Vertex Contact Graphs of Paths on a Grid

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Abstract

Contact and intersection representations of graphs and particularly of planar graphs have been studied for decades. The by now best known result in the area may be the Koebe-Andreev-Thurston circle packing theorem. A more recent highlight in the area is a result of Chalopin and Gonçalves: every planar graph is an intersection graph of segments in the plane. This boosted the study of intersection and contact graphs of restricted classes of curves. In this paper we study planar graphs that are VCPG, i.e. graphs admitting a representation as Vertex Contact graph of Paths on a Grid. In such a representation the vertices of G are represented by a family of interiorly disjoint grid-paths. Adjacencies are represented by contacts between an *endpoint* of one grid-path and an *interior point* of another grid-path. Defining $u \to v$ if the path of u ends on path of v we obtain an orientation on G from a VCPG representation. To get hand on the bends of the grid path the 2-orientation is not enough. We therefore consider pairs (α, ψ): a 2-orientation α and a flow ψ in the angle graph. The 2-orientation describes the contacts of the ends of a grid-path and the flow describes the behavior of a grid-path between its two ends. We give a necessary and sufficient condition for such a pair (α, ψ) to be realizable as a VCPG.

Using realizable pairs we show that every planar (2,2)-tight graph can be represented with at most 2 bends per path and that this is tight (i.e. there exist (2,2)-tight graphs that cannot be represented with at most one bend per path). Using the same methodology it is easy to show that loopless planar (2,1)-sparse graphs have a 4-bend representation and loopless planar (2,0)-sparse graphs have 6-bend representation. We do not believe that the latter two are tight, we conjecture that loopless planar (2,0)-sparse graphs have a 3-bend representation.

1 Introduction

Outline of results. In this paper we consider the question whether a planar graph G admits a VCPG, i.e. a representation as a Vertex Contact graph of Paths on a Grid. In such a representation the vertices are represented by a family of interiorly disjoint grid-paths. An *endpoint* of one grid-path coincides with an *interior point* of another grid-path if and only if the two represented vertices are adjacent.

A VCPG induces a unique orientation of the edges of G: Orienting the edge uv as $u \to v$ if the grid-path of u ends on grid-path of v we obtain an orientation of G. As each grid-path has two ends, in the induced orientation each vertex has outdegree at most two. We denote such an orientation simply 2-orientation.

On the other hand, for a planar graph, every 2-orientation induces a VCPG (Section 1.1). However, a 2-orientation of G defines the representation of the edges in a VCPG but not how the grid-paths behave (e.g. how many bends a grid-path has). To get a description of the behavior of the grid-paths between its endpoints, we introduce a flow network in the angle graph (Section 2.1). A flow in this network represents the bends of a grid-path, e.g. consider v, a degree two vertex separating two faces, each *bend* of the grid-path p_v is a *convex* corner on the boundary of one of the faces and a *concave* corner on

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the boundary of the other face. Such a relation can be represented as one unit of flow leaving one face through a convex corner of a grid-path and entering the other face through the concave corner.

To obtain a full combinatorial description of a VCPG we consider a pair (α, ψ): a 2-orientation α in the graph and a flow ψ in the angle graph. Our main contribution is a necessary and sufficient condition for such a pair (α, ψ) to be realizable as a VCPG. We will then use such realizable pairs to give bounds on the number of bends needed for certain graph classes.

When the number of bends of each path is at most k we denote the representation by B_k*-VCPG* and when every path has precisely k bends we speak about *strict* B_k*-VCPG*.

Related results. In [KUV13] Kobourov, Ueckerdt and Verbeek show that all planar Laman graphs admit an L-contact representation, i.e. a strict B_1-VCPG. A graph $G = (V, E)$ is Laman if $|E| = 2|V| - 3$ and every subset of k vertices induces at most $2k - 3$ edges. It is immediate that every subgraph of a planar Laman graph also has a strict B_1-VCPG. There are graphs that are not Laman that have a strict B_1-VCPG. In [KUV13] the question was posed which conditions are necessary and sufficient for a graph to have such a representation.

Vertex intersection graphs of paths on a grid (VPG-graphs) have been investigated by Asinowski et al. [ACG+12]. They showed that all planar graphs are B_3-VPG, i.e., each vertex is represented by a path with at most three bends and the edges are intersections of two grid-paths. They conjectured that this bound was tight. Chaplick and Ueckerdt disproved this by showing that every planar graph is B_2-VPG [CU13].

In orthogonal graph drawing there have been many results on minimizing bend numbers, i.e. vertices are points in the plane and edges are grid-paths between these points and the number of bends is minimized. Note that in this setting vertices have at most degree four, or as a workaround, the vertices can be represented as boxes. An early result of Tamassia gives an algorithm to obtain an orthogonal drawing with minimal bend number which preserves the embedding [Tam87]. Optimizing the bend number locally (for each path) has gotten much attention too, Schäffter gives an algorithm to draw 4-regular graphs in a grid with at most two bends per edge (which is tight when not restricted to planar graphs) [Sch95]. For orthogonal drawings without degree restriction, Fößmeier, Kant and Kaufmann have shown that every plane graph has an orthogonal drawing preserving the embedding with at most one bend per edge [FKK96].

Outline of the paper. The remainder of this Section we will give the definitions and show some necessary conditions based on 2-orientations. In Section 2 we will introduce the flow network. We then give the necessary and sufficient condition for a pair, a 2-orientation and a flow, to be realizable as a VCPG. In Section 3 and 4 we show how to use realizable pairs to give bounds on the number of bends in a VCPG.

1.1 Preliminaries: on (2,l)-sparse graphs

A necessary condition for a planar graph to admit a VCPG follows from the 2-orientation. In an orientation that is induced by a VCPG, all edges of a graph $G = (V, E)$ are oriented and each vertex has outdegree at most two, therefore the number of edges of G is at most twice the number of vertices: $|E| \leq 2|V|$. Moreover, this bound must hold for all induced subgraphs as well.

Definition 1.1 (Sparse and Tight Graphs)**.** Graphs that satisfy $\forall W \subseteq V : |E_W| \leq k|W| - l$ are denoted (k, l)-sparse graphs. If also $|E| = k|V| - l$ holds, the graph is called (k, l)-tight.

Graphs that admit a VCPG must be planar and (2,0)-sparse. In this paper we focus on (2,0)-tight graphs, (2,1)-tight graphs and (2,2)-tight graphs which are simple and planar. Note that for every (2,l)-sparse graph H there exists a (2,l)-tight graph G such that H is a subgraph of G.

First we show that every planar $(2, l)$-tight graph, $l \geq 0$, has a VCPG.

Lemma 1.2. Every planar $(2, l)$-tight graph has a 2-orientation.

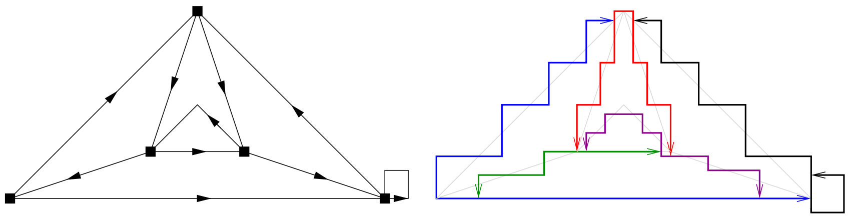

Figure 1: A (2,0)-tight plane graph with a 2-orientation and a VCPG of this graph.

Proof. Let $G = (V, E)$ a planar $(2, l)$-tight graph. Suppose there is a subset W of the vertices of G that has less than $2|W|$ incident edges. Then $G[V - W]$ must induce at least $2|V| - l - (2|W| - 1) = 2|V - W| - l + 1$ edges, which contradicts $(2, l)$-tightness. Hence every subset W of the vertices of G has at least $2|W|$ incident edges. Now we construct a bipartite graph. The first vertex class, V_1, consists of two copies of all but l of the vertices of G, and the remaining l vertices are only added once. The second class, V_2, contains all edges. The edge set is defined by the incidences in G: two vertices are connected if the corresponding vertex of G is an endpoint of the corresponding edge of G. By $(2, l)$-tightness of G this bipartite graph satisfies Hall's marriage condition and hence it has a perfect matching. A perfect matching defines a 2-orientation of G. □

When a planar graph has a 2-orientation it easily follows that it has a VCPG. An example is shown in Figure 1.

Lemma 1.3. Let $l \geq 0$. Every planar $(2, l)$-tight graph is VCPG.

Proof. Consider an embedding of a planar (2,l)-tight graph G and a 2-orientation α of G. Subdivide each loop twice. Every pair of vertices is connected by at most two edges (since the graph is $(2, l)$-tight and $l \geq 2$) and if so one of the multiple edges is subdivided. The result is a simple plane graph, which has a straight line drawing by Fàry's theorem. Replace each straight line edge in such a drawing by an axis-aligned grid-path leaving the start and endpoint intact and such that two grid-paths starting in the same point only coincide in this point. The subdivided edges are merged without changing the grid-paths. A vertex is identified with its outgoing edge(s). The last step is to perturb the last straight part of a grid-path p_v that ends on a grid-path p_w in such a way that this point is not used by any grid-path other than p_v and p_w. This procedure gives a VCPG of G that realizes the chosen embedding. □

An obvious question is: how many bends are needed in a VCPG of a certain graph. As mentioned before, Kobourov, Ueckerdt and Verbeek have shown that all planar (2,3)-tight graphs (Laman graphs) admit a strict B_1-VCPG [KUV13]. They also gave the following bound on the number of edges of a graph that admits a (strict) B_1-VCPG.

Proposition 1.4. If $G = (V, E)$ admits a B_1-VCPG then

$$\forall W \subseteq V : |E_W| \leq 2|W| - 2 \; . \tag{1}$$

Proof: cf. appendix.

Therefore the candidate graphs that have a B_1-VCPG are planar (2,2)-sparse graphs. In this paper we show that planar (2,2)-tight graphs are B_2-VCPG and that this is tight. Thus condition 1.4 is necessary but not sufficient. The proof is based on realizable pairs.

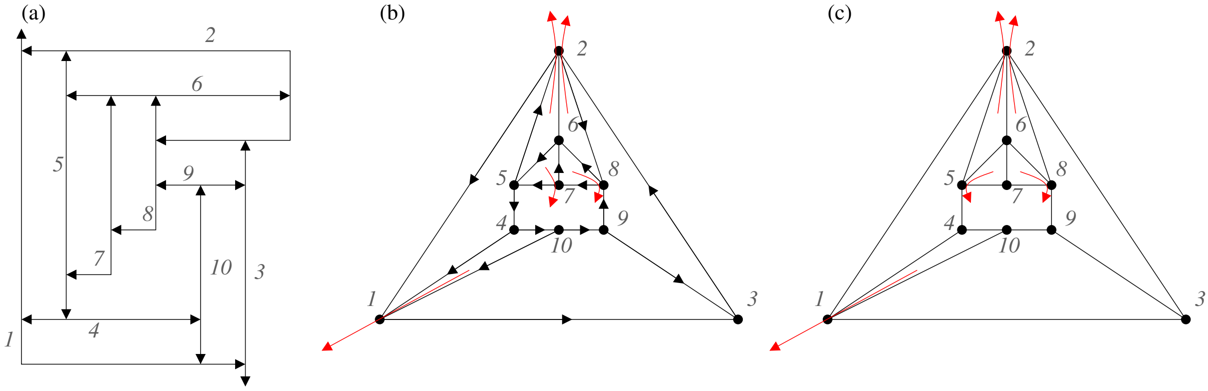

Figure 2: Given a graph G, the figure shows (a) a VCPG of G, the arrows correspond to the orientation of the edges; (b) the agreeing combinatorial VCPG, the orientation is given on the edges, the flow is given by the red arrows between faces through vertices; (c) a flow ψ such that there is no 2-orientation α such that (α, ψ) is realizable, as the orientation must orient $5 \to 7$ and $8 \to 7$ hence 7 can only have outdegree one.

2 Realizable Pairs

A VCPG is not completely described by a plane graph G and a 2-orientation. Therefore we introduce a flow network. We will use a flow in such a network to obtain a full description of a VCPG. We denote the 2-orientation by α and the flow by ψ. In this section we identify a property of a pair (α, ψ) that comes from a VCPG of a plane graph G, hence this property is necessary. On the other hand, not every pair (α, ψ) on G induces a VCPG of G (an example is shown in Figure 2 (c)). We call a pair (α, ψ) *realizable* when it does. We will prove that the necessary property is also sufficient, hence realizable pairs are in bijection to VCPGs. Our proof method is algorithmic, it shows how one can construct a VCPG (the geometric setting) from a realizable pair (the combinatorial setting).

2.1 The Flow Network

From here on, we consider the graph to be simple and 2-connected. Note that any $(2, l)$-tight graph can easily be extended to a 2-connected $(2, l)$-tight graph by adding an appropriate number of degree two vertices. The angle graph $A(G)$ of a plane 2-connected graph G is a plane bipartite graph that arises from G by setting the union of the vertices and faces of G as the vertices of $A(G)$ and the edges of $A(G)$ are the pairs vf, $v \in V(G), f \in F(G)$, such that v is a vertex on f in G. The angle graph is a plane maximal bipartite graph.

Intuitively, a unit of flow in the angle graph from f_1 to f_2 through v is a bend of p_v (the grid-path that represents v) such that the convex angle of this bend lies in f_1 and the concave angle lies in f_2 (see e.g. Figure 2 (a) and (b)). More precise, a flow ψ is a weighted directed graph, with as underlying graph the angle graph $A(G)$. The face-vertices of $A(G)$ can be a source or a sink, depending on the degree. The vertex-vertices of $A(G)$ are neither sources nor sinks. The capacity of the edges is unbounded. The number of bends prescribed by the flow ψ for a vertex v is denoted with $\psi(v)$, which is the sum of incoming flow, which in turn is equal to the sum of outgoing flow. We define $c(f)$ to be the excess of an interior face-vertex f of $A(G)$. The excess prescribes the amount of outgoing flow minus the incoming amount of incoming flow of this face.

Following the boundary of an interior region of a VCPG and adding the changes in direction one should obtain 2π. Each edge is represented as a proper contact and therefore changes the direction with $\pi/2$. A convex angle changes the direction with $\pi/2$ as well and a concave angle changes the direction with $-\pi/2$. By the following equation: $2\pi = \pi/2 \cdot |f| - \pi/2 \cdot c(f)$ where $|f|$ is the number of edges on the boundary of f, the excess of each interior face is:

$$c(f) = 4 - |f|.$$

For the outer face f_∞ of a $(2, l)$-tight graph we set the excess $c(f_\infty) = (2l - 4) - |f_\infty|$. With $|f_\infty|$ we denote the number of vertices on the outer face.

Let ψ be a flow in $A(G)$ that satisfies the excess of each face, then the value of the flow ψ is

$$w(\psi) = \sum_{v \in V(G)} \psi(v).$$

The sum of the excess over all faces cancels out and there is no capacity restraint on the edges, therefore there exists a flow that satisfies the excess of every face. A vertex cannot absorb any flow, as having a convex corner means having a concave corner on the other side. Therefore the minimum value of a flow that satisfies the facial excesses is a lower bound on the number of bends needed for a VCPG[1]. As shown by Figure 2 (c) not every flow that satisfies the facial excesses is related to a VCPG.

2.1.1 Necessary and Sufficient Condition

Given a simple, plane, 2-connected $(2,l)$-tight graph, a 2-orientation α and a flow ψ that satisfies the facial excesses. We will give a necessary and sufficient condition on the pair (α, ψ). When this condition is satisfied, there exists a VCPG that maintains the embedding such that:

(a) The grid-path of u ends on the grid-path of v if and only if the edge uv is oriented from u to v in α, and,

(b) The grid-path of v has precisely $\psi(v)$ bends.

We denote a pair that satisfies the condition *realizable.* Let $A[N_{A(G)}[v]]$ denote the angle graph induced by the closed neighborhood of a vertex v, i.e. induced by v and all its neighbors in $A(G)$.

Definition 2.1 (Realizability Condition)**.** The pair (α, ψ) satisfies the realizability condition at vertex v if and only if given $A[N_{A(G)}[v]]$ and the flow in this subgraph: (see Figure 4)

(a) each incoming unit of flow at v can be matched to an outgoing unit of flow at v, such that any two units of flow proceed without crossing through v and,

(b) if the outdegree of the vertex is two in α, then every unit of flow is cut by the two outgoing edges of v under α, or,

(b') if the outdegree of the vertex is one, there is an incoming edge such that every unit of flow is cut by this incoming edge together with the outgoing edge of v under α.

When the pair (α, ψ) satisfies the realizability condition at each vertex we say that the pair is realizable.

Theorem 2.2. The realizable pairs are in bijection with VCPGs.

The remainder of this section is dedicated to the proof of Theorem 2.2. First we will show that a VCPG induces a realizable pair (α, ψ) and then the converse, i.e. we will construct a VCPG from a realizable pair.

Lemma 2.3. A pair (α, ψ) that comes from a VCPG is realizable.

Proof. First note that a VCPG of G describes an embedding of G. If there is a grid-path with one free end, then before proceeding we reduce all unnecessary bends, i.e. if a grid-path has bends between its last neighbor and its free end, these bends are removed. A 2-orientation can be constructed from a VCPG by orienting an edge $u \to v$ if and only if the grid-path of u ends on the grid-path of v. Consider the grid-path that represents a vertex v. If this path has no bends, the realizability condition is satisfied at this vertex. Suppose the path has k bends. Draw an arrow from the face containing a convex corner to the face in which the associated concave corner lies. Now the set of arrows represents the flow $\psi(v)$. This flow is non-crossing through v and every unit of flow is cut by the the grid-path of v. When these arrows are introduced for all bends of all grid-paths, the flow given by these arrows satisfies the excess

[1]Note that there might be different bounds for different embeddings of the same graph.

of each face. Contract the strictly interior steps of the grid-path to a vertex and every unit of the non-crossing flow through v is now cut by the outermost two segments of the grid-path, which correspond to the outgoing edges of v, or to the outgoing edge and the location of the last incoming edge before the free end of the grid-path. Hence the realizability condition is satisfied at each vertex, therefore the pair obtained from the VCPG is realizable. □

To prove the converse we will show how to construct a VCPG given an realizable pair. Note that an embedding follows from the map $A(G)$ (in which the flow ψ is defined). Consider a realizable pair (α, ψ) (see Figure 3 (a)). The proof consists of four steps, which we first outline here:

Step 1: First we expand the vertices that have k units of flow going through them, to a path of length k. We obtain a bipartite graph.

Step 2: We introduce help edges and vertices in the bipartite graph to construct a quadrangulation (see Figure 3 (b)).

Step 3: We then find a segment contact representation of the quadrangulation. It has been shown that the 2-orientations of maximal bipartite planar graphs are in bijection with separating decompositions of this graph (e.g. [dFdM01]). In turn a separating decomposition induces a segment contact representation (cf. [HNZ91, dFdMP95, Fel13]). Hence we can construct a segment contact representation where the representation of the edges is in bijection with the given 2-orientation. An example is shown in Figure 3 (c).

Step 4: Last we will show that the extra edges that have been introduced to make a quadrangulation of the bipartite graph can be deleted in order to obtain a VCPG of G (see Figure 3 (d)).

Step 1. Given a realizable pair (α, ψ) for G. We expand all vertices with non-zero flow. The plane graph we obtain is denoted $\tilde{G}$. For every vertex v for which $\psi(v) \neq 0$, expanding v denotes the following steps (see Figure 4):

1. Expand v to a circle, we will denote this the *bag* of v.
2. Between the two outgoing edges of v, or the special incoming and the outgoing edge of v if v has outdegree one, inside the circle, add a path with $\psi(v) + 1$ vertices.
3. Connect the edges that end on the circle to the path vertex in such a way that the flow between two faces only crosses an edge of the path.

Step 2. After all the expansions have been done we obtain a graph where all faces have even length. Each face gets $|4 - |f|| + 2k$ extra vertices due to the expanding step, where k is the amount of flow proceeding through the face. The resulting faces in $\tilde{G}$ have size $|f| + |4 - |f|| + 2k$, for $|f| = 3$ this gives $4 + 2k$ and for $|f| > 3$ this gives $2|f| - 4 + 2k$, both are even. So in $\tilde{G}$ all faces have even length and therefore $\tilde{G}$ is a bipartite graph. Now we add help edges to extend $\tilde{G}$ to a quadrangulation. We denote the quadrangulation G_Q. We will also orient the new edges to obtain a 2-orientation of G_Q. In order to explain how the help edges are added, we need the following lemma.

Lemma 2.4. Every interior face $\tilde{f}$ of $\tilde{G}$ has $(|\tilde{f}| - 4)/2$ units of incoming flow.

Proof. Let $\psi^+(f)$ (resp. $\psi^-(f)$) denote the incoming (resp. outgoing) flow in face f. Let f be the equivalent face of $\tilde{f}$ in the original graph G.

The excess $c(f)$ of f is the difference between incoming and outgoing flow:

$$\psi^+(f) - \psi^-(f) = c(f) = |f| + 4\ .$$

Now we use that the size of the extended face $\tilde{f}$ is the size of f plus the incoming and the outgoing flow.

$$|\tilde{f}| = |f| + \psi^+(f) + \psi^-(f) = |f| + \psi^+(f) + \psi^+(f) - |f| + 4 = 2\psi^+(f) + 4$$

Hence we find

$$\psi^+(f) = \frac{|\tilde{f}| - 4}{2}\ . \tag{2}$$

□

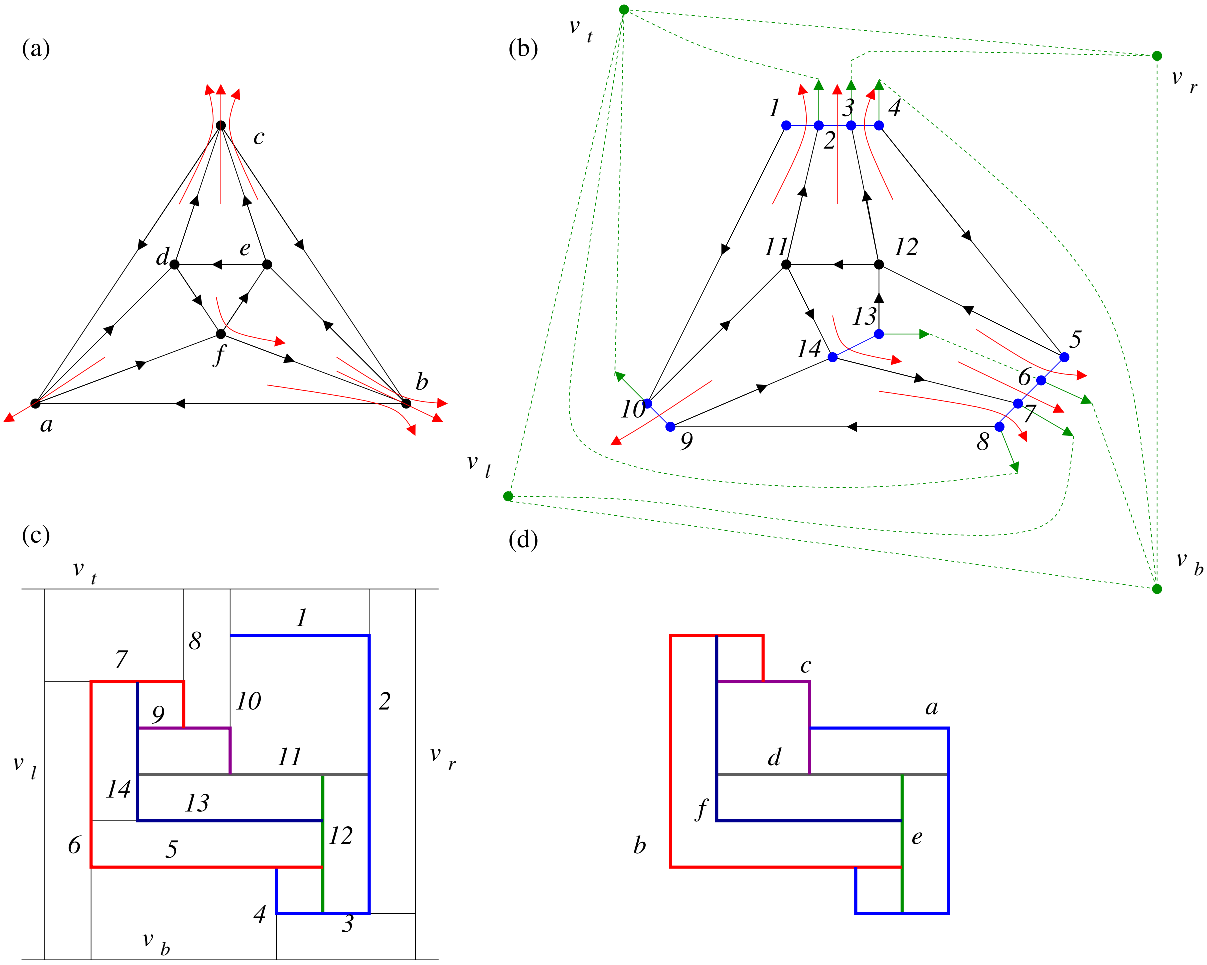

Figure 3: From a combinatorial VCPG to a geometric VCPG: (a) a plane (2,0)-tight graph with a realizable pair (ψ in red); (b) expanding the vertices according to the flow (in blue) and extending the bipartite graph to a quadrangulation (in green); (c) a segment contact representation of the quadrangulation with the segments that belong to the orignal graph highlighted; (d) a geometric VCPG.

Using $(|\tilde{f}| - 4)/2$ edges one can quadrangulate $\tilde{f}$. The help edges should be added in such a way that every bag (vertex expansion) gets as many help edges as it has flow going into $\tilde{f}$ in the flow ψ. Informally, a concave corner is arises from two segments that both end in one point. The theory of segment contact representations that we will use, there are only proper contacts or free ends. Each help edge represents a segment of a concave corner that proceeds into the face, this part will then later be removed. Such a help edge will be oriented outgoing from the bag.

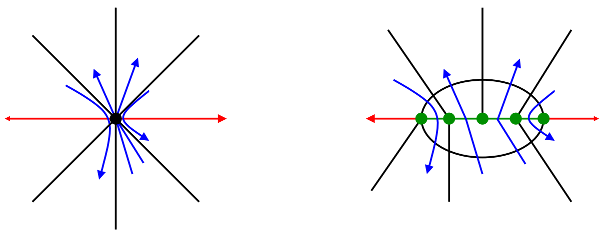

Figure 4: Expanding a vertex. The flow ψ is depicted by blue arcs, the red arcs represent the outgoing edges of the vertex in α, the black lines are the incoming edges.

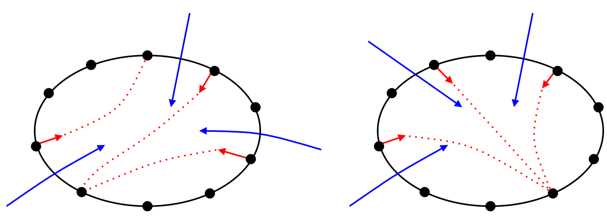

Figure 5: Adding help edges in a face. The flow ψ is depicted by blue arcs. The red half-arcs together with the dashed extensions represent the help edges.

Later the segment contact representation is constructed and for this it is necessary that every interior vertex has outdegree two. Therefore the help-edges must be added in such a way that this is possible for all vertices. Each interior bag vertex should precisely two outgoing arcs (which are not edges in the original graph) and the vertices on the boundary of the bag get precisely one outgoing arc that is not in the original graph. The help edges are added along flow from a vertex into a face, this will give the correct amount of new edges for every bag.

Lemma 2.5. Each inner face $\tilde{f}$ of $\tilde{G}$ can be quadrangulated in such a way that each bag through which k units of flow enter $\tilde{f}$ gets k new outgoing arcs. The outer face of $\tilde{G}$ can be quadrangulated using four help-vertices (v_t, v_r, v_b, v_l), in such a way that each bag through which k units of flow enter the outer face gets k new outgoing arcs.

An example of quadrangulating an interior face is depicted in Figure 5. The flow ψ is given by the blue arrows. First half-arcs are added, the red solid arcs. Then these half-arcs are subsequently connected in such a way that they close one 4-face (red dashed lines). The quadrangulation of the outer face is based on the same idea.

Proof. Assign a half-arc into $\tilde{f}$ from each vertex clockwise after a unit of incoming flow, see Figure 5. There are $(|\tilde{f}| - 4)/2$ vertices who get a half-arc thus $(|\tilde{f}| + 4)/2$ without a half-arc. Hence there exists a vertex with a half arc which is followed by two consecutive vertices u, v without a half-arc. Consider the half-arc clockwise before u, v and connect it to the vertex after u, v. Now we have constructed a 4-face and completed one half-arc. This step can be repeated considering the resulting face and its half-arcs. By counting it follows that all half-arcs can be extended into arcs with this method and that the result is a quadrangulation of $\tilde{f}$ in such a way that each bag through which k units of flow enter $\tilde{f}$ gets k new outgoing arcs.

To take care of the outer face we add a quadrilateral around our graph and quadrangulate the new inner face between the graph and the quadrangle.

We distinguish the four cases, (1) there exists a vertex s in the original graph G which has no outgoing arcs under α, (2) there exist two vertices s, t in the original graph G which both have precisely one outgoing arc under α, (3) there exists precisely one vertex s which has precisely one outgoing arc under α, (4) all vertices have outdegree 2 under α. We add a quadrilateral around our graph and construct an inner face where the units of incoming flow is k and its size is $2k + 4$. Then we can use the same method as for the interior faces.

(1) Note that $l = 2$. First add a quadrangle around the graph with vertices v_l, v_t, v_r, v_b in clockwise order. Now s is expanded[2], we label the expansion in counterclockwise order following the boundary of $\tilde{f_\infty}$ into $s_1, \ldots, s_k$. Add arcs (s_1, v_t) and (s_1, v_b). Now the bounded face f^* containing s_1, v_t, v_r, v_b on its boundary has the following properties: $|f^*| = |\tilde{f_\infty}| + 4$, it has $\frac{1}{2}|\tilde{f_\infty}|$ incoming flow. We consider the same method as for an inner face, add half-arcs and consecutively make 4-faces.

(2) Note that $l = 2$. First add a quadrangle around the graph with vertices v_l, v_t, v_r, v_b in clockwise order. If s and t are expanded, we label the expansion vertices in the respective bags such that s_1 and t_1 have no outgoing arc under α and they are end vertices of the extension path. If s resp. t is not expanded, we label $s = s_1$ resp. $t = t_1$. Add arc (s_1, v_t) and let f^* be the face between the quadrangle and $\tilde{f_\infty}$. Let ψ_{t_1,s_1} denote the incoming flow to f^* between t_1 and s_1 clockwise around $\tilde{f_\infty}$. Assign the label q to the vertex at distance $2\psi_{t_1,s_1} + 3$ from t_1 walking counterclockwise around f^*. Add the arc (t_1, q). Now we have obtained two faces f_u, f_d, for which the incoming flow $\psi^+(f_u), \psi^+(f_d)$ is equal to $|f_u|/2 - 2$, $|f_d|/2 - 2$. We consider the same method as for an inner face, add half-arcs and consecutively make 4-faces.

(3) Note that $l = 1$. Add a quadrangle around the graph with vertices v_l, v_t, v_r, v_b in clockwise order. The vertex s has outdegree precisely one under α. Label the vertex in the bag of s that has no outgoing edge and is an end vertex of the extension path s_1 or if s is not expanded we label $s = s_1$. Add arc (s_1, v_t). We obtain a face f^* between the quadrangle and $\tilde{f_\infty}$ for which the incoming flow is of size $\frac{1}{2}|\tilde{f_\infty}| + 1$ and $|f^*| = |\tilde{f_\infty}| + 6$. We consider the same method as for an inner face, add half-arcs and consecutively make 4-faces.

[2] If s is not expanded then we label $s = s_1$.

(4) Note that $l = 0$. Add a quadrangle around the graph with vertices v_l, v_t, v_r, v_b in clockwise order and let f^* be the face between the quadrangle and $\tilde{f_\infty}$. We will use one unit of flow to connect $\tilde{f_\infty}$ to the quadrangle. First add a half-arc into f^* from each vertex clockwise after a unit of incoming flow. Choose any half-arc and connect it to v_t. We obtain a face f with $\frac{1}{2}|\tilde{f_\infty}| + 1$ incoming flow and $|f| = \tilde{f_\infty} + 6$. We consider the same method as for an inner face, add half-arcs and consecutively make 4-faces. □

To obtain a quadrangulation G_Q with a 2-orientation, we still need to orient the edges that are strictly inside the bags and the four boundary edges. The orientation of all other edges comes from α. Each bag b_v contains $|b_v| - 1 = \psi(v)$ edges which are not yet oriented, all others are oriented and such that b_v has outdegree $|b_v| + 1$.

Lemma 2.6. Each bag b_v in G_Q has precisely outdegree $|b_v| + 1$ and each vertex ($\in b_v$) has outdegree at most two.

Proof. If a bag b_v comes from a vertex v which has outdegree 2 in α, then the expansion results in a bag of size $\psi(v) + 1$. According to the flow, $\psi(v)$ outgoing arcs are added, so in total we find outdegree $|b_v| + 1$.

If a bag b_v comes from a vertex v which has outdegree 1 in α, then the expansion results in a bag of size $\psi(v) + 1$ and the quadrangulation step has assigned another outgoing arc to this bag, hence again we find we find outdegree $|b_v| + 1$.

If a bag b_v comes from a vertex v which has outdegree 0 in α, then the expansion results in a bag of size $\psi(v) + 1$ and the quadrangulation step has assigned two outgoing arcs to this bag, hence again we find we find outdegree $|b_v| + 1$.

Suppose one of the bag-vertices has outdegree three or more. At most one of the arcs can be an edge of the original graph. When a vertex gets an outgoing arc from the flow, this means that it must be clockwise after a flow into an adjacent face. If a vertex is twice clockwise after a flow, it must be incident to two faces on each side of the path, hence this vertex is not an end vertex of the path and it can not have an outgoing arc that is an edge of the original face. □

Orient $(v_l, v_t), (v_r, v_t), (v_l, v_b)$ and (v_r, v_b) towards v_t respectively v_b, the two poles of the 2-orientation. The orientation of the path edges is now trivial.

Lemma 2.7. The path edges can be oriented (greedily) such that the resulting orientation is a 2-orientation of G_Q.

Proof. We subsequently orient the edges of a path towards a vertex which has outdegree two and show that in each step, all neighbors of a vertex with outdegree two, have outdegree at most one. Hence we can continue orienting edges until all edges are oriented.

First we note that a neighbor of a path vertex with outdegree two can not also have outdegree two, as through every path edge at most one unit of flow proceeds and hence a neighboring pair in the path has together outdegree at most three. Consider a vertex of a path v which has outdegree two. Let u be a neighbor of v on the path, v has at most outdegree one. We need to show that if it has outdegree one, its next neighbor has at most outdegree one or it is an end vertex of the path. Suppose u has outdegree one and is not an end vertex of the path, let w be the next vertex on the path. Vertex v is the clockwise subsequent vertex for the flow through (u, v), so the outgoing arc of u must come from the flow through (u, w). Hence for w the only possibility is an arc that comes from flow through the next edge of the path or w is an end vertex. Either way, w has outdegree at most one. Hence we orient the path edge from u to v and again consider a vertex with outdegree two.

As the number of path edges is precisely the number of outarcs needed for every bag, this process will end and each bag vertex has outdegree precisely two (except for the two poles: v_t and v_b. All vertices that are not in a bag, already had outdegree two in α and did not get new outgoing arcs. Hence we have a 2-orientation $\hat{\alpha}$ of G_Q. □

Step 3. From G and the realizable pair (α, ψ) we constructed the quadrangulation G_Q with a 2-orientation $\hat{\alpha}$ (v_t and v_b are the only two vertices with outdegree zero instead of outdegree two). We construct a segment contact representation, i.e. the vertices of the two color classes become horizontal respectively vertical segments and the edges are proper contacts between the segments satisfying $\hat{\alpha}$ (cf. [HNZ91, dFdMP95, Fel13]).

Step 4. Last to show is that this segment representation of G_Q is equivalent to a VCPG of G where the path of a vertex v is given by its outgoing arcs in α and $\psi(v)$ denotes the number of bends of the path of v. For this we need the following lemma, which shows that the sets of segments ending on different sides of a segment s can be moved indepently.

Lemma 2.8. Given a horizontal segment in a segment contact representation, then its (vertical) bottom neighbors can be shifted independently from its (vertical) top neighbors. The same holds for a vertical segment and its left resp. right incoming neighbors.

Proof. Suppose the vertical segment v_1 ends on the bottom and v_2 ends on the top of the horizontal segment h. The segment v_1 is left of the segment v_2 and we want it to be right of v_2. Consider a cutline as follows: from h just left of v_1 to the top of the drawing such that it intersects only horizontal segments; from h just to the right of v_2 to the bottom, such that it only intersects horizontal segments; and the part on h between the these two vertical cutlines. We cut the graph along the cutline. We move the half containing v_1 to the right such that v_1 now is right of v_2. Extend all the horizontal lines that are cut. We have obtained a new, equivalent, segment contact representation where v_1 is right of v_2. Figure 6 depicts such a move. □

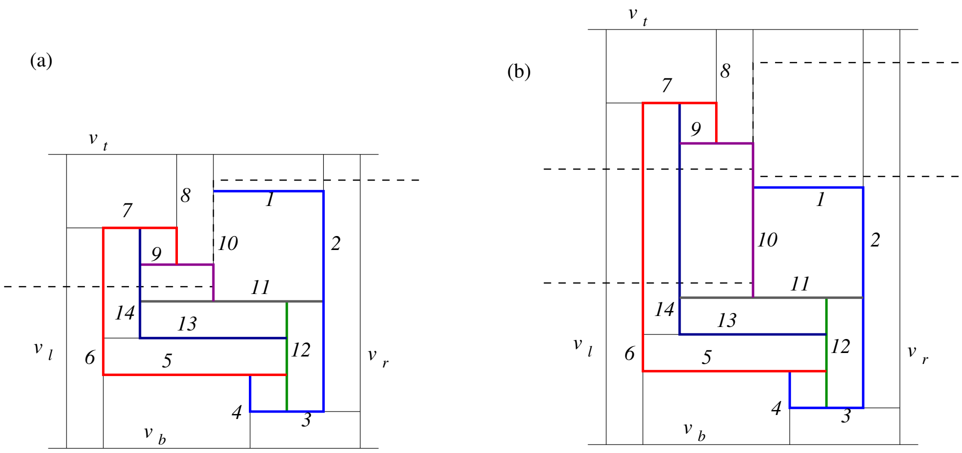

Figure 6: Cutting open and shifting shows that neighbors on each side can be moved independently: (a) the graph of Figure 1 (c) where highlighted part of 1 does not end on the highlighted part of 10, and a cutting line (dashed); (b) the graph is cut and the top is pulled upwards extending the vertical segments that are cut.

Theorem 2.9. The segment representation of G_Q obtained from an realizable pair (α, ψ) induces a VCPG of G.

Proof. We apply identification, shifting and deletion to the segment representation of G_Q and will then show that the result is a VCPG of G.

Identification. For each vertex v that is not expanded we color the segment with color v. For each bag b_v, select the segments representing the bag vertices $v_1, \ldots, v_k$. For each bag vertex v_i color the part of the segment between v_{i-1} and v_{i+1} with color v. For the end vertices of the bag, v_1 (resp. v_k) color the part of the segment between v_2 (resp. v_{k-1}) and the outgoing neighbor of v_1 (resp. v_k) that is not in the bag. Now the grid-path of vertex v is highlighted among the selected segments. When $l = 1, 2$ we have added arcs while quadrangulating the outer face, that do not correspond to the flow nor to original edges. These edges represent the free ends of grid-paths. Hence we prolongue these to be just further than the last neighbor ending on this segment.

Shifting. It may occur that for an arc of G, say (u, v), the endpoint from u on v appears on a non-highlighted part of v, in this case we need to shift. We let v_i be the vertex represented by the segment on which u ends, and v_{i-1}, v_{i+1} the neighbors of v_i in the bag. We need that around v_i, there are consecutively one outgoing edge, at most one incoming edge from a bag neighbor, say v_{i-1}, incoming other edges, one outgoing edge, at most one incoming edge from a bag-neighbor, say v_{i+1}, incoming other edges. In other words, in either clockwise or counter clockwise order, there is no incoming edge of the original graph between the outgoing edge and the incoming bag edge. This ensures (by Lemma 2.8) that there is a segment representation in which the contacts of edges of the original graph to v_i lie between the contacts of v_{i-1} and v_{i+1} with v_i. The statement follows from the construction of G_Q. If the outgoing edges of v_i are to v_{i+1} and v_{i-1} it is trivial. Suppose not, w.l.o.g. the outgoing edge north of v_i can only be induced by flow through (v_{i+1}, v_i) and south of v_i by (v_{i-1}, v_i). If both appear, then these outgoing edges are just before the incoming bag edges in cw or ccw order, hence we are done. If at most one appears, say v_i has one outgoing edge to v_{i+1}, then the other outgoing edge is just before the incoming edge from v_{i-1}, i.e. it comes from the flow going through the edge (v_i, v_{i-1}).

Hence we find that there exists a segment representation in which the contacts of edges of the original graph to v_i lie between the contacts of v_{i-1} and v_{i+1} with v_i. Moreover in the representation we have, we can shift v_{i-1} and v_{i+1} such that all other contacts to v_i lie between them. The coloring extends trivially along the shifting.

Deletion. After the shifting, all endpoints representing edges of G occur between highlighted segments. Therefore we can delete all non-highlighted parts of the segments without losing edges of the original graph.

Conclusion. It follows from the three steps that each edge (u, v) of G is represented by a non-degenerate contact. If this edge is oriented from u to v then the path of u ends on v. Moreover each vertex v is represented by $\psi(v) + 1$ segments, hence it is a grid-path with $\psi(v)$ bends. Hence the result is a VCPG of G that agrees with α and ψ. □

With the four steps we have obtained a VCPG from an realizable pair. This completes the proof of Theorem 2.2. In the remainder of this paper we will use realizable pairs to give bounds on the number of bends for certain graph classes.

3 Not all (2,2)-tight graphs are B_1-VCPG

In a B_1-VCPG each vertex is represented by a grid-path with at most one bend. Strict B_1-VCPG is a proper subclass of B_1-VCPG (this is shown in the full version of this paper [AF]). Suppose a graph has a B_1-VCPG, then there must be an realizable pair (α, ψ) such that $0 \leq \psi(v) \leq 1$ for all vertices v. Consider the plane graph on the right of Figure 7. Suppose it has a flow ψ which satisfies $0 \leq \psi(v) \leq 1$ for all vertices v. The two grey-colored K_4 subgraphs both have excess $c(K_4) = 3$. So there must be 6 units of flow going out of the two K_4 subgraphs, however there are only 5 different vertices bounding the two K_4 subgraphs. We conclude that there is no flow that satisfies all excesses such that there is at most one unit of flow going through each vertex.

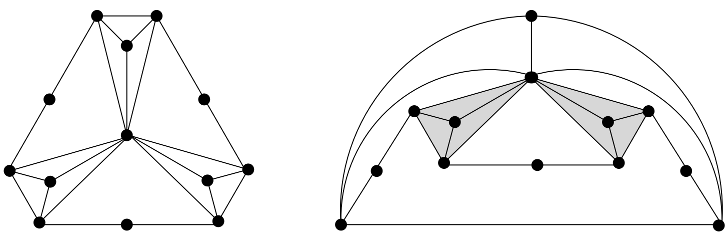

Figure 7: Two embeddings of a planar (2,2)-tight graph that is not B_1-VCPG.

On the other hand, graphs that do not have a vertex that is incident to many (almost-disjoint) critical sets, might be B_1-VCPG. In particular, it follows easily from the construction of strict B_1-VCPGs of planar Laman graphs that planar Laman-plus-one graphs have a strict B_1-VCPG. For the proof we need the following lemma. In a *monotone* VCPG each grid-path may have many bends, but it is monotone, i.e. there is no ⊔-shape.

Proposition 3.1. Let G be a (2,2)-tight graph, and $\mathcal{E}_G$ a monotone VCPG of G. If two vertices u, v contribute a straight line segment to the boundary of an interior face f in $\mathcal{E}_G$, then u and v do not bound f on the same side.

Proof. Consider two vertices u and v, to contribute a straight line segment to the boundary of an interior face f in such a way that both bound f on the north side. Since there is no free end inside f, u and v cannot be neighbors (as in this case there would not be a proper contact between them). Suppose there is a path on the boundary of f between u and v and let w be the neighbor of u in this sequence. Then, w induces either no change in direction (c) or a change of $\frac{1}{2}\pi$ (e) or a change of π (a). However to introduce v at some point, we need to be at angle $-\frac{1}{2}\pi$ or $-\pi$ w.r.t. to the angle at u. It is clear that in a plane represention this can only be achieved using free ends. □

A Laman-plus-one graph G is a (2,2)-tight graph, such that there exists an edge e in G for which $G - e$ is (2,3)-tight, i.e. a Laman graph. Note that (2,2)-tight graphs that are not Laman-plus-one but also do not have a vertex that is incident to many (almost-disjoint) critical sets might be B_1-VCPG but they are not necessarily *strict* B_1-VCPG, as shown by the example in Figure 8.

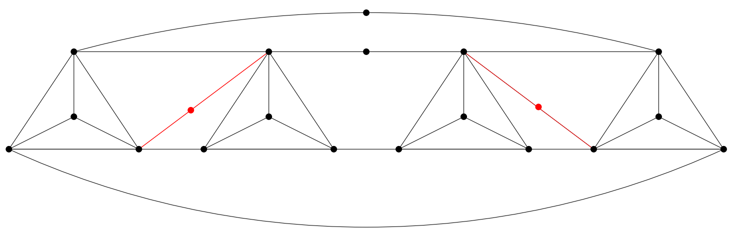

Figure 8: A graph that has a B_1 representation such that the red vertices must be represented by segments (i.e. a grid path without bends).

Theorem 3.2. Every planar Laman-plus-one graph has a strict B_1-VCPG. In this representation precisely one face has two convex corners, the outer face has no convex corner, all other faces have one convex corner[3].

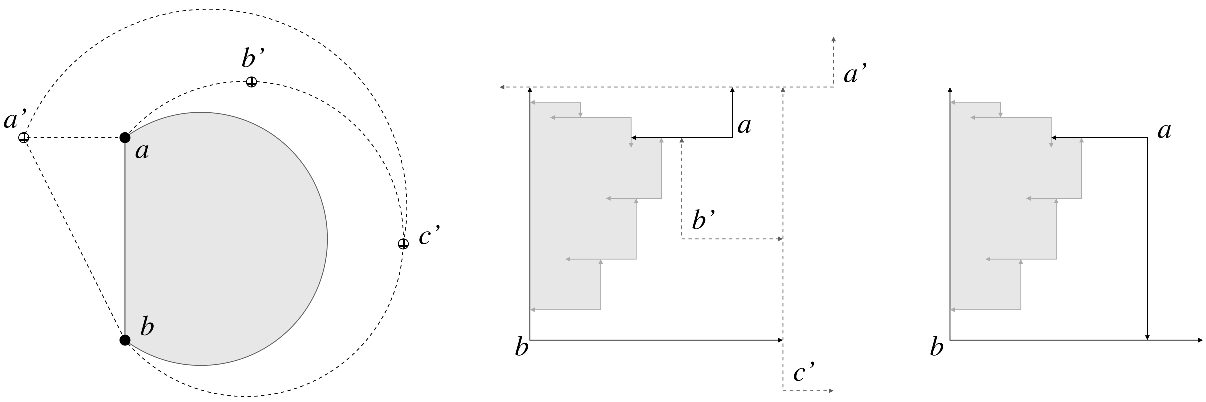

Figure 9: Extension of $G - e$ and adding e.

[3]In the construction of Kobourov et al. each face has precisely one convex corner, they denote this *proper L-contact graph*.

Proof. Let G a planar Laman-plus-one graph and $e = ab$ an edge such that $G - e$ is a Laman graph. Consider an embedding of G such that e is incident to the outer face. We construct an extended graph which has a triangular outer face (see Figure 9). First add a' and connect it to a and b, then add b', connect it to a and b in such a way that the outer face now consists of a, a', b and b'. Subdivide the edge bb', call the new vertex c' and connect it to a' such that the outer face now consits of a', b and c'. Note that the addition of a', b' and c' are Henneberg construction steps and the graph $G - e$ with the extension, denoted $G' - e$ is a Laman graph. We construct a strict B_1-VCPG of $G' - e$ according to the method of Kobourov, Ueckerdt and Verbeek [KUV13], in such a way that a' and c' are the special vertices. Consider the face $a'ac'b'$. Suppose that a is the unique convex corner for this face in the strict B_1-VCPG. Then b' must admit a concave corner to this face and has its convex corner in the other incident face, i.e. the face with b, c', b', a and a path from a to b on its boundary. It follows that a and b both admit a straight line segment to this face, in such a way that the face is on the same side of a and b, this is a contradiction of Proposition 3.1. Therefore b' is the unique convex corner for this face. Removing a', b' and c' from the representation leaves a strict B_1-VCPG in which one leg of a has no other vertices ending on it. Moreover, no vertex but one of the free ends of b is to the right of this leg of a. Hence we move this leg in such a way that it ends on b, the result is a strict B_1-VCPG of G. It follows from the construction of Kobourov, Ueckerdt and Verbeek that all faces except the two incident to the bend of a have precisely one convex corner, the face incident to the convex corner of a has also a convex corner from b and the outer face has no convex corners. □

Note that a Laman-plus-one graph has precisely one critical set, namely the whole graph. When all minimal critical sets are disjoint it may be possible to use a structure as we used above to obtain a 1-bend VCPG of all minimal critical sets and glue them together (using 1-bend or 0-bend vertices). However when the critical sets intersect this is no longer possible.

4 Locally Minimizing Bends

For simple (2,2)-tight graphs we show that for every 2-orientation with one sink α, there exists a flow ψ such that $\psi(v) \leq 2$ for all vertices v and the pair (α, ψ) is realizable.

Theorem 4.1. Every planar (2,2)-tight graph is B_2-VCPG.

Proof. Let G a planar (2,2)-tight graph and $\mathcal{E}$ a planar embedding of G. Hence we have a dual graph G^*. The excess of the face-vertices in the dual is given by $c(f) = 4 - |f|$ and for the outer face it is $c(f_\infty) = -|f_\infty|$.

For every subset of face-vertices H there are at least $|\sum_{f \in H} c(f)|$ edges leaving H in the dual graph. Let b the number of edges leaving H, i.e. the number of boundary edges in the primal w.r.t. H. Note that we only count interior faces (when we use Euler's formula).

$$\left|\sum_{f \in H} c(f)\right| = \left|\sum_{f \in H} 4 - |f|\right| = |4|H| - 2e_H + b| \leq |4|H| - 4|H| + b| = b$$

Hence we can satisify all excess in the dual graph by using every edge at most once. Given an edge-disjoint flow in the dual graph, such that the excess of all faces is satisfied and the capacity of each edge is one. Consider any orientation α of G such that every vertex has outdegree at most 2. In such a way that there is either one vertex with outdegree 0 incident to the outer face, or two vertices with outdegree 1 incident to the outer face. We construct the flow ψ in the angle graph as follows: If there is a flow from f_1 to f_2 crossing edge uv, then if $u \to v$ we add $f_1 \to u$ and $u \to f_2$ to ψ. If $v \to u$ we add $f_1 \to v$ and $v \to f_2$ to ψ.

Since the flow in the dual graph is edge-disjoint and each vertex has at most two outgoing edges we have $\psi(v) \leq 2$ for all vertices v. At each vertex the flow cuts off the outgoing edge, hence the expanding condition is satisfied at each vertex.

We conclude that the pair (α, ψ) is realizable. □

Theorem 4.2. Every loopless planar (2,1)-tight graph is B_4-VCPG.

Proof. We claim that there is a flow in the dual such that each edge has capacity two. Similarly to the proof of Theorem 4.1 we obtain that a loopless planar (2,1)-tight graph is B_4-VCPG.

For every subset of face-vertices H there are at least $|\sum_{f\in H} c(f)|$ edges leaving H in the dual graph. Let b the number of edges leaving H, i.e. the number of boundary edges in the primal w.r.t. H. Note that we only count interior faces (when we use Euler's formula).

$$\left|\sum_{f\in H} c(f)\right| = \left|\sum_{f\in H} 4-|f|\right| = |4|H|-2e_H+b| \leq |4|H|-4|H|+2+b| = 2+b$$

As $b > 1$ we can satisfy the flow using b edges with capacity 2. □

We have only been able to show that the lower bound is three (e.g. the octahedron deleted an interior edge), hence we conjecture that loopless planar (2,1)-tight graphs are B_3-VCPG.

Theorem 4.3. Every loopless planar (2,0)-tight graph is B_6-VCPG.

Proof. We claim that there is a flow in the dual such that each edge has capacity three. Similarly to the proof of Theorem 4.1 we obtain that a loopless planar (2,0)-tight graph is B_6-VCPG.

For every subset of face-vertices H there are at least $|\sum_{f\in H} c(f)|$ edges leaving H in the dual graph. Let b the number of edges leaving H, i.e. the number of boundary edges in the primal w.r.t. H. Note that we only count interior faces (when we use Euler's formula).

$$\left|\sum_{f\in H} c(f)\right| = \left|\sum_{f\in H} 4-|f|\right| = |4|H|-2e_H+b| \leq |4|H|-4|H|+4+b| = 4+b$$

As $b > 1$ we can satisfy the flow using b edges with capacity 3. □

Again, the only lower bound we have been able to show is three (e.g. the octahedron), hence we conjecture that loopless planar (2,0)-sparse graphs are B_3-VCPG.

4.1 A Laman-Plus-One Variant of (k,l)-tight

In the case of planar (2,2)-tight graphs we have obtained a result on a subclass, namely Laman-plus-one graphs. This subclass is denoted Laman-plus-one by Haas et al. in [HOR+05] and it is such that there exists an edge such that the graph deleted this edge is a Laman graph. We have shown that Laman-plus-one graphs are B_1-VCPG (actually we have shown that they are strict B_1-VCPG). In this section we consider similar subclasses of $(2,l)$-tight graphs.

Planar $(2,2)$*-tight-plus-one graphs* are (2,1)-tight graphs such that there exists an edge e such that the graph deleted e is a (2,2)-tight graph.

Theorem 4.4. Loopless planar $(2,2)$-tight-plus-one graphs are B_3-VCPG such that precisely one grid-path has three bends.

Proof. Let G be a loopless planar (2,2)-tight-plus-one graph. Consider an edge e such that $G-e$ is $(2,2)$-tight. Let $\mathcal{E}_e$ an embedding of G such that e lies in the outer face. Now $\mathcal{E}_e - e$ has a B_2-VCPG for every 2-orientation. Let x a vertex of e and α a 2-orientation of $G-e$ such that x has outdegree 0. Now there is a B_2 representation of $G-e$ such that x is represented by a segment with two free ends in the outer face. The other end of e, say y, also lies in the outer face. Now the edge e can be added by taking a free end of x and let it end on y using at most three bends. Hence we find a B_3-VCPG such that all vertices but x have at most two bends and x has at most three. □

Planar $(2,1)$*-tight-plus-one graphs* are (2,0)-tight graphs such that there exists an edge e such that the graph deleted e is a (2,1)-tight graph.

Theorem 4.5. Loopless planar $(2,1)$-tight-plus-one graphs are B_5-VCPG such that precisely one grid-path has five bends.

Proof. Let G be a loopless planar (2,1)-tight-plus-one graph. Consider an edge e such that $G - e$ is $(2,1)$-tight. Let $\mathcal{E}_e$ an embedding of G such that e lies in the outer face. Now $\mathcal{E}_e - e$ has a B_4-VCPG for every 2-orientation. Let x a vertex of e and α a 2-orientation of $G - e$ such that x has outdegree 1. Now there is a B_4 representation of $G - e$ such that x is represented by grid-path with at most two bends and with one free end in the outer face. The other end of e, say y, also lies in the outer face. Now the edge e can be added by taking the free end of x and let it end on y using at most three bends. Hence we find a B_5-VCPG such that all vertices but x have at most four bends and x has at most five. □

4.2 Constructive Argument for (2,2)-tight graphs

A constructive argument can be build upon the construction of simple (2,2)-tight graphs given by Nixon [Nix11] based on earlier work e.g. [NOP12].

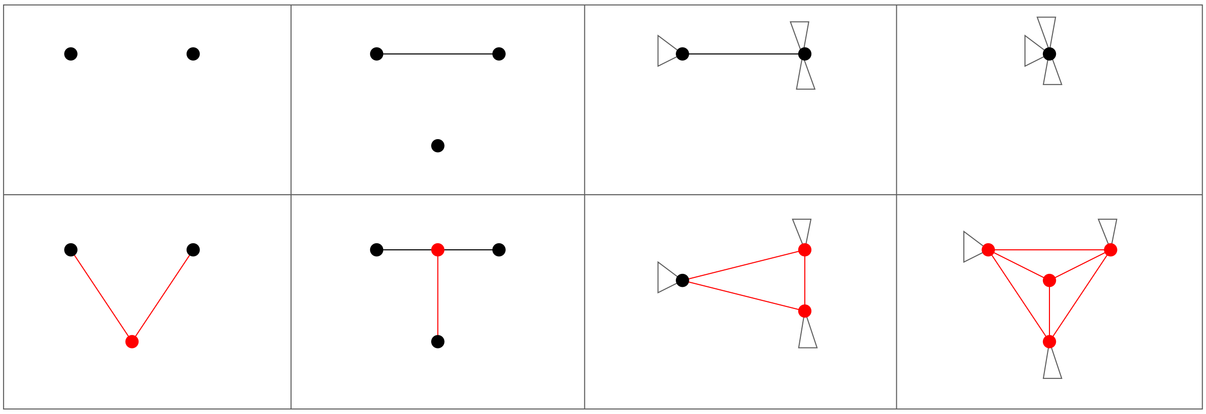

Figure 10: The four steps: Henneberg type 1, Henneberg type 2, edge-to-K_3 and vertex-to-K_4. In the latter two the sets of neighbors are given by the grey colored triangles.

4.2.1 Construction of a B_2-VCPG for (2,2)-tight planar graphs.

We will build an edge-disjoint flow in the dual graph. Then given any orientation, the flow of the dual graph can be mapped to a flow in the angle graph such that this flow together with the orientation is realizable.

Nixon has shown that every simple (2,2)-tight graph G is derivable from K_4 by the Henneberg type 1, Henneberg type 2, vertex-to-K_4 and edge-to-K_3 moves. Moreover a planar (2,2)-tight graph has such a construction in which all intermediate graphs are planar. We will build a flow in the dual graph along the plane construction of G.

We start with K_4 and the flow in the dual graph that leaves the three inner faces through the edges that are incident to the outer face (as given in Figure 11). Throughout the construction we maintain an edge-disjoint flow in the dual graph.

Extension along Henneberg type 1. Vertex x is added in face f and connected to two vertices of f say u and v. The face f is now splitted into two faces f_1, f_2 and the discrepency is at most two, i.e. in the worst case f_1 now has excess 2 and f_2 has excess -2. To prove this we consider the dual graph, and in particular the vertex f and its neighbors. The excess of f is 0 before the Henneberg type 1 step. The degrees of f, f_1 and f_2 in the dual graph satisfy $\deg(f) = \deg(f_1) + \deg(f_2) - 4$, moreover the excess of

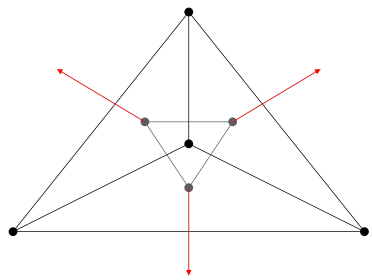

Figure 11: K_4 an a flow in the dual graph.

f is equal to the sum of the excesses of f_1 and f_2, so at most for one of the faces the excess is positive, say f_1. We also know that the current flow is edge-disjoint. Suppose the excess of f_1 is larger than 2, hence it has more than $\deg(f_1) - 2$ incoming edges, but the two edges to f_2 have not been used in the current flow. Therefore f_1 has at most $\deg(f_1) - 2$ incoming edges and thus at most excess 2. Therefore at most two edges need to be directed from f_1 to f_2. As there are two edges not in yet used in the flow, that connect f_1 with f_2, we can extend the flow in an edge-disjoint way.

Extension along Henneberg type 2. Edge uv is subdivided and the new vertex x is connected to w. The face f is splitted in f_u and f_v, the face f_x on the other side of the edge uv has gotten one more vertex, x. Consider the faces in the dual graph and the flow, deleted the flow that uses the edge (f, f_x). Now the excess of f_x is at least 0 and at most 2 and the sum of the excesses $c(f_x) + c(f_u) + c(f_v)$ is zero.

Suppose $c(f_x) = 0$, then in the worst case $c(f_1) = -c(f_2) = 2$ and we add flow $f_1 \to f_x, f_2$ and $f_x \to f_2$.

Suppose $c(f_x) = 1$, then in the worst case $c(f_1) = 1, c(f_2) = -2$ and we add flow $f_1 \to f_2$ and $f_x \to f_2$.

Suppose $c(f_x) = 2$, then $c(f_1) = c(f_2) = -1$ and we add flow $f_x \to f_1, f_2$.

Extension along vertex-to-K_4. The neighbors of a vertex x are splitted into three (possibly empty) sets, maintaining the rotation system at x. Remove x, add a K_4 such that the graph stays plane, and the three outermost vertices of the K_4 are connected with the respective neighborsets of x. The extension of the flow is trivial, the negative excess of the faces between the splitted neighbor sets of x is satisfied with flow coming from the three new faces of the K_4.

Extension along edge-to-K_3. The neighbors of a vertex x are splitted into three (possibly empty) sets N_1, N_2, N_3, one of the three sets contains only one neighbor, say $N_3 = \{u\}$. The rotation system at x is maintained. Remove x and u, add a K_3. Two vertices of the K_3 are connected with N_1 and N_2 respectively. The third vertex of the K_3 gets the neighbors of u. The extension of the flow is trivial, the negative excess of the face between the splitted neighbor sets N_1 and N_2 is satisfied with flow coming from the new face of the K_3.

Now we have obtained an edge-disjoint flow in the dual graph, such that the excess of all faces is satisfied. Similarly as before, given any 2-orientation α, this flow can transfered to a flow ψ in the angle graph such that there is at most two units of flow going through a vertex. The pair (α, ψ) is realizable, this gives a B_2-VCPG.

5 Conclusion

We have shown that the class of planar (2,2)-tight graphs is not equal to the class of B_1-VCPG graphs. However the only type of (2,2)-tight, planar graph that we found not to be B_1-VCPG has at least one vertex which is the intersection of "many" proper critical subsets. Are all planar (2,2)-tight graphs that have no such vertex B_1-VCPG?

We have also obtained bounds for loopless (2,1)-tight and (2,0)-tight planar graphs, however we believe that these bounds are not tight. Lower bounds of three bends are given by the octahedron minus one edge and the octahedron.

Conjecture 5.1. Loopless planar (2,0)-tight graphs are B_3-VCPG.

The bounds that we have shown do not depend on a chosen 2-orientation (i.e. the bounds hold for every 2-orientation). There are easy examples of a (2,0)-tight graph with a particular 2-orientation such that there is a vertex represented by a 4-bend path for every flow. Hence it would be interesting to find a sufficient condition on a flow such that, when satisfied, there exists a 2-orientation such that the pair is realizable. Is there a way to construct an realizable pair simultaneously? Is there a way to find that minimal flow that belongs to an realizable pair?

Local Flow Decreasing Steps There exist some (trivial) flow decreasing steps. Given a graph G, an realizable pair (α, ψ), if one of the steps is possible, we construct an realizable pair (α, ψ') such that $w(\psi') < w(\psi)$. We will also describe cases where none of the steps is possible, yet the flow is not the least possible for this graph. It is obvious that given an edge (u, v), two adjacent faces f_1, f_2, if there is flow from f_1 to f_2 as well as from f_2 to f_1 that this cancels out (no matter which vertex it goes through). This is a trivial flow decreasing step which leaves an realizable pair. The same holds for cyclic flow that consists of more than two units of flow.

Detour removal. Given a graph G and an realizable pair (α, ψ). Given an edge (u, v), the two faces adjacent to this edges, f_1 and f_2, and a face f_u adjacent to u. If the following flow is in ψ: $f_1 \to v \to f_2$ and $f_2 \to u \to f_u$, then it can be replaced by $f_1 \to u \to f_u$. The orientation does not change, and since $f_1 \to v \to f_2 \in \psi$, the edge is oriented from v to u in α. Hence the expansion condition is not violated by this change. Moreover, if the following flow is in ψ: $f_2 \to v \to f_1$ and $f_u \to u \to f_2$, similarly this flow can be replaced by $f_u \to u \to f_1$.

In general flow decreasing steps might not give an realizable pair. There are examples such that given a minimum flow ψ in the angle graph that satisfies the facial demands, there is no 2-orientation which together with ψ is an realizable pair. And given an realizable pair, there are examples for which the cycle and detour removals do not give the minimum flow.

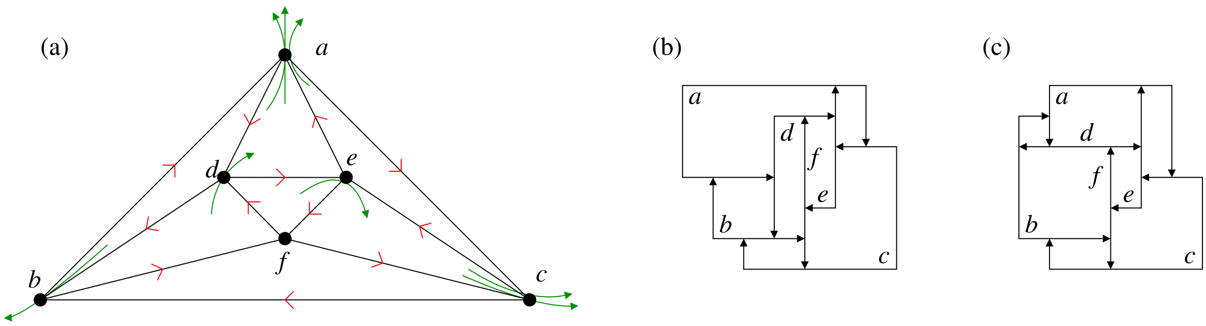

Figure 12: A graph G given a flow that cannot be reduced by the given steps (a) and the VCPG induced by this flow and the given 2-orientation (b); a VCPG of G with the minimal number of bends (c).

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