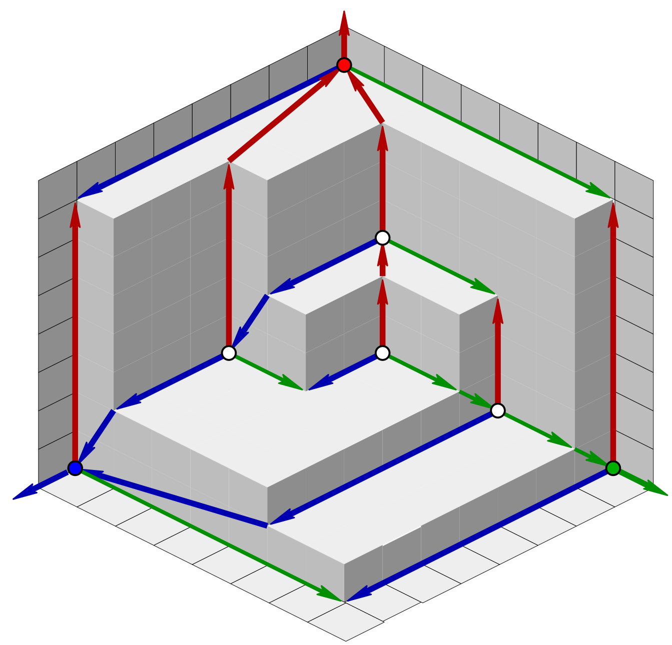
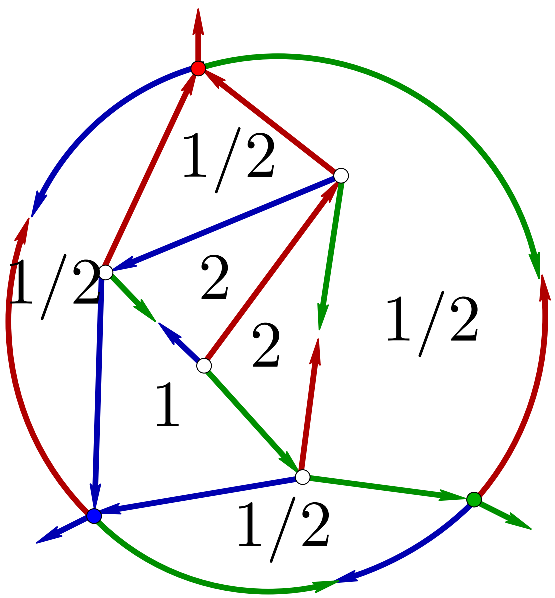
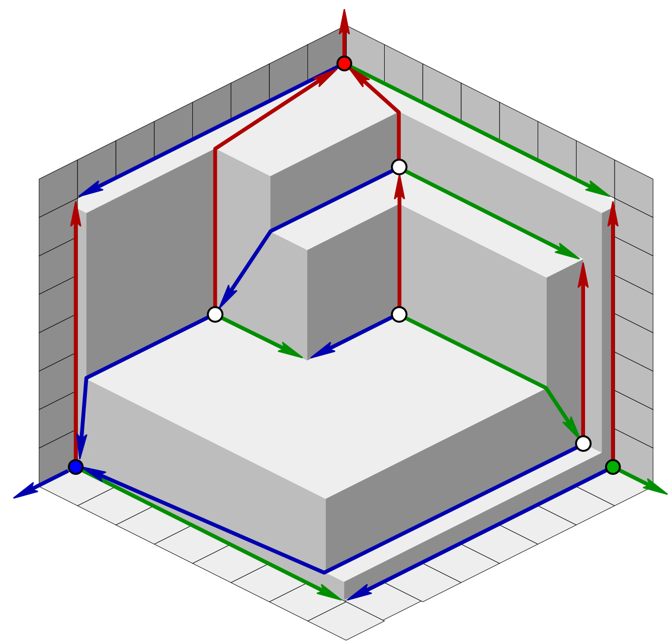




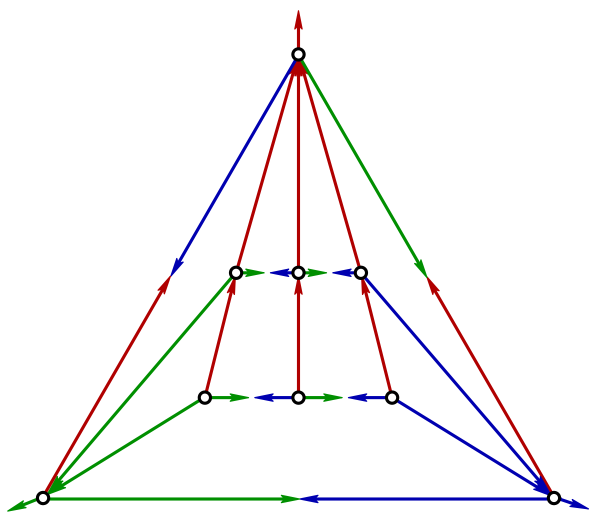
Weighted Count

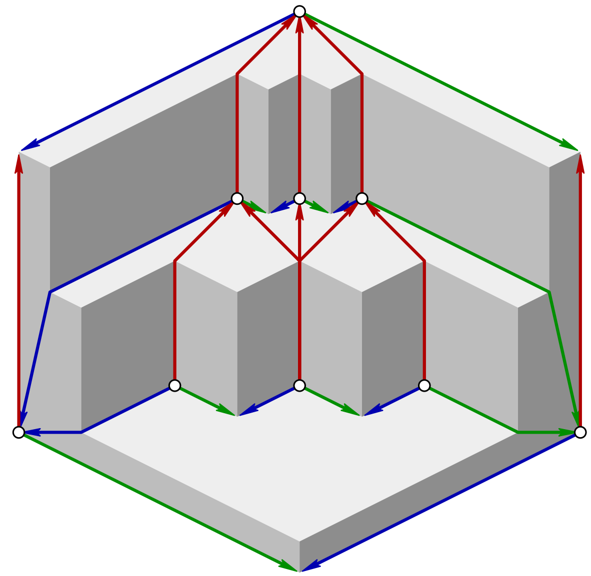
Theorem. Every coplanar orhogonal surface supporting a Schnyder wood S can be obtained from weighted regions.





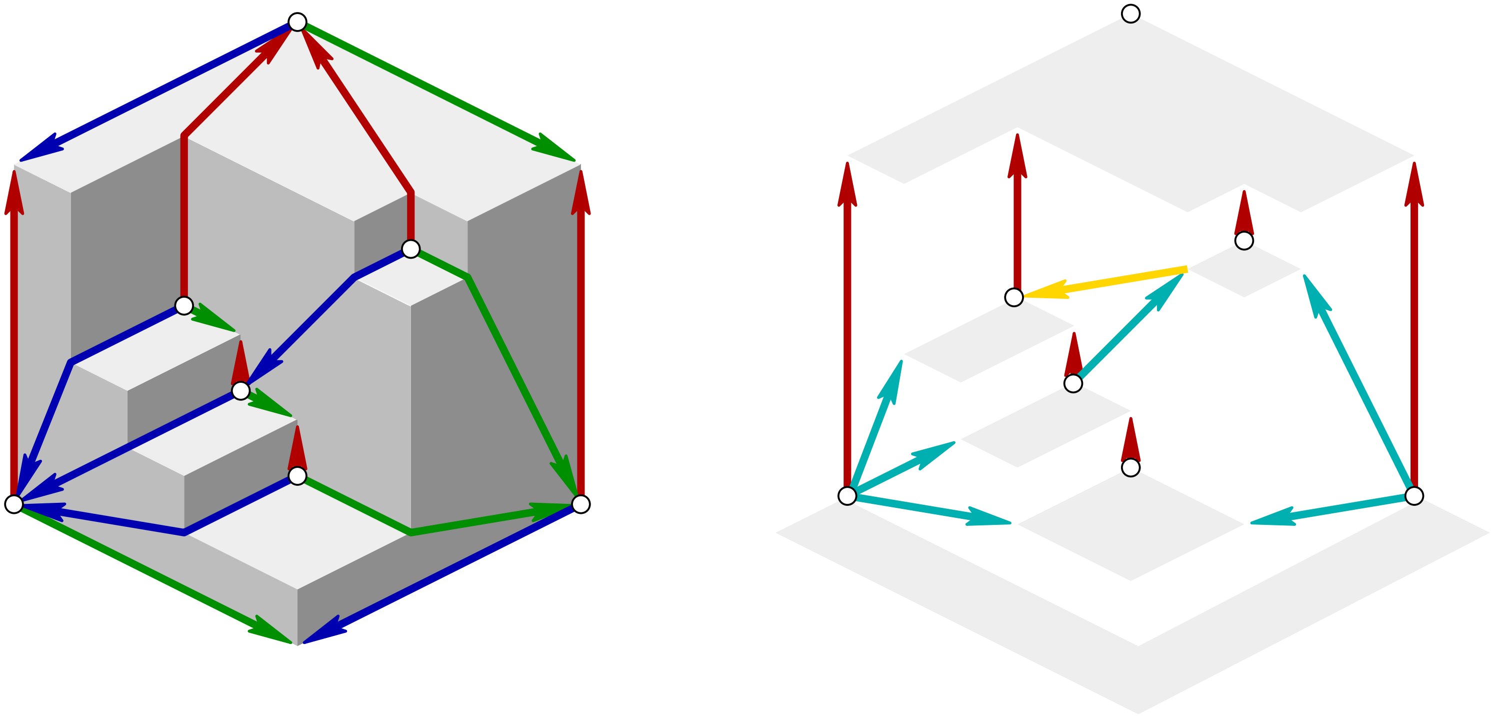
Non-rigid Surfaces





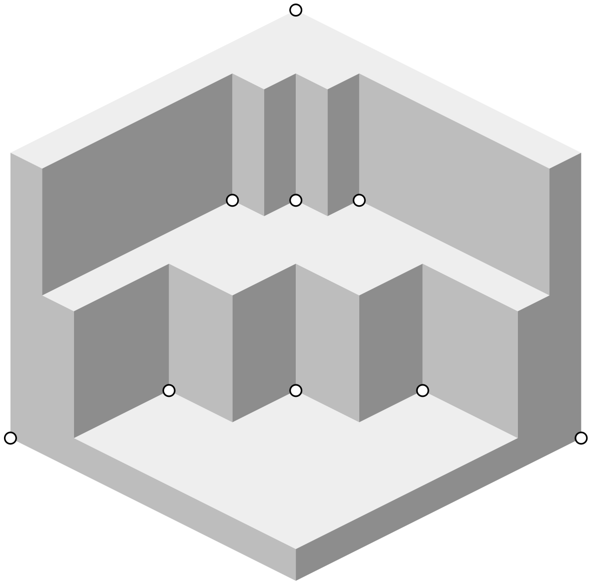
Counting faces doesn't yield an order preserving embedding of $\mathcal{F}_G \setminus F_\infty$ into $\mathbb{R}^3$.

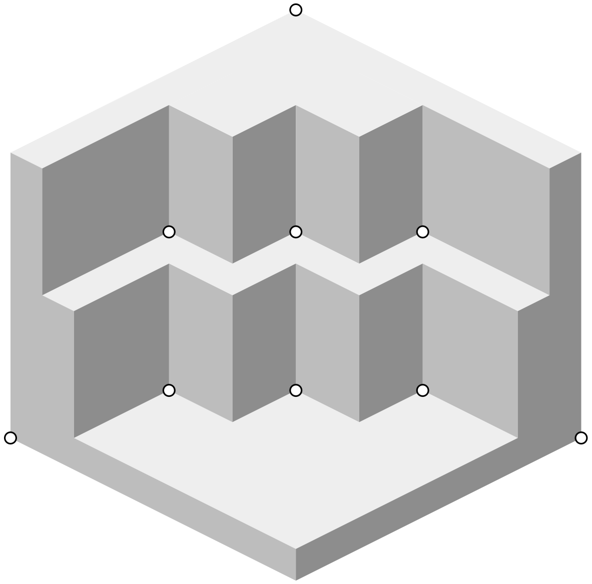
Relations for Flats

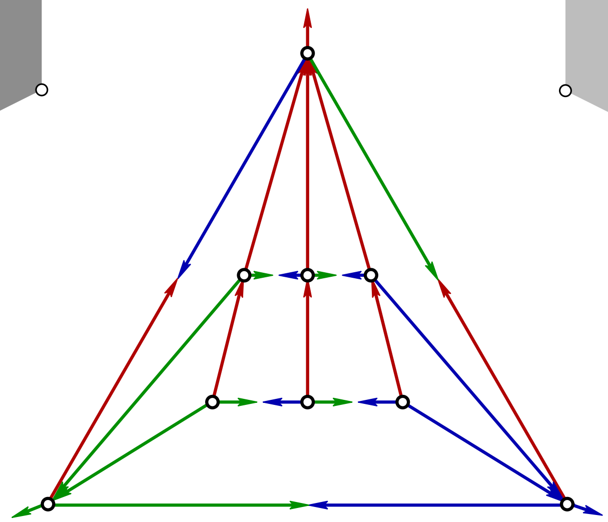


Lemma. The arrow-relation on flats of one color is acyclic.

Shifting Flats

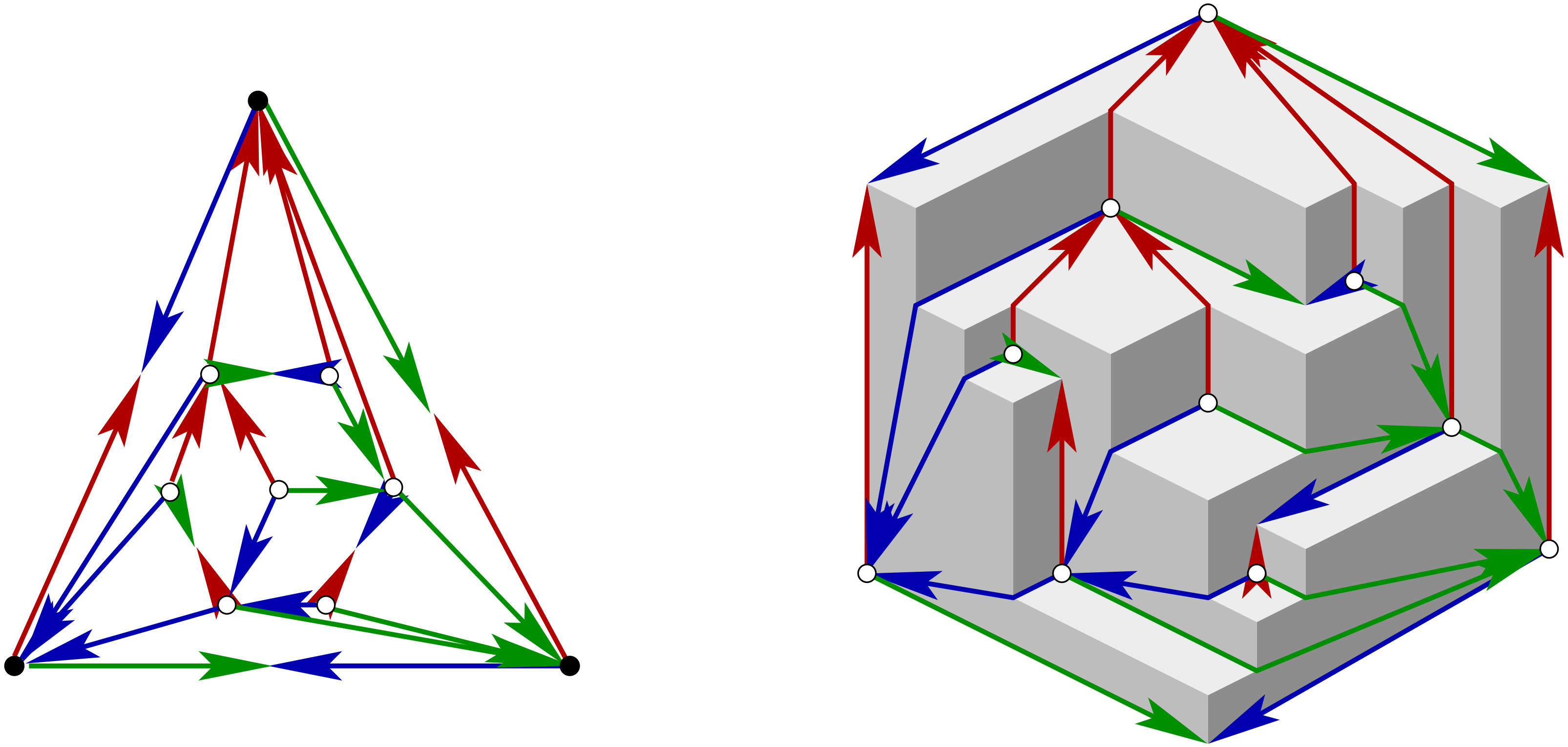






$\Longrightarrow$ The Brightwell-Trotter Theorem.

Rigid or Coplanar



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Thank you.