Zonotopes Associated with Higher Bruhat Orders

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Abstract

The higher Bruhat orders $B(n,k)$ are combinatorially defined partial orders (and hence graphs) that "look like" the graphs of $(n-k)$-dimensional zonotopes — and they are, for small parameters. Here we explain that this is since they *contain* the graphs of zonotopes of this dimension, but that in general they are not covered by these zonotopal graphs, and they are not polytopal in general.

As a special case, this applies to the graph G_n of all arrangements of n pseudolines connected by flips, since this graph is the graph of the higher Bruhat order $B(n,2)$.

1 Introduction.

Suppose you want to generate a pseudoline arrangement uniformly at random, for example with the goal of estimating the average number of triangular regions. A natural way to approach this is to set up a Markov chain on the arrangements. The transition graph for the Markov chain can be chosen to be the graph whose vertices are all combinatorially different simple arrangements of n pseudolines and edges corresponding to triangular flips.

If the arrangements "live" in the Euclidean plane we can orient the edges from ∇ to Δ (see Figure 1 where two arrangements of pseudolines are displayed by their wiring diagrams). This

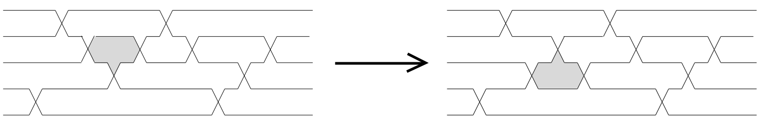

Figure 1: Elementary flip at a triangular region.

directed graph G_n is the diagram of a partial order on arrangements. The graph G_5 is shown in Figure 2. In the picture arrangements of 5 lines are represented by their "dual" zonotopal tilings of a 10-gon (see Theorem 2.1 for a discussion of this correspondence). In Corollary 2.2

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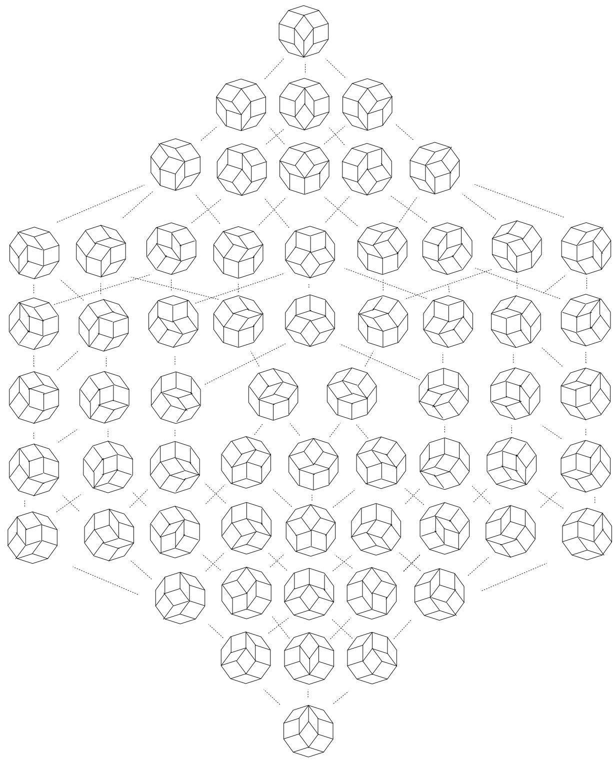

Figure 2: The graph G_5.

we will get that this partial order P_n on the arrangements of n pseudolines (whose graph is G_n) coincides with the "higher Bruhat order" $B(n, 2)$ introduced by Manin & Schechtman [10] and further studied in [17] [7] [8]. The graph of Figure 2 is the graph of a polytope; Figure 3 is another picture of the same graph emphasizing this aspect.

To what extent is this a special case of a general pattern? Can the higher Bruhat orders always be obtained as graphs of polytopes? We show that the graphs of the higher Bruhat orders are *not* polytopal in general. However, there are "fiber zonotopes" closely related to the higher Bruhat orders. In particular, the higher Bruhat orders contain large zonotopal subgraphs. Our analysis of different parameter sets shows that for some parameter sets the zonotopal subgraphs cover all vertices of the graph $B(n, k)$, while in others they don't.

2 The Setting.

We refer to [4] and to [18, Chapters 6 and 7] for background about oriented matroids, and about zonotopes and their tilings. $C^{n,k}$ denotes the "cyclic" oriented matroid [4, Sect. 9.4] of rank k on n elements that can be represented e. g. by the columns of the $(k \times n)$-matrix

$$V : \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & \cdots & n-1 & n \\ 1 & 4 & \cdots & (n-1)^2 & n^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2^{k-1} & \cdots & (n-1)^{k-1} & n^{k-1} \end{pmatrix}.$$

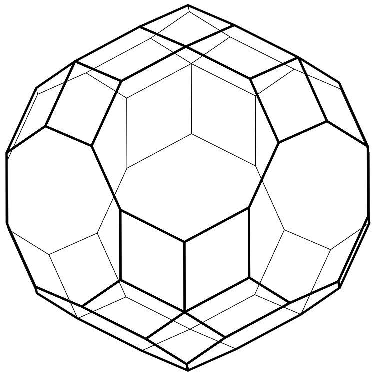

Figure 3: The graph G_5 as graph of a zonotope.

The cyclic oriented matroid $C^{n+1,k}$ provides us with a "canonical" extension of $C^{n,k}$ by a single element, and other single element extensions $C^{n,k}+e$ of $C^{n,k}$ can be compared with respect to their "distance" from $C^{n+1,k}$. To measure this distance we introduce the sets

$$C_+^*(M+e) \quad := \quad \{ \text{ the cocircuits of } M+e \text{ that contain } e \text{ positively } \}.$$

The *higher Bruhat order* $B(n,k)$ of Manin & Schechtman is in [17] characterized as follows: it consists of all the single element extensions $C^{n,k}+e$ of $C^{n,k}$, ordered by single-step inclusion of the *difference sets*

$$S(C^{n,k}+e) \quad := \quad C_+^*(C^{n,k}+e) \setminus C_+^*(C^{n+1,k}).$$

(Single-step inclusion requires a sequence of single element inclusions such that exactly one element is added to the difference set when proceeding from one extension in the chain to the next. It is shown in [17] that this is more restrictive in general than just considering inclusion of difference sets.)

The higher Bruhat orders thus defined are graded partial orders of length $\binom{n}{k}$, with minimal element $C^{n+1,k}$ and maximal element $\overline{{}_{n+1}}C^{n+1,k}$. A cover relation for the partial order $B(n,k)$, and adjacency in its graph, corresponds to reversal of e in a single cocircuit of $M+e$. (Compare Las Vergnas' characterization of single element extensions in terms of cocircuit signatures [4, Sect. 7.1].)

Every vector configuration $V=(v_1,\ldots,v_n)\subseteq\mathbb{R}^k$ determines a zonotope

$$Z(V) \quad := \quad \Big\{ \sum_{i=1}^n \lambda_i v_i : \lambda_i \in [-1,+1] \Big\},$$

referred to as *the zonotope of* V. If the vectors in V are nonzero, pairwise linearly independent, and span $\mathbb{R}^k$, then $Z(V)$ is a zonotope of dimension k with n distinct zones. A *zonotopal tiling* of $Z(V)$ is a tiling by translates of zonotopes $Z(W_j)$, where the configurations W_j are subsets of V. The tiling is *tight* if the zonotopes $Z(W_j)$ are parallelotopes, that is, if the sets W_j are bases of $\mathbb{R}^k$.

Theorem 2.1 *There are canonical bijections between the following four sets*

$$\begin{aligned} B(n,k) \quad &\overset{\text{def}}{\longleftrightarrow} \quad \{ \text{ the } 1\text{-element oriented matroid extensions of } \ C^{n,n-k-1} \ \} \\ &\overset{*}{\longleftrightarrow} \quad \{ \text{ the } 1\text{-element oriented matroid liftings of } \ C^{n,k} \ \} \\ &\overset{\text{BD}}{\longleftrightarrow} \quad \{ \text{ the combinatorial types of tight tilings of } \ Z(C^{n,k}) \ \}. \end{aligned}$$

These bijections preserve the partial orders and thus adjacency on these sets.

Proof. The first equality is the geometric interpretation of the higher Bruhat orders achieved in [17]. In the following, we take this as the *definition* of the higher Bruhat order $B(n,k)$.

The next bijection is oriented matroid duality — we refer to [4, Sect. 3.4]. In particular, in the current situation we use that the dual oriented matroid to $C^{n,k}$ is a reorientation of $C^{n,n-k}$, where reorientation does not affect the set of single element extensions or liftings of an oriented matroid. Because of $(\mathcal{M}/e)^* = \mathcal{M}^* \backslash e$, extensions and liftings are dual concepts. (The partial order, and adjacency, on the set of single element liftings of $C^{n,n-k}$ are defined in terms of the circuits that positively contain the lifting element e, in complete analogy to the definition for single element extensions described above.)

The last bijection is the Bohne-Dress theorem [5] [13] in its pure form, applied to a special oriented matroid and its zonotope. □

Corollary 2.2 *The partial order P_n on the wiring diagrams of n pseudolines defined in the introduction coincides with the higher Bruhat orders $B(n,2)$.*

This is since the wiring diagrams of pseudoline arrangements can be interpreted as single element liftings of the oriented matroid $C^{n,2}$, which is represented by the order $1, 2, \ldots, n$ in which the pseudolines intersect the line ℓ_e at infinity. The minimal element of $B(n,2)$ corresponds to the pseudoline arrangement (wiring diagram) in which any three pseudolines induce a ∇-triangle.

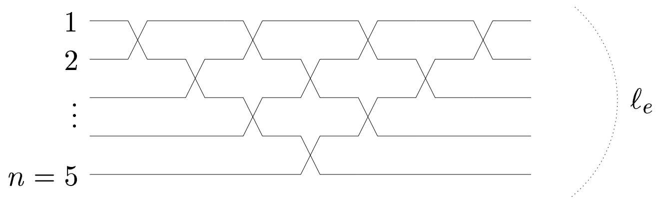

Figure 4: The minimal element of $B(5,2)$

3 The Main Theorem.

Given any spanning vector configuration $V = (v_1, \ldots, v_n) \subseteq \mathbb{R}^k$, a *dual vector configuration* is a spanning configuration $V^* = (v_1^*, \ldots, v_n^*) \subseteq \mathbb{R}^{n-k}$ such that the row space of the matrix V^* is the orthogonal complement to the row space of V. (Thus the dual configuration lives in complementary dimension, but it has the same number of vectors as the original configuration. The oriented matroid $M(V^*)$ is the dual of $M(V)$.)

The *adjoint* of a spanning vector configuration $W = (w_1, \dots, w_n) \subseteq \mathbb{R}^k$ is another vector configuration W^{ad} in $\mathbb{R}^k$, which consists of exactly one non-zero vector orthogonal to each hyperplane in $\mathbb{R}^k$ that is spanned by a subset of W. (Thus the adjoint lives in the same dimension, but typically it has many more vectors than the original configuration. A key observation is that in general the oriented matroid of W^{ad} is not determined by the oriented matroid of W: it depends on the particular vector configuration representing W.)

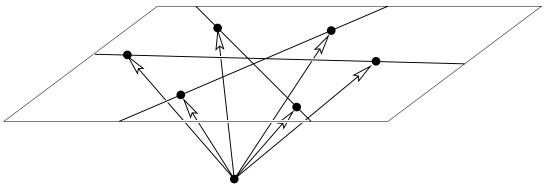

Figure 5: The adjoint of $C^{6,3}$ is not unique!

Theorem 3.1 *Let V be a configuration of n vectors in $\mathbb{R}^k$ with cyclic oriented matroid $M = M(V) \cong C^{n,k}$, and let $Z = Z(V)$ be its zonotope (of dimension k, with n zones). Let $\pi : C_n \longrightarrow Z$ be a projection of an n-dimensional cube with $\pi(C_n) = Z$.*

Then the fiber polytope associated with this polytope projection is a zonotope $\widehat{Z}$ of dimension $n-k$, which is generated by the adjoint of the dual vector configuration $V^* \subseteq \mathbb{R}^{n-k}$.

In this situation we have a graph inclusion

$$G(\widehat{Z}) \hookrightarrow B(n,k),$$

here $G(\widehat{Z})$ is the graph of $\widehat{Z}$ and $B(n,k)$ represents the diagram of the higher Bruhat order as undirected graph. The two graphs are equal if and only if the oriented adjoint of $M(V^*)$ is unique (that is, independent of the particular representation V chosen for $C^{n,k}$ in the first place).

Proof. According to Billera & Sturmfels [3, Theorem 4.1], the fiber polytope of the polytope projection $\pi : C_n \longrightarrow Z$ is a zonotope $\widehat{Z}$ generated by the circuits E_ν of V, which by oriented matroid duality coincide with the cocircuits of V^*.

Specifically, the (non-zero) circuits E_ν as constructed by Billera & Sturmfels are elementary (minimal non-zero) vectors in the rowspace of V^*. That is, we may describe them in the form

$$E_\nu = y^t V^* = (y^t v_1^*, \dots, y^t v_n^*)$$

where $y^t = (y_1, \dots, y_{n-d})$ is a vector of minimal (non-zero) support in the rowspace of V^*. The corresponding cocircuit of $M(V^*)$ is defined by the hyperplane

$$H_\nu = \{x \in \mathbb{R}^{n-k} : y^t x = 0\},$$

which is the hyperplane in $\mathbb{R}^{n-k}$ spanned by exactly those $v_i^* \in V^*$ such that $y^t v_i^* = 0$.

Now the vertices of $\widehat{Z}$ are in bijection with the regions of the hyperplane arrangement $\mathcal{A}(M(V^*)) = \{H_\nu : \nu \in \mathcal{C}(C^{n,k})\}$, and these are in turn in bijection with those single element extensions of $C^{n,n-k}$ that can be realized by extending the given realization V^* of $M(V^*) \cong C^{n,n-k}$ (compare [3, Cor. 4.2] [2, Thm. 2.3]). These extensions naturally form a subset of the set of *all* single element extensions of $C^{n,n-k}$, which in turn is in natural bijection with $B(n,k)$, by Theorem 2.1.

Thus we obtain an inclusion

$$\text{vertices}(\widehat{Z}) \hookrightarrow B(n,k).$$

Adjacent vertices of $\widehat{Z}$ correspond to adjacent regions of the hyperplane arrangement $\mathcal{A}(M(V^*))$, that is, to regions whose points are separated only by one hyperplane of the arrangement, and thus we are considering single element extensions of $C^{n,k}$ that differ in the signature of exactly one cocircuit, that is, adjacent elements of $B(n,k)$.

Thus we have an inclusion

$$G(\widehat{Z}) \hookrightarrow B(n,k)$$

which preserves adjacency (that is, an embedding as an induced subgraph).

The map that we have thus obtained is surjective if and only if *all* single-element extensions of $C^{n,n-k}$ can be obtained by extending the particular realization given by $M(V^*)$. Two effects could prevent this: the first one is if there is a single-element extension of $M(V^*)$ that is realizable but appears only in a different realization of $M(V^*)$. Then we see that the adjoint of $M(V^*)$ is not unique.

The second bad case is if $M(V^*)$ does have some non-realizable single-element extensions. However, we will see in Proposition 4.2 that this does not happen in the cases where the adjoint of $M(V^*)$ is unique. □

The combinatorial structure of the adjoint cannot be derived from the (oriented) matroid $M = M(V^*)$, but (except for small k and $n-k$) it depends on the precise coordinates of V. In Proposition 4.1 we will determine the precise range of "small" parameters for this.

4 Unique Adjoints.

In Theorem 3.1 we have shown that the graph of $\widehat{Z}$ is contained in $B(n,k)$, where $\widehat{Z}$ is generated by the adjoint of the dual of a vector configuration V with $M(V) \cong C^{n,k}$. In this section we determine in which cases the oriented matroid adjoint is unique. We also verify that in all these cases there are no non-realizable extensions, as needed to complete the proof of Theorem 3.1.

Recall from Section 2 that the dual of $C^{n,k}$ is a reorientation of $C^{n,n-k}$. For notational convenience let $r := n-k$.

Proposition 4.1 *The (oriented) matroid of the adjoint of $C^{n,r}$ is unique for $r \le 2$, for $n-r \le 1$ and for $(n,r) = (5,3)$, but not for any other values.*

Proof. In the following V^* denotes a vector configuration that represents $C^{n,r}$.

We first deal with the cases where the adjoints are unique. The case $r \le 2$ is trivial: for $r = 1$ there is only one hyperplane and for $r = 2$ the adjoint of V^* is equivalent to V^*. For $n - r \le 1$ we have projective uniqueness of the configuration V^*: the oriented matroid in this case consists of a single circuit, i. e. of a projective basis.

In the case $(n,r) = (5,3)$ we may projectively transform V^* so that it is given by the matrix

$$V^* : \quad \begin{array}{c} \begin{matrix} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ -s & -1 & 0 & 1 & 0 \\ t & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

all of whose maximal minors are positive for $1 < s < t$. The adjoint of V^* is given by

$$A: \quad \begin{array}{c} \begin{matrix} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{matrix} \\ \begin{pmatrix} t-s & 0 & -t & -s & 0 & -1 & -1 & 0 & 0 & 1 \\ t-1 & t & 0 & -1 & 1 & 0 & -1 & 0 & -1 & 0 \\ s-1 & s & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}.$$

One can now verify that the signs of 3×3 minors of A are completely determined by the condition $1 < s < t$.

In the case $(n,r) = (6,3)$ we have a classical configuration of six points in convex position whose adjoint is not unique (see Figure 5, and below). The three main diagonals may or may not intersect in one point. Thus the adjoint arrangement is not combinatorially determined by (the oriented matroid of) the six point configuration.

For the case $(n,r) = (6,4)$ consider the vector configuration given by the matrix

$$V^*: \quad \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -t & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

all of whose (4×4)-minors are positive if $t > 3$. The adjoint of V^* contains the columns of the matrix

$$\begin{array}{c} \begin{matrix} 234 & 135 & 146 & 256 \end{matrix} \\ \begin{pmatrix} & & -3 & -1 \\ & t & & -1 \\ -1 & & 1 & \\ -1 & -2 & & \end{pmatrix} \end{array},$$

whose determinant $6-t$ is positive, negative or zero, for different choices of $t > 3$.

The examples above may be used to construct two representations of $C^{n,r}$ with different adjoints, for each of the remaining cases. For this we note that if the adjoint of an oriented matroid M is not unique, then the adjoint cannot be unique either for any oriented matroid N that has M as a minor. $\square$

Proposition 4.2 *The oriented matroid $C^{n,r}$ has no non-realizable extensions for the pairs (n,r) such that the adjoint of $C^{n,r}$ is unique (as given by Proposition 4.1).*

Proof. See [4, Cor. 8.3.3]. $\square$

Problem 4.3 *Characterize those pairs (n,r) such that the oriented matroid $C^{n,r}$ has a non-realizable single element extension.*

This problem is non-trivial: Since $C^{8,2}$ has non-realizable uniform single element liftings (given by Ringel's non-stretchable pseudoline arrangement), duality yields non-realizable single element extensions for $C^{8,6}$. (However, single element extensions of a cyclic oriented matroid cannot be too badly non-realizable, as indicated by [14, Thm. 4.12].) On the other hand, for $(n,r) = (8,3)$ we find that all uniform extensions of $C^{8,3}$ are realizable — this is since the non-realizable oriented matroid of rank 3 on 9 points is unique [4, Thm. 8.3.4(1)], and its single element deletions are not cyclic. More generally, Richter-Gebert [11, Thm. 8.3] has shown that all uniform single element extensions of $C^{n,3}$ are realizable, for all $n \geq 3$.

5 The Higher Bruhat Graphs.

In this section we look at the graphs of the higher Bruhat orders in some more detail, and try to identify zonotopal subgraphs $G(\widehat{Z})$. Three different cases are considered separately.

Case $k = 2$.

This is the case of pseudoline arrangements. For $n \leq 5$ we have $G(\widehat{Z}) = B(n, 2)$, which is a single edge for $n = 3$, an 8-gon for $n = 4$, and the graph of Figures 2 and 3 for $n = 5$.

For $n \geq 6$ there are pseudoline arrangements so that the arrangement is not representable with certain fixed slopes [12] [4, p. 42]. Prescribing the slopes of the arrangement corresponds to fixing a realization V for the oriented matroid $C^{n-1,2}$. Thus there are choices of V such that not all elements of $B(6,2)$ appear as vertices of $G(\widehat{Z})$. With the following proposition we show that for *every* choice of V the vertices of $G(\widehat{Z})$ form a proper subset of $B(6,2)$.

Proposition 5.1 *For any set of prescribed slopes $s_1 > s_2 > s_3 > s_4 > s_5 > s_6$ at most one of the two arrangements A_1, A_2 of Figure 6 is realizable.*

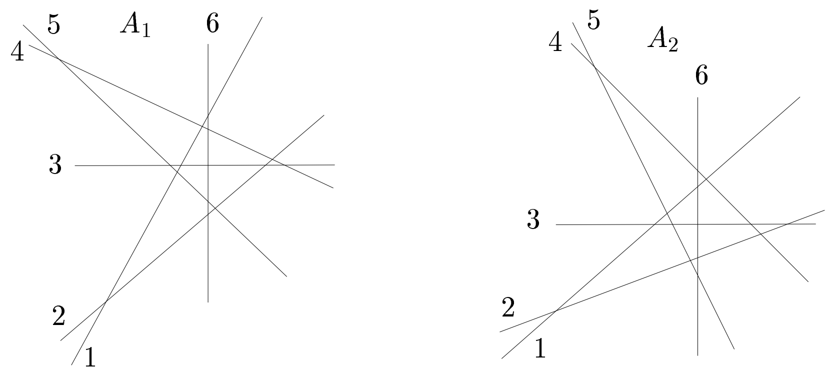

Figure 6: Arrangements A_1 and A_2.

Proof. Suppose that there is a set of six different slopes such that both arrangements are realizable with these slopes. Let $\ell_j(A_i)$ denote line ℓ_j in such a representation of arrangement A_i. Using appropriate similarity transformations we may assume that

- $s_3 = 0$ and $s_6 = \infty$,
- $\ell_i(A_1) = \ell_i(A_2)$ for $i = 1, 2, 4$.

Let p_i be the crossing point of lines ℓ_3 and ℓ_6 in arrangement A_i for $i = 1, 2$. By considering the "orientation" of the triangles formed by ℓ_1, ℓ_4, ℓ_6 and ℓ_2, ℓ_3, ℓ_4 the areas containing point p_i in arrangement A_i are restricted to the triangles shown in Figure 7.

Taking into account the orientation of the triangle ℓ_1, ℓ_3, ℓ_5 we find the following order of crossings from left to right on line ℓ_1:

$$\ell_2, \ell_5(A_1), \ell_3(A_1), \ell_3(A_2), \ell_5(A_2), \ell_6(A_2), \ell_4, \ell_6(A_1).$$

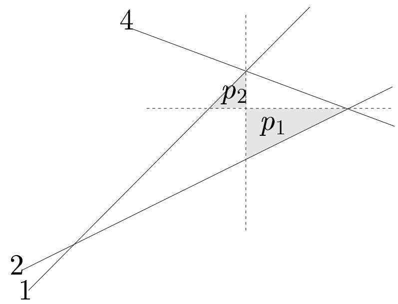

Figure 7: The triangle of lines 1,2 and 4.

Similarly, the triangle ℓ_2, ℓ_5, ℓ_6 forces the following order of crossings on ℓ_2:

$$\ell_1, \ell_5(A_2), \ell_6(A_2), \ell_6(A_1), \ell_5(A_1), \ell_3(A_1), \ell_4, \ell_3(A_2).$$

This shows that $\ell_5(A_1)$ and $\ell_5(A_2)$ have a crossing between ℓ_1 and ℓ_2, contradicting the assumption that they have the same slope. $\square$

Corollary 5.2 *The zonotope graphs $G(\widehat{Z})$ taken together cover all the vertices of $B(6,2)$, but none of them covers $B(6,2)$ by itself.*

For $n \geq 9$, some elements of $B(n,2)$ do not appear as vertices of $G(\widehat{Z})$ for any choice of V. They correspond to non-stretchable pseudoline arrangements.

Nevertheless, we did not decide for any $n \geq 6$ whether the graph of $B(n,2)$ is polytopal. Already for $n = 6$ this seems to be a non-trivial problem, since it is known [9] [17] that $B(6,2)$ has 908 elements. (In this case the polytope in question would necessarily be 4-dimensional.)

Case $n - k = 2$.

The $2n$ elements of $B(n,k)$, for $n - k = 2$, correspond to the one element extensions of $C^{n,1}$. They are realizable and, hence, are vertices of $G(\widehat{Z})$ for some V. However, the adjoint is unique in this case and, therefore, $G(\widehat{Z})$ is independent of the choice of V. This proves $G(\widehat{Z}) = B(n,k)$.

Case $n - k = 3$.

Recall from Section 2 that the elements of $B(n,k)$, $n - k = 3$ correspond to the uniform single element extensions of cyclic pseudoline arrangements. As mentioned above, all these extensions are realizable [11, Thm. 8.3].

The first non-trivial case is $n = 6$; this is the classical case of a non-unique adjoint! See Figure 5 above, but also e. g. [1, p. 301] [4, p. 340] [16, Example 8.7] [15, Example 2.2] [6]. Consequently there is a V such that $G(\widehat{Z}) \neq B(6,3)$. The stronger conclusion that $G(\widehat{Z}) \neq B(6,3)$ for *all* V can be obtained from considerations on the graph of $B(6,3)$, given in Figure 8, as follows.

Observation 5.3 *The graph of $B(6,3)$ is not polytopal.*

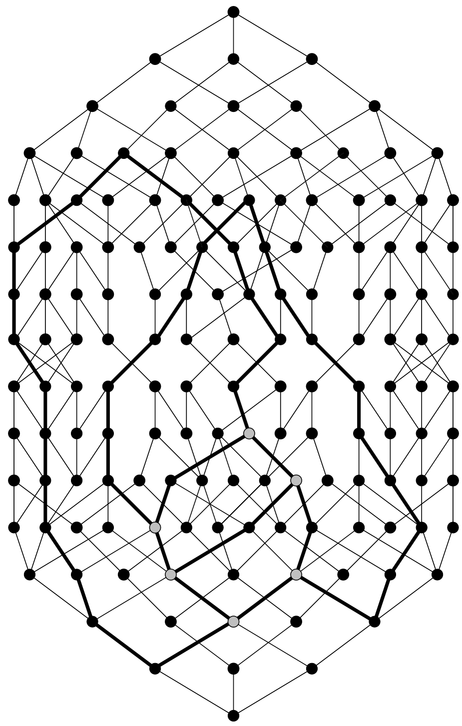

Figure 8: The graph of $B(6,3)$, with a highlighted subdivision of $K_{3,3}$

Proof. The graph has vertices of degree 3; thus it is not the graph of a polytope of dimension $d \geq 4$. Moreover, the graph is not planar: using the "small cubes" that appear due to non-unique adjoint it is easy to find a $K_{3,3}$ minor; see Figure 8. Thus the cover graph of $B(6,3)$ is not the graph of a polytope of dimension $d \leq 3$. $\square$

In Figure 9 a part of the cover graph of $B(6,3)$ is given again with the corresponding one element extensions of a cyclic line arrangement with six lines. The remaining parts of $B(6,3)$ can be obtained from this quarter by reflection and complementation.

References

[1] A. Bachem & W. Kern: *Adjoints of oriented matroids,* Combinatorica **6** (1986), 299–308.

[2] M. Bayer & K.A. Brandt: *Discriminantal arrangements, fiber polytopes and formality,* J. Algebr. Comb. **6** (1997), 229–246.

[3] L. J. Billera & B. Sturmfels: *Fiber polytopes,* Annals of Math. **135** (1992), 527–549.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White & G. M. Ziegler: *Oriented Matroids, Encyclopedia of Mathematics*, Vol. 46, Cambridge University Press 1993.

[5] J. Bohne: *Eine kombinatorische Analyse zonotopaler Raumaufteilungen,* Dissertation, Bielefeld 1992; Preprint 92-041, SFB 343, Universität Bielefeld, 100 pages.

[6] M. J. Falk: *A note on discriminantal arrangements,* Proc. Amer. Math. Soc. **122** (1994), 1221–1227.

[7] S. Felsner & H. Weil: *Sweeps, arrangements and signotopes,* Discr. Appl. Math, to appear.

[8] S. Felsner & H. Weil: *A theorem on higher Bruhat orders,* Discrete Comput. Geometry, to appear.

[9] D. E. Knuth: *Axioms and Hulls,* Lecture Notes in Computer Science **606**, Springer 1992.

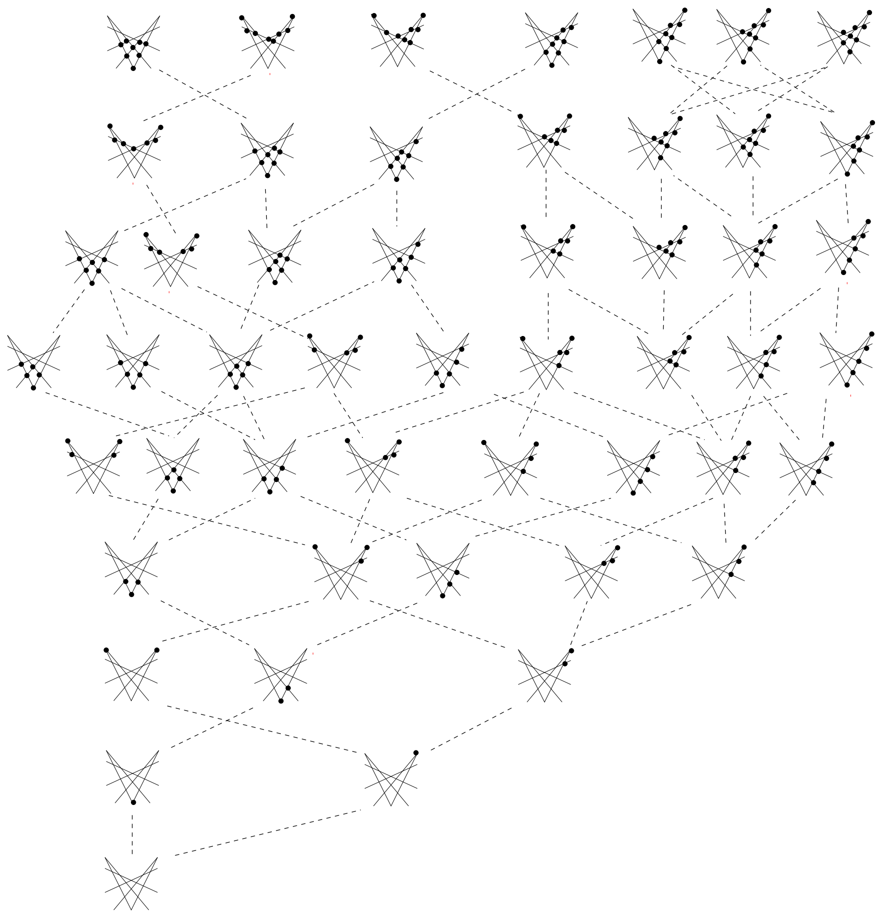

Figure 9: A quarter of $B(6,3)$ with the corresponding one element extensions of a cyclic line arrangement.

[10] Yu. I. Manin & V. V. Schechtman: *Arrangements of hyperplanes, higher braid groups and higher Bruhat orders,* in: "Algebraic Number Theory" (J. Coates et al., eds.) *Advanced Studies in Pure Mathematics* **17** (1989), Academic Press, pp. 289–308.

[11] J. Richter-Gebert: *On the Realizability Problem for Combinatorial Geometries — Decision Methods,* Dissertation, Darmstadt 1992.

[12] J. Richter & B. Sturmfels: *On the topology and geometric construction of oriented matroids and convex polytopes,* Transactions Amer. Math. Soc. **325** (1991), 389–412.

[13] J. Richter-Gebert & G. M. Ziegler: *Zonotopal tilings and the Bohne-Dress theorem,* in: Proc. *Jerusalem Combinatorics '93* (H. Barcelo, G. Kalai, eds.), *Contemporary Math.* **178**, Amer. Math. Soc. 1994, 211–232.

[14] B. Sturmfels & G. M. Ziegler: *Extension spaces of oriented matroids,* Discrete Comput. Geometry **10** (1993), 23–45.

[15] S. Yuzvinsky: *The first two obstructions to the freeness of arrangements,* Transactions Amer. Math. Soc. **335** (1993), 231–244.

[16] G. M. Ziegler: *Combinatorial construction of logarithmic differential forms,* Advances in Mathematics **76** (1989), 116–154.

[17] G. M. Ziegler: *Higher Bruhat orders and cyclic hyperplane arrangements,* Topology **32** (1993), 259–279.

[18] G. M. Ziegler: *Lectures on Polytopes,* Graduate Texts in Mathematics **152**, Springer-Verlag, New York 1995; *Updates, corrections, and more*, electronic preprint available per WWW from URL `http://www.math.tu-berlin.de/~ziegler`