Contact Representations of Planar Graphs: Combinatorial Structure and Algorithm $\mathcal{X}$

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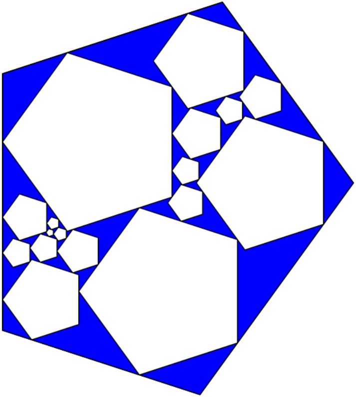
Perugia



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Outline

Segment Contacts – Quadrangulations

Algorithm $\mathcal{X}$ does its job

Algorithm $\mathcal{X}$

The abstract view

Homothetic Triangle Contact Representations

Monster packing and Algorithm $\mathcal{X}$

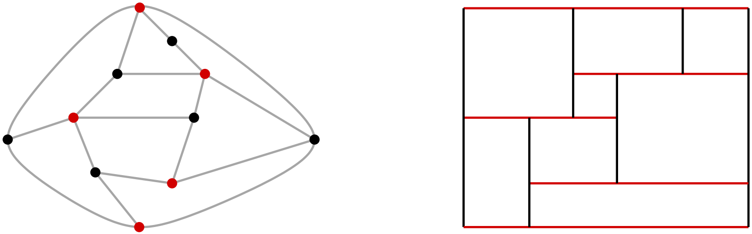
Square Contact Representations

Including an existence proof

More Contact Representations – Pentagons

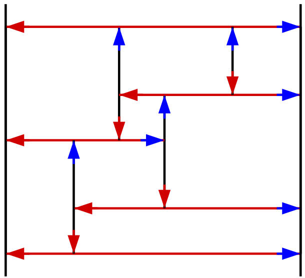
5-color trees and the coin theorem

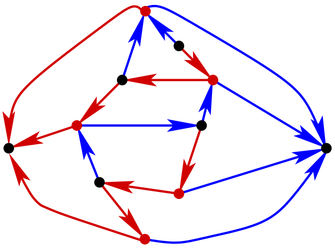
Segment Contact Representations



A quadrangulation with a segment contact representation.

Induced separating Decompositions



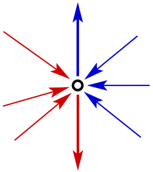


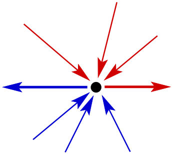
The contact representation induces a separating decomposition on the quadrangulation.

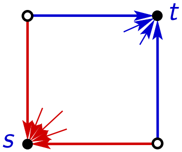
Separating Decompositions

$G = (V, E)$ a plane quadrangulation with outer face $F = \{s, x, t, y\}$.

A coloring and orientation of the edges of G with colors red and blue is a separating decomposition of G iff





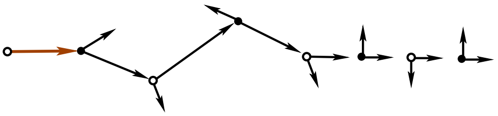


Separating Decompositions and 2-Orientations

Theorem.
Separating decompositions and 2-orientations are in bijection.

Proof.

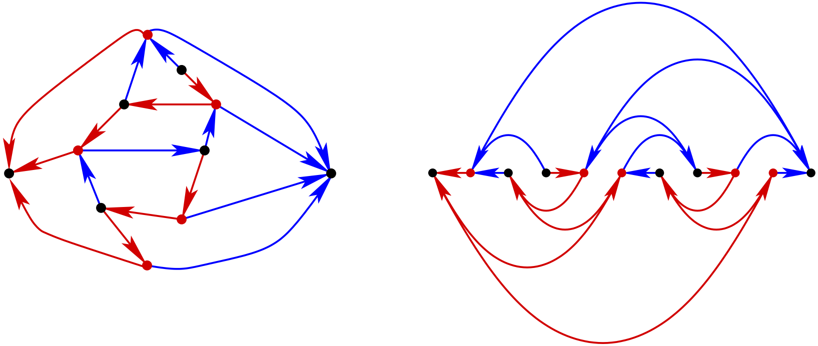
- Define the path of an edge:



- The path is simple (Euler), hence, ends in a sink:
  in the red s or in the blue t.

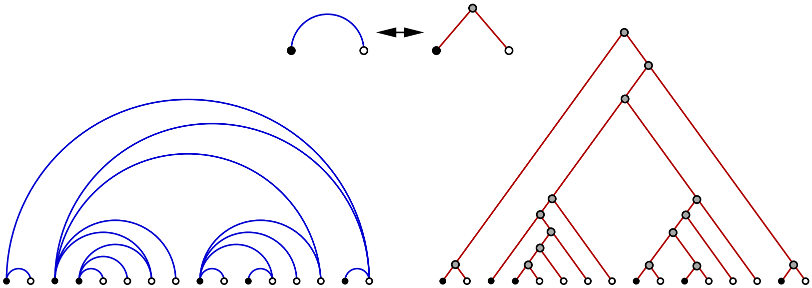
Sketch: Compute Segment Contact Representations

- Compute a separating decomposition.
- Separate the two trees — 2-page book embedding.



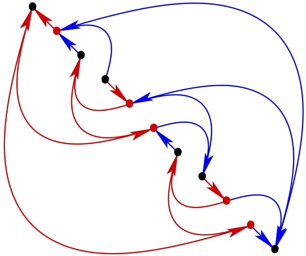
Alternating and Full Binary Trees

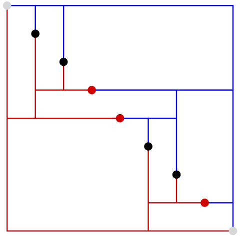
Proposition. There is bijection between on-sided and binary trees that preserves types (left/right) of nodes.



Sketch: Compute Segment Contact Representations

- The two binary trees obtained from the separating decomposition fit together.



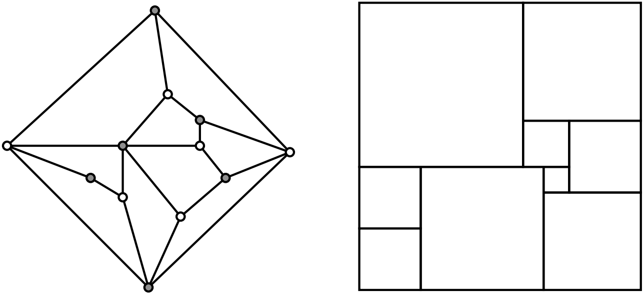


More Problems for Segment Contact Representations

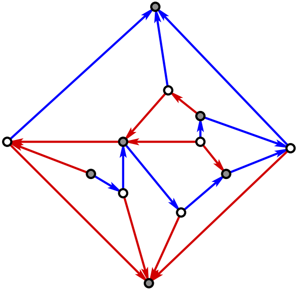
Add some conditions.

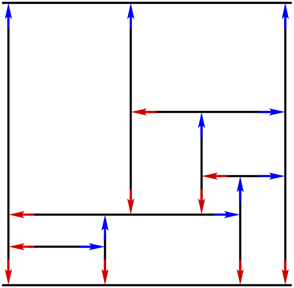
- Find a representation in a square such that all intersect the diagonal.
  (We just saw a solution)
- Find a representation such that all inner rectangles are squares.
  The dissection of rectangles into squares
  Brooks, Smith, Stone and Tutte 1940.
- Find a representation such that each inner rectangle has its own prescribed aspect ratio.
  (aspect ratio universality)

Segment Contact Squarings



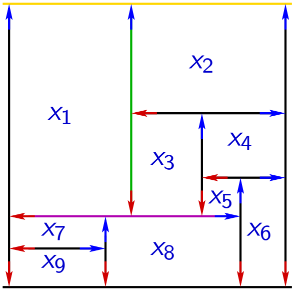
Computing a Segment Contact Squaring





Step I: Compute a separating decomposition on Q.
This describes the contacts of the segments.
(combinatorial or abstract contact representation)

Equations



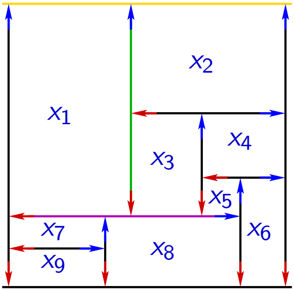
- $x_1 + x_2 = 1$
- $x_1 = x_2 + x_3$
- $x_1 + x_3 + x_5 = x_7 + x_8$

  $x_2 = x_3 + x_4$

  $\vdots$

Step II: Set up a linear system of equations: $A_S \cdot x = e_1$

Equations



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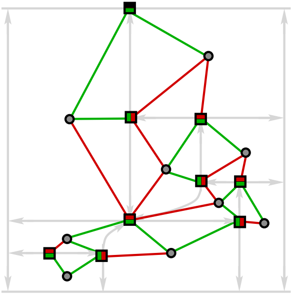
$\vdots$

Step II: Set up a linear system of equations: $A_S \cdot x = e_1$

- A_S is a square matrix.

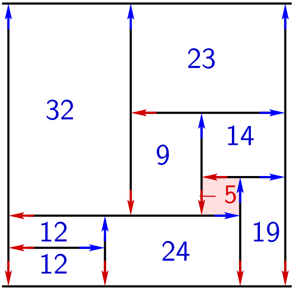
Theorem. $\det(A_S) \neq 0$.

The Determinant

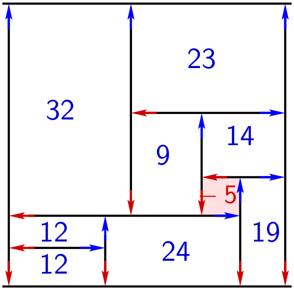


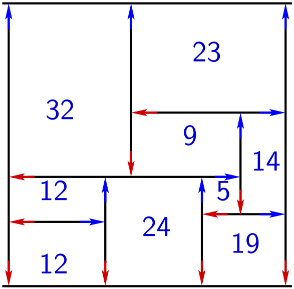
- A_S is a bipartite adjacency matrix of a bipartite graph H.
- $\det(A_S) = \sum_\pi sign(\pi) \ \prod a_{i,\pi(i)}$
- $\det(A_S) = \sum_M sign(M)$, with M perfect matching of H.
- $sign(M) = sign(M')$ for all M and M' perfect matching.
  Face cycles: length 4, odd number of minus signs.
- H has a perfect matching.

Solve the System



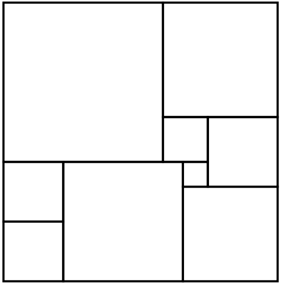
Flip the Negative





- The boundary of the negative contains no complete segment.
- Hence, the boundary is a directed cycle in the separating decomposition
- Flip boundaries of negative areas to get the good separating decomposition and the squaring.

The Squaring



- The method also works for given aspect ratios.
- Alternative algorithms exist.
  Example: electric flow in bipolar networks.

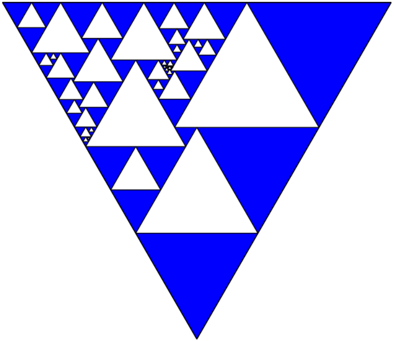
Algorithm $\mathcal{X}$

Algorithm $\mathcal{X}$ a Coarse Description

Aim for a contact representation of G.

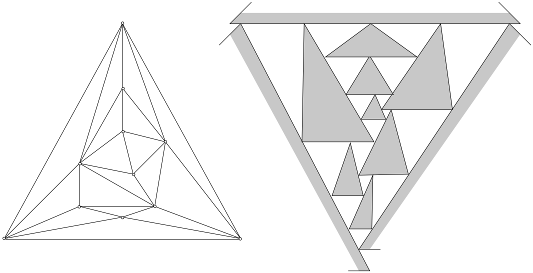
- Compute a combinatorial representation – directed graph D.
- Extract linear equations: $A_D = e_1$.
  - Show that A_D is square and $\det A_D \neq 0$.
- Solve the system.
  Solution positive – Done.
  - Show that negative variables in the solution correspond to directed cycles in D which can be flipped ($D \to D'$).
- Try again with D'.

Homothetic Triangle Contact Representations



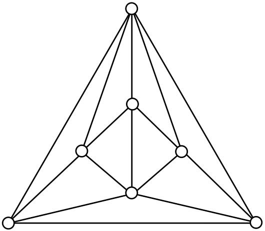
Triangle Contact Representation

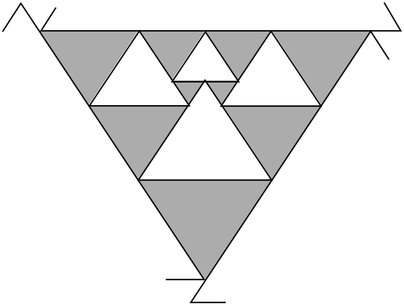
Theorem [de Fraysseix, Ossona de Mendez and Rosenstiehl].
Triangulations have triangle contact representations.



Homothetic Triangle Contact Representations

Theorem [Gonçalves, Lévêque, Pinlou 2010].
Every 4-connected triangulation has a triangle contact representation with homothetic triangles.





Triangle Contact Representations

Proof uses Schramm's "Monster Packing Theorem".

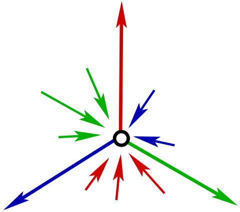
Theorem. Let T be a planar triangulation with outer face $\{a, b, c\}$ and let C be a simple closed curve partitioned into arcs $\{C_a, C_b, C_c\}$. For each interior vertex v of T prescribe a convex set P_v containing more than one point. Then there is a contact representation of T with homothetic copies.

Remark.
In general homothetic copies of the P_v can degenerate to a point. This is impossible if T is 4-connected and all the P_v are homothetic triangles.

Schnyder Woods

$G = (V, E)$ a plane triangulation,
$F = \{a_1, a_2, a_3\}$ the outer triangle.
A coloring and orientation of the interior edges of G with colors 1,2,3 is a Schnyder wood of G iff

- Inner vertex condition:



- Edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

Schnyder Woods and 3-Orientations

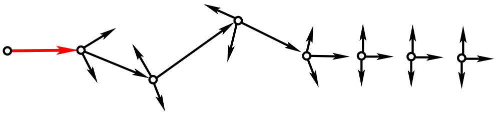
Theorem. Schnyder wood and 3-orientation are in bijection.

Proof.

- All edges incident to a_i are oriented $\rightarrow a_i$.

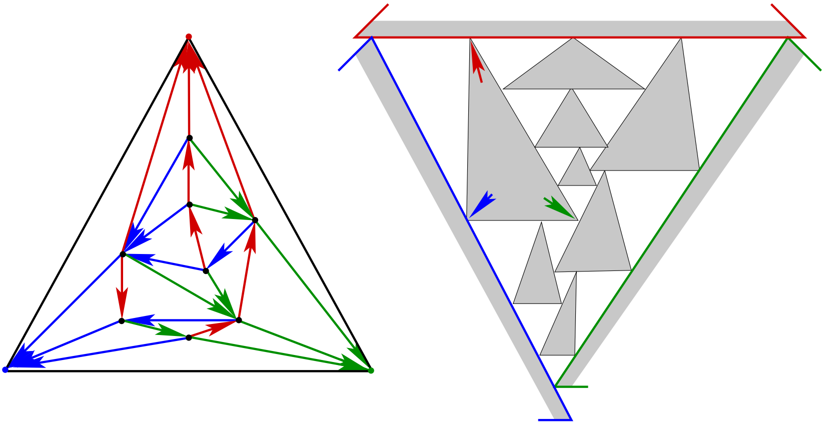
  G has $3n-9$ interior edges and $n-3$ interior vertices.

- Define the path of an edge:

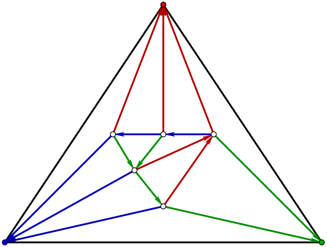


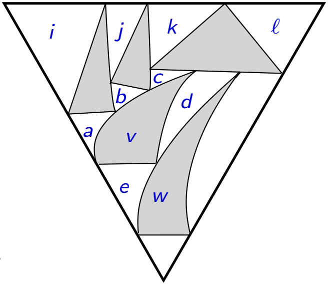
- The path is simple (Euler), hence, ends at some a_i.
- Two path starting at a vertex do not meet again (Euler).

Schnyder Woods as Abstract Triangle Contact Representations



Triangle Contacts and Equations



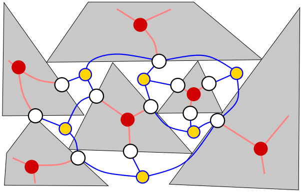


The abstract triangle contact representation implies equations for the sidelength:

$x_i + x_j + x_k + x_\ell = 1$ and $x_a + x_b + x_c = x_v$ and $x_d = x_v$ and $x_e = x_v$ and $x_d + x_e = x_w$ and $\ldots$

The System of Equations

- The matrix A_S is square.
- A_S corresponds to a bipartite graph



Every face has length 6 and two negative signs.

- The bipartite graph has a perfect matching.

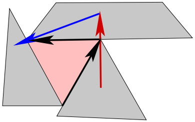
Theorem. $\det A_S \neq 0$.

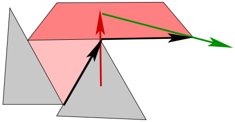
- The system of equations has a unique solution.

Negatve Variables

In the solution some variables may be negative.

- The boundary of the negative variables induces a directed cycle in the Schnyder wood.



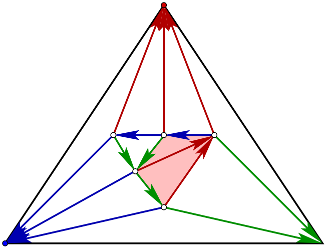


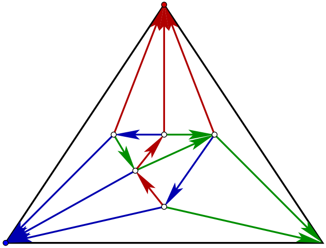
Flipping Cycles

From the bijection

Schnyder woods $\Longleftrightarrow$ 3-orientations

it follows that cycles can be reverted (flipped).





The Status of Algorithm $\mathcal{X}$ in this Case

- Homothetic triangle contact representations are an instance for Algorithm $\mathcal{X}$.
- We have not been able to prove that the algorithm stops.
  In practice, however, it does!

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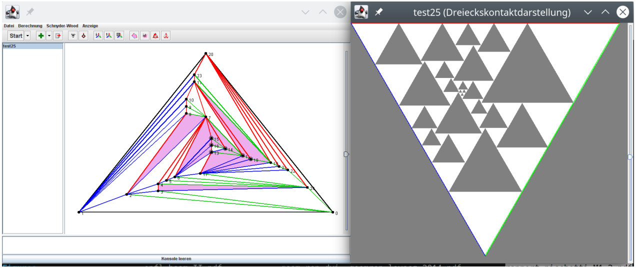
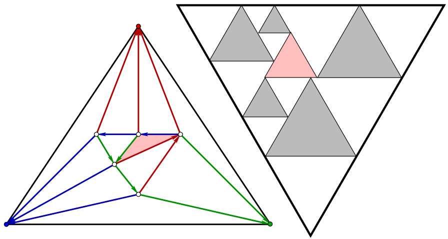


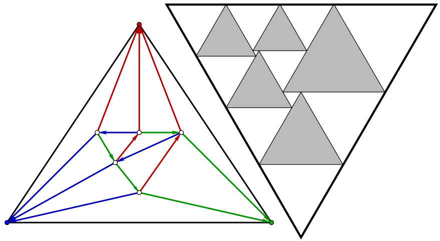
Program written by Julia Rucker.

The Status of Algorithm $\mathcal{X}$ in this Case

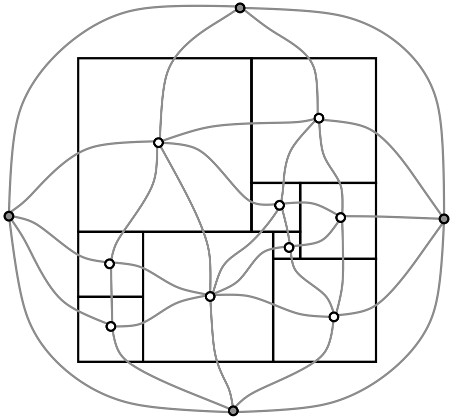
Theoretical support:

Theorem. A negative triangle becomes positive by flipping.



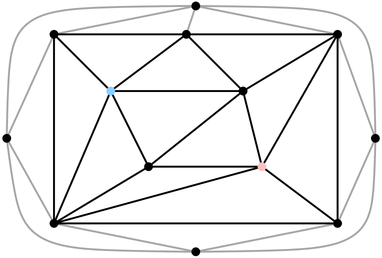


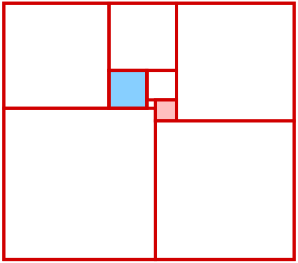
Squarings of Inner Triangulations.



Squarings for Inner Triangulations

Theorem [O. Schramm 1993]. The squaring of a 5-connected inner triangulation exists and it is is unique.



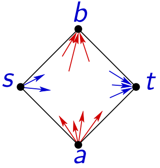


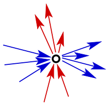
O. Schramm *Square Tilings with prescribed Combinatorics.* 1993
Schramm uses extremal lengths,

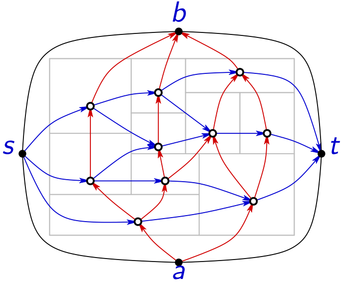
Lovász gave a proof using convex corners.

Transversal structures

Proposition. 4-connected inner triangulations of a quadrangle admit transversal structures (a.k.a. regular edge labeling).

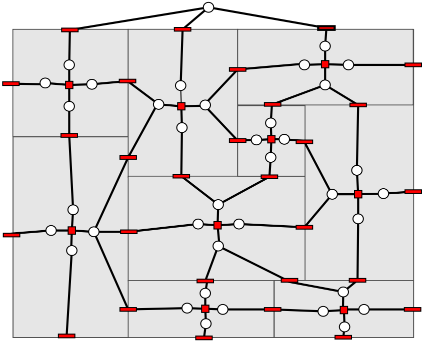






A Systems of Equations

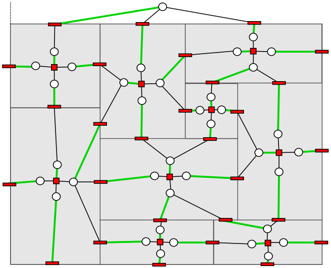
transversal structure $\Longrightarrow$ rectangular dissection
(abstract squaring).



- red rectangles and red circles $=$ variables
- white circles $=$ equations

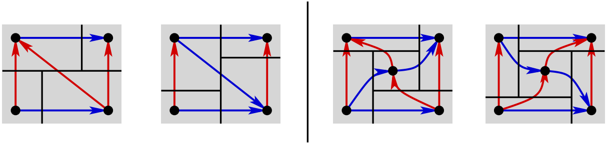
Properties of the System

- The matrix A_T is a square matrix.
- The bipartite graph has facial cycles of length 10 with four negative signs $\implies$ perfect matchings have the same sign.
- The graph has a perfect matching

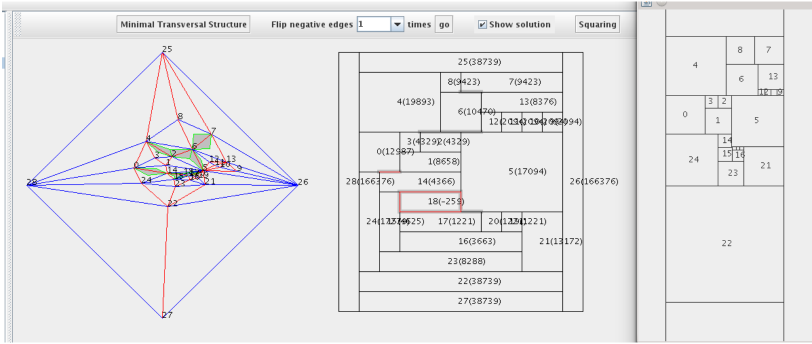


Flips on Transversal Structures

- It is possible to associate a digraph D to a inner triangulation with a transversal structure such that negative variables induce a directed boundary cycle which can be reverted.
- We describe the elementary flips directly:



A Program



Written by Thomas Picchetti

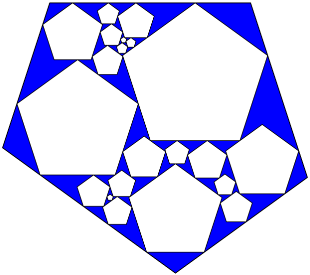
Existence Reproved (Hendrik Schrezenmaier)

Theorem.
Every inner triangulation G of a 4-gon admits a squaring.

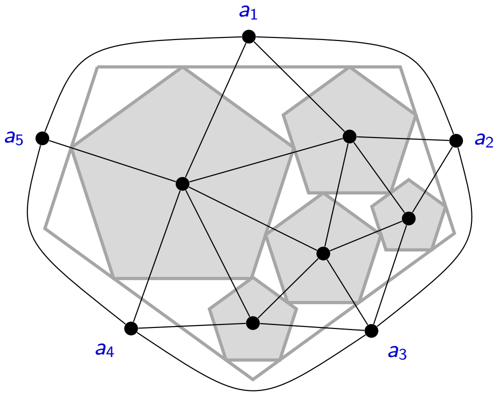
Proof (Sketch)

- Let R be a rectangulation of G with aspect ratio vector α_0 and transversal structure T_0.
- Let $\alpha_1 = \mathbf{1}$ be the aspect ratio vector of a squaring and let $\ell = \{\alpha_t \,:\, t \in [0,1]\}$ be the line from α_0 to α_1.
- The set A_0 of all aspect ratio vectors β representable by T_0 is a subset of $\mathbb{R}^n$ containing α_0. The set is defined by polynomial inequalities (positivity) with polynomials of bounded degree (determinant, Cramer's rule).
- When ℓ leaves A_0 some variables change their sign in T_0 this set corresponds to a flippable set, this defines T_1 and A_1.
- Continue until α_1 is reached.

Pentagon contact representations



Pentagon Contact Representations



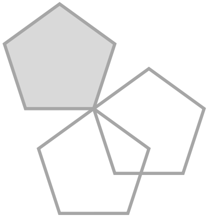
G an inner triangulation of the 5-gon $a_1, \ldots, a_5$

- Existence
- Uniqueness
- Combinatorial structure
- Computation

Homothetic Pentagon Contact Representations

Theorem.
Every triangulation of a 5-gon has a contact representation with homothetic pentagons.

Proof: Use Schramm's "Monster Packing Theorem".

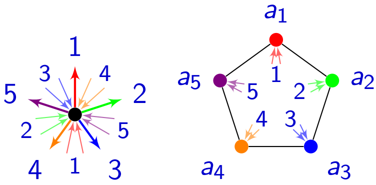


With pentagons there are no degeneracies.

The combinatorial structure: five color forests

Definition (Five color forest)

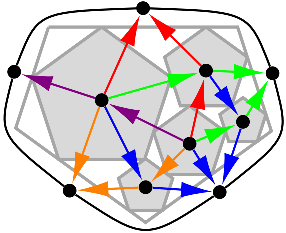
Orientation and coloring of inner edges of inner triangulation of 5-gon $a_1, \ldots, a_5$, s.t.

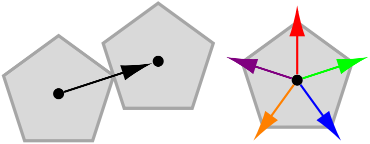


- no incoming edge of color i $\Rightarrow$ outgoing edge of color $i-2$ or $i+2$ exists

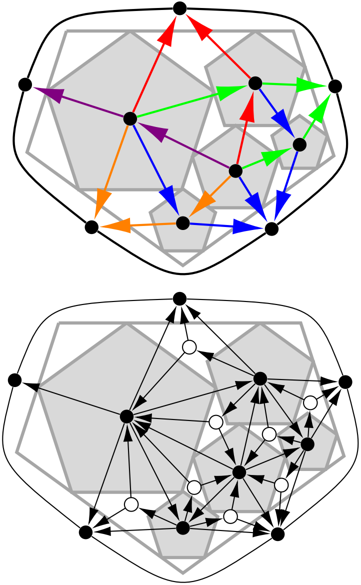
Theorem

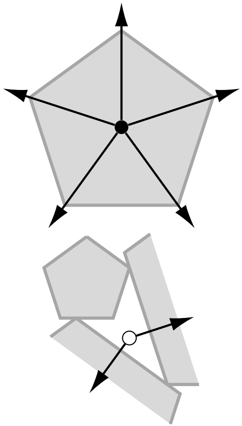
Regular pentagon contact representation induces five color forest on its contact graph.





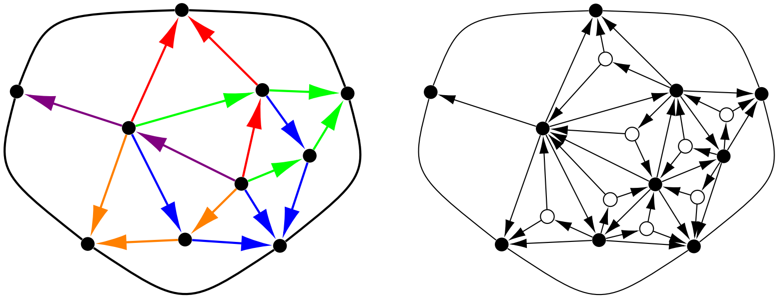
Five color forests $\leftrightarrow$ $(5,2)$-orientations





- $\text{outdeg}(\bullet) = 5$
- $\text{outdeg}(\circ) = 2$

Five color forests $\leftrightarrow$ $(5, 2)$-orientations

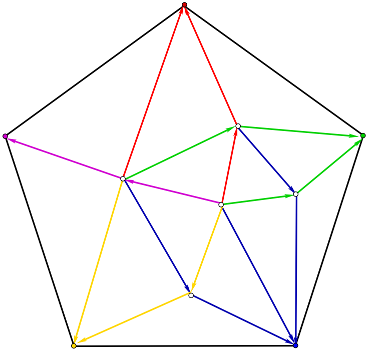


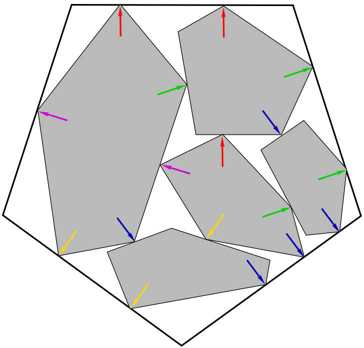
Theorem

There is a bijection between the five color forests and $(5, 2)$-orientations of a graph G.

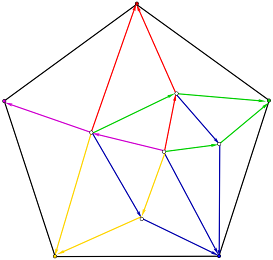
Abstract contact representations

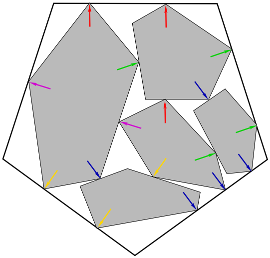
- Compute a five color forest.
- This yields an abstract contact representation.





A system of linear equations





Variables:

- one side length for each vertex: x_v
- four side lengths for each face: $x_f^{(1)}, \ldots, x_f^{(4)}$

Equations:

- five for each vertex: $x_v =$ sum of touching face side lengths
- two for each face: $x_f^{(3)} = x_f^{(1)} + \phi x_f^{(2)}$, $x_f^{(4)} = \phi x_f^{(1)} + x_f^{(2)}$
- one inhomogeneous: length of upper segment $= 1$

System of linear equations

Computing a regular pentagon contact representation induced by a fixed five color forest

Lemma

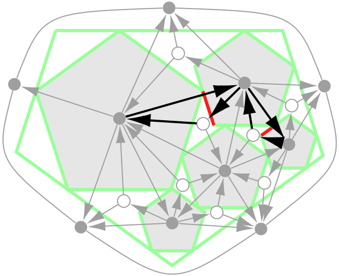
The system $A_F x = \mathbf{e_1}$ is uniquely solvable.

Lemma

$x \geq 0 \;\Leftrightarrow\;$ *there is a regular pentagon contact repr. inducing* F

Algorithm $\mathcal{X}$ for Pentagons

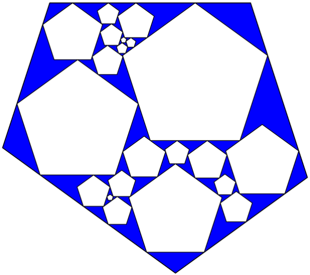
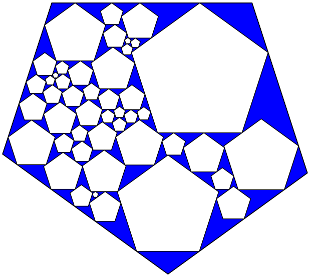
- Guess a five color forest F
- **Case 1**: solution of $A_F x = \mathbf{e_1}$ is nonnegative
  - construct contact repr. from solution
- **Case 2**: solution contains negative and nonnegative variables
  - **Lemma**: neg. and nonneg. variables are separated by oriented cycles in the $(5, 2)$-orientation



  - change orientation of these cycles
  - restart with new $(5, 2)$-orientation, resp. five color forest

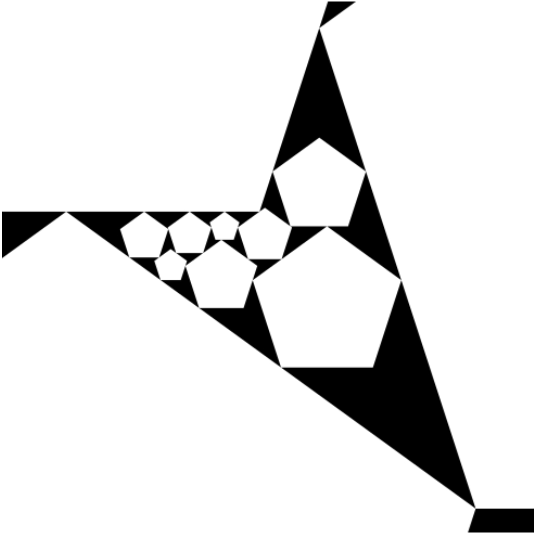
The Status

It works!

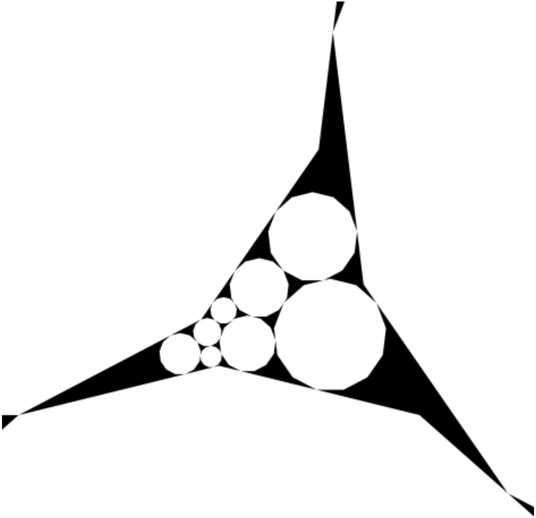
 

But, we have no proof that the process always ends.

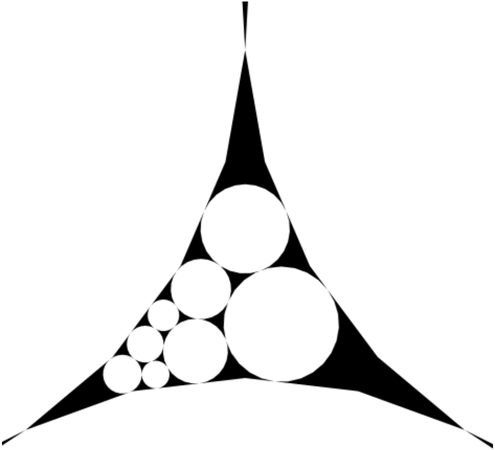
5-gons



13-gons

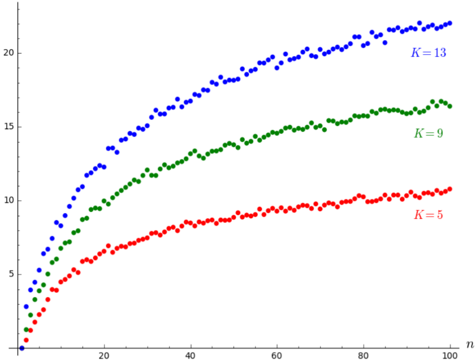


27-gons



Before you ask:
Yes circle contact representations are accumulation points.

The Efficiency of Algorithm $\mathcal{X}$

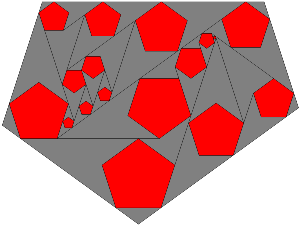


- For each pair (n, K) the dot is the average over 100 random graphs.

The End

Visit and enjoy:

www3.math.tu-berlin.de/diskremath/research/kgon-representations/index.html



History and acknowledgments:

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- Important contributions came from Torsten Ueckerdt, Hendrik Schrezenmaier and Raphael Steiner.
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