Topological Drawings meet Classical Theorems from Convex Geometry*

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Abstract

In this article we discuss classical theorems from Convex Geometry in the context of topological drawings and beyond. In a simple topological drawing of the complete graph K_n, any two edges share at most one point: either a common vertex or a point where they cross. Triangles of simple topological drawings can be viewed as convex sets. This gives a link to convex geometry.

As our main result, we present a generalization of Kirchberger's Theorem that is of purely combinatorial nature. It turned out that this classical theorem also applies to "generalized signotopes" – a combinatorial generalization of simple topological drawings, which we introduce and investigate in the course of this article. As indicated by the name they are a generalization of signotopes, a structure studied in the context of encodings for arrangements of pseudolines.

We also present a family of simple topological drawings with arbitrarily large Helly number, and a new proof of a topological generalization of Carathéodory's Theorem in the plane and discuss further classical theorems from Convex Geometry in the context of simple topological drawings.

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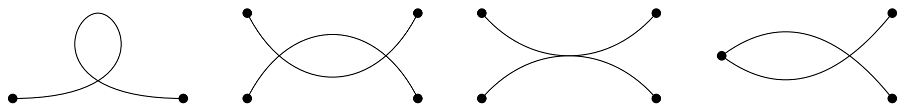

Figure 1: Forbidden patterns in topological drawings: self-crossings, double-crossings, touchings, and crossings of adjacent edges.

1 Introduction

A point set in the plane (in general position) induces a straight-line drawing of the complete graph K_n. In this article we investigate simple topological drawings of K_n and use the triangles of such drawings to generalize and study classical problems from the convex geometry of point sets. Since we only deal with simple topological drawings we omit the attribute *simple* and define a *topological drawing* D of K_n as follows:

- vertices are mapped to distinct points in the plane,
- edges are mapped to simple curves connecting the two corresponding vertices and containing no other vertices, and
- every pair of edges has at most one common point, which is either a common vertex or a crossing (but not a touching).

Figure 1 illustrates the forbidden patterns for topological drawings. Moreover, we assume throughout the article that no three edges cross in a single point. Topological drawings are also known as "good drawings" or "simple drawings".

In this article, we discuss classical theorems such as Kirchberger's, Helly's, and Carathéodory's Theorem in terms of the *convexity* hierarchy of topological drawings developed by Arroyo, McQuillan, Richter, and Salazar [AMRS17a], which we introduce in Section 2. In that section, we also define *generalized signotopes*, a combinatorial generalization of topological drawings. The connection between generalized signotopes and topological drawings is deferred to Section 6. Our proof of a generalization of Kirchberger's Theorem in Section 3 makes use of this combinatorial structure. Section 4 deals with a generalization of Carathéodory's Theorem. In Section 5, we present a family of topological drawings with arbitrarily large Helly number. We conclude this article with Section 7, where we discuss some open problems.

2 Preliminaries

Let D be a topological drawing and v a vertex of D. The cyclic order π_v of incident edges around v is called the *rotation* of v in D. The collection of rotations of all vertices is called the *rotation system* of D. Two topological drawings are *weakly isomorphic* if there is an isomorphism of the underlying abstract graphs which preserves the rotation system or reverses all rotations.

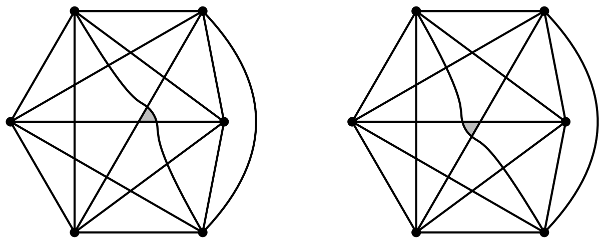

Figure 2: Two weakly isomorphic drawings of K_6, which can be transformed into each other by a triangle-flip.

A triangular cell, which has no vertex on its boundary, is bounded by three edges. By moving one of these edges across the intersection of the two other edges, one obtains a weakly isomorphic drawing; see Figure 2. This operation is called *triangle-flip*. Gioan [Gio05], see also Arroyo et al. [AMRS17b], showed that any two weakly isomorphic drawings of the complete graph can be transformed into each other with a sequence of triangle-flips and at most one reflection of the drawing.

Besides weak isomorphism, there is also the notion of strong isomorphism: two topological drawings are called *strongly isomorphic* if they induce homeomorphic cell decompositions of the sphere. Every two strongly isomorphic drawings are also weakly isomorphic.

2.1 Convexity Hierarchy

Given a topological drawing D, we call the induced subdrawing of three vertices a *triangle*. Note that the edges of a triangle in a topological drawing do not cross. The removal of a triangle separates the plane into two connected components – a bounded component and an unbounded component. We call the closure of these connected components *sides*. A side of a triangle is *convex* if every edge that has its two end-vertices in the side is completely drawn in the side. We are now ready to introduce the "convexity hierarchy" of Arroyo et al. [AMRS17a]). For $1 \leq i < j \leq 6$, drawings with property (j) also have property (i).

(1) topological drawings;

(2) *convex* drawings: each triangle has a convex side;

(3) *hereditary-convex* drawings: if a triangle $\triangle_1$ is fully contained in the convex side of another triangle $\triangle_2$, then also its convex side is;

(4) *face-convex* drawings: there is a special face f_∞ such that, for every triangle, the side not containing f_∞ is convex;

(5) *pseudolinear* drawings: all edges of the drawing can be extended to bi-infinite curves – called *pseudolines* – such that any two cross at most once[1];

(6) straight-line drawings: all edges are drawn as straight-line segments connecting their endpoints.

Arroyo et al. [AMRS18] showed that the face-convex drawings where the special face f_∞ is drawn as the unbounded outer face are precisely the pseudolinear drawings (see also [ABR20] and [AHP$^+$15]).

Pseudolinear drawings are generalized by pseudocircular drawings. A drawing is called *pseudocircular* if the edges can be extended to pseudocircles (simple closed curves) such that any pair of non-disjoint pseudocircles has exactly two crossings. Since stereographic projections preserve (pseudo)circles, pseudocircularity is a property of drawings on the sphere. Pseudocircular drawings were studied in a recent article by Arroyo, Richter, and Sunohara [ARS20]. They provided an example of a topological drawing which is not pseudocircular. Moreover, they proved that hereditary-convex drawings are precisely the *pseudospherical* drawings, i.e., pseudocircular drawings with the additional two properties that

- every pair of pseudocircles intersects, and
- for any two edges $e \neq f$ the pseudocircle γ_e has at most one crossing with f.

The relation between convex drawings and pseudocircular drawings remains open.

Convexity, hereditary-convexity, and face-convexity are properties of the weak isomorphism classes. To see this, note that the existence of a convex side is not affected by changing the outer face or by transferring the drawing to the sphere, moreover, convex sides are not affected by triangle-flips. Hence, these properties only depend on the rotation system of the drawing. For pseudolinear and straight-line drawings, however, the choice of the outer face plays an essential role.

2.2 Generalized Signotopes

Let D be a topological drawing of a complete graph in the plane. Assign an *orientation* $\chi(abc) \in \{+,-\}$ to each ordered triple (a,b,c) of vertices. The sign $\chi(abc)$ indicates whether we go counterclockwise or clockwise around the triangle if we traverse the edges $(a,b),(b,c),(c,a)$ in this order.

If D is a straight-line drawing of K_n, then the underlying point set $S = \{s_1,\dots,s_n\}$ has to be in general position (no three points are on a line). Assuming that the points are sorted from left to right, then for every 4-tuple s_i,s_j,s_k,s_l with $i<j<k<l$ the sequence $\chi(ijk),\chi(ijl),\chi(ikl),\chi(jkl)$ (index-triples in lexicographic order) is *monotone*, i.e., there is at most one sign-change. A *signotope* is a mapping $\chi : \binom{[n]}{3} \to \{+,-\}$

[1] Arrangements of pseudolines obtained by such extensions are equivalent to *pseudoconfigurations of points*, and can be considered as oriented matroids of rank 3 (cf. Chapter 5.3 of [BLW$^+$99]).

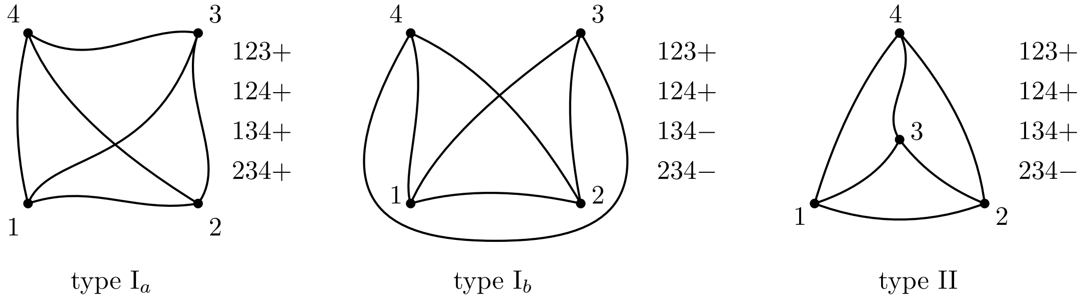

Figure 3: The three types of topological drawings of K_4 in the plane.

with the above monotonicity property, where $[n] = \{1, 2, \ldots, n\}$. Signotopes are in bijection with Euclidean pseudoline arrangements [FW01] and can be used to characterize pseudolinear drawings [BFK15, Theorem 3.2].

When considering topological drawings of the complete graph we have no left to right order of the vertices, i.e., no natural labeling. Exchanging the labels of two vertices reverts the orientation of all triangles containing both vertices. This suggests to look at the *alternating* extension of χ. Formally $\chi(i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}) = \mathrm{sgn}(\sigma) \cdot \chi(i_1, i_2, i_3)$ for any distinct labels i_1, i_2, i_3 and any permutation $\sigma \in S_3$. This yields a mapping $\chi : [n]_3 \to \{+, -\}$, where $[n]_3$ denotes the set of all triples (a, b, c) with pairwise distinct $a, b, c \in [n]$. To see whether the alternating extension of χ still has a property comparable to the monotonicity of signotopes, we have to look at 4-tuples of vertices, i.e., at drawings of K_4. On the sphere there are two types of drawings of K_4: type-I has one crossing and type-II has no crossing. Type-I can be drawn in two different ways in the plane: in type-$\mathrm{I_a}$ the crossing is only incident to bounded faces and in type-$\mathrm{I_b}$ the crossing lies on the outer face; see Figure 3.

A drawing of K_4 with vertices a, b, c, d can be characterized in terms of the sequence of orientations $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$. The drawing is

- of type-$\mathrm{I_a}$ or type-$\mathrm{I_b}$ iff the sequence is $++++$, $++--$, $+--+$, $-++-$, $--++$, or $----$; and
- of type-II iff the number of $+$'s (and $-$'s respectively) in the sequence is odd.

Therefore there are at most two sign-changes in the sequence $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ and, moreover, any such sequence is in fact induced by a topological drawing of K_4. Allowing up to two sign-changes is equivalent to forbidding the two patterns $+-+-$ and $-+-+$.

If χ is alternating and avoids the two patterns $+-+-$ and $-+-+$ on sorted indices, i.e., $\chi(ijk), \chi(ijl), \chi(ikl), \chi(jkl)$ has at most two sign-changes for all $i < j < k < l$, then it avoids the two patterns in $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise distinct $a, b, c, d \in [n]$. We refer to this as the *symmetry property* of the forbidden patterns.

The symmetry property allows us to define *generalized signotopes* as alternating mappings $\chi\colon [n]_3 \to \{+, -\}$ with at most two sign-changes on $\chi(abc), \chi(abd), \chi(acd), \chi(bcd)$ for any pairwise different $a, b, c, d \in [n]$. We conclude:

Proposition 1. *Every topological drawing of K_n induces a generalized signotope on n elements.*

We defer the structural investigation of generalized signotopes to Section 6, where we show that there are more generalized signotopes than topological drawings. Hence generalized signotopes extend the convexity hierarchy introduced above.

3 Kirchberger's Theorem

Two closed sets $A, B \subseteq \mathbb{R}^d$ are called *separable* if there exists a hyperplane H separating them, i.e., $A \subset H_1$ and $B \subset H_2$ with H_1, H_2 being the two closed half-spaces defined by H. It is well-known that, if two non-empty compact sets A, B are separable, then they can also be separated by a hyperplane H containing points of A and B. *Kirchberger's Theorem* (see [Kir03] or [Bar02]) asserts that two finite point sets $A, B \subseteq \mathbb{R}^d$ are separable if and only if for every $C \subseteq A \cup B$ with $|C| = d+2$, $C \cap A$ and $C \cap B$ are separable.

Goodman and Pollack [GP82] proved duals of Kirchberger's Theorem and further theorems like Radon's, Helly's, and Carathéodory's Theorem for arrangements of pseudolines. Their results also transfer to pseudoconfigurations of points and thus to pseudolinear drawings. To be more precise, they proved a natural generalization of Kirchberger's Theorem to pseudoline-arrangements in the plane which, by duality, is equivalent to a separating statement on pseudoconfigurations of points in the plane (cf. Theorem 4.8 and Remark 5.2 in [GP82]).

The 2-dimensional version of Kirchberger's Theorem can be formulated in terms of triple orientations. We show a generalization for topological drawings using generalized signotopes. Two sets $A, B \subseteq [n]$ are *separable* if there exist $i, j \in A \cup B$ such that $\chi(i, j, x) = +$ for all $x \in A \setminus \{i, j\}$ and $\chi(i, j, x) = -$ for all $x \in B \setminus \{i, j\}$. In this case we say that *ij separates* A from B and write $\chi(i, j, A) = +$ and $\chi(i, j, B) = -$. Moreover, if we can find $i \in A$ and $j \in B$, we say that A and B are *strongly separable.* As an example, consider the 4-element generalized signotope of the type-$\mathrm{I_b}$ drawing of K_4 in Figure 3. The sets $A = \{1, 2\}$ and $B = \{3, 4\}$ are strongly separable with $i = 2$ and $j = 3$ because $\chi(2, 3, 1) = +$ and $\chi(2, 3, 4) = -$.

Theorem 1 (Kirchberger's Theorem for Generalized Signotopes)**.** *Let $\chi : [n]_3 \to \{+, -\}$ be a generalized signotope, and let $A, B \subseteq [n]$ be two non-empty sets. If for every $C \subseteq A \cup B$ with $|C| = 4$, the sets $A \cap C$ and $B \cap C$ are separable, then A and B are strongly separable.*

Note that, since every topological drawing yields a generalized signotope, Theorem 1 generalizes Kirchberger's Theorem to topological drawings of complete graphs. The converse statement of the classical version of Kirchberger's Theorem is trivially true. Concerning Theorem 1, also a stronger version of the converse of the theorem is true: If A and B are separable, then for every $C \subseteq A \cup B$ with $|C| = 4$, the sets $A \cap C$ and $B \cap C$ are separable. This can be verified by an elaborate case distinction on generalized signotopes on at most 6 elements because, if there exists a separator ij for $A \cup B$ and a non-separable 4-tuple C, then the configuration can be restricted to $C \cup \{i, j\}$.

Proof of Theorem 1. First we prove that all 4-tuples $C \subseteq A \cup B$ with $C \cap A$ and $C \cap B$ non-empty which are separable are also strongly separable. This can be verified looking at Tables 1 and 2, which show that, in every separable generalized signotopes on $\{a, b_1, b_2, b_3\}$ and $\{a_1, a_2, b_1, b_2\}$, respectively, there is a strong separator of the sets $\{a\}$ and $\{b_1, b_2, b_3\}$ or $\{a_1, a_2\}$ and $\{b_1, b_2\}$, respectively. Hence in the following we assume that all such 4-tuples from $A \cup B$ are strongly separable.

$\chi(a, b_1, b_2)$	$\chi(a, b_1, b_3)$	$\chi(a, b_2, b_3)$	$\chi(b_1, b_2, b_3)$	list of separators
$+$	$+$	$+$	$+$	$\underline{ab_3}, b_1a, b_1b_3$
$+$	$+$	$+$	$-$	$\underline{ab_3}, b_1a, b_1b_2, b_2b_3$
$+$	$+$	$-$	$+$	$\underline{ab_2}, b_1a, b_1b_3, b_3b_2$
$+$	$+$	$-$	$-$	$\underline{ab_2}, b_1a, b_1b_2$
$+$	$-$	$+$	$+$	(no separator)
$+$	$-$	$-$	$+$	$\underline{ab_2}, b_3a, b_3b_2$
$+$	$-$	$-$	$-$	$\underline{ab_2}, b_1b_2, b_3a, b_3b_1$
$-$	$+$	$+$	$+$	$\underline{ab_3}, b_1b_3, b_2a, b_2b_1$
$-$	$+$	$+$	$-$	$\underline{ab_3}, b_2a, b_2b_3$
$-$	$+$	$-$	$-$	(no separator)
$-$	$-$	$+$	$+$	$\underline{ab_1}, b_2a, b_2b_1$
$-$	$-$	$+$	$-$	$\underline{ab_1}, b_2a, b_2b_3, b_3b_1$
$-$	$-$	$-$	$+$	$\underline{ab_1}, b_2b_1, b_3a, b_3b_2$
$-$	$-$	$-$	$-$	$\underline{ab_1}, b_3a, b_3b_1$

Table 1: Separators for generalized signotopes on $\{a, b_1, b_2, b_3\}$. Strong separators are underlined.

By symmetry we may assume $|A| \leq |B|$. First we consider the cases $|A| = 1, 2, 3$ individually and then the case $|A| \geq 4$.

Let $A = \{a\}$, let B' be a maximal subset of B such that B' is strongly separated from $\{a\}$, and let $b \in B'$ be such that $\chi(a, b, B') = -$. Suppose that $B' \neq B$, then there is a $b^* \in B \backslash B'$ with

$$\chi(a, b, b^*) = +. \tag{1}$$

By maximality of B' we cannot use the pair a, b^* for a strong separation of $\{a\}$ and $B' \cup \{b^*\}$. Hence, for some $b' \in B'$:

$$\chi(a, b^*, b') = +. \tag{2}$$

Since χ is alternating (1) and (2) together imply $b' \neq b$. Since $b' \in B'$ we have $\chi(a, b, b') = -$. From this together with (1) and (2) it follows that the four-element set $\{a, b, b', b^*\}$ has no separator. This is a contradiction, whence $B' = B$.

$\chi(a_1a_2b_1)$	$\chi(a_1a_2b_2)$	$\chi(a_1b_1b_2)$	$\chi(a_2b_1b_2)$	list of separators
$+$	$+$	$+$	$+$	$a_2a_1, \underline{a_2b_2}, b_1a_1, b_1b_2$
$+$	$+$	$+$	$-$	$a_2a_1, \underline{a_2b_1}, b_1a_1$
$+$	$+$	$-$	$+$	$a_2a_1, \underline{a_2b_2}, b_2a_1$
$+$	$+$	$-$	$-$	$a_2a_1, \underline{a_2b_1}, b_2a_1, b_2b_1$
$+$	$-$	$+$	$+$	$\underline{a_1b_2}, b_1a_1, b_1b_2$
$+$	$-$	$-$	$+$	(no separator)
$+$	$-$	$-$	$-$	$\underline{a_2b_1}, b_2a_2, b_2b_1$
$-$	$+$	$+$	$+$	$\underline{a_2b_2}, b_1a_2, b_1b_2$
$-$	$+$	$+$	$-$	(no separator)
$-$	$+$	$-$	$-$	$\underline{a_1b_1}, b_2a_1, b_2b_1$
$-$	$-$	$+$	$+$	$a_1a_2, \underline{a_1b_2}, b_1a_2, b_1b_2$
$-$	$-$	$+$	$-$	$a_1a_2, \underline{a_1b_2}, b_2a_2$
$-$	$-$	$-$	$+$	$a_1a_2, \underline{a_1b_1}, b_1a_2$
$-$	$-$	$-$	$-$	$a_1a_2, \underline{a_1b_1}, b_2a_2, b_2b_1$

Table 2: Separators for generalized signotopes on $\{a_1, a_2, b_1, b_2\}$. Strong separators are underlined.

As a consequence we obtain:

- ▶ Every one-element set $\{a\}$ with $a \in A$ can be strongly separated from B. Since χ is alternating there is a unique $b(a) \in B$ such that $\chi(a, b(a), B) = -$.

Now consider the case that $A = \{a_1, a_2\}$. Let $b_i = b(a_i)$, i.e., $\chi(a_i, b_i, B) = -$ for $i = 1, 2$. If $\chi(a_1, b_1, a_2) = +$ or if $\chi(a_2, b_2, a_1) = +$, then a_1b_1 or a_2b_2, respectively, is a strong separator for A and B. Therefore, we may assume that $\chi(a_1, b_1, a_2) = -$, $\chi(a_2, b_2, a_1) = -$ and therefore $b_1 \neq b_2$. We get the sequence $+--+$ for the four-element set $\{a_1, a_2, b_1, b_2\}$ which has no strong separator (cf. Table 2), a contradiction.

Let $A = \{a_1, a_2, a_3\}$. Suppose that A is not separable from B. Let $b_i = b(a_i)$, i.e., $\chi(a_i, b_i, B) = -$ for $i = 1, 2, 3$. For $i, j \in \{1, 2, 3\}$, $i \neq j$ we define $s_{ij} = \chi(a_i, b_i, a_j)$.

If $s_{ij} = +$ for some i and all $j \neq i$, then a_ib_i separates A from B. Hence, for each i there exists $j \neq i$ with $s_{ij} = -$.

If $s_{ij} = s_{ji} = -$ for some i, j, then since χ is alternating $b_i \neq b_j$ and $\{a_i, a_j, b_i, b_j\}$ corresponds to the row $+--+$ in Table 2, i.e., there is no strong separator. Hence, at least one of s_{ij} and s_{ji} is $+$.

These two conditions imply that we can relabel the elements of A such that $s_{12} = s_{23} = s_{31} = +$ and $s_{13} = s_{21} = s_{32} = -$. Suppose that $b_i = b_j = b$ for some $i \neq j \in \{1, 2, 3\}$, then the four elements $\{b, a_1, a_2, a_3\}$ have the pattern $-+-*$. Avoiding the forbidden pattern, we get $-+--$ in Table 1, i.e., there is no strong separator. This contradiction shows that b_1, b_2, b_3 must be pairwise distinct.

From $s_{32} = -$ and $s_{31} = +$ we find that $\{b_3, a_1, a_2, a_3\}$ corresponds to a row of type $*+-*$ in Table 1. We conclude that the strong separator of $\{b_3, a_1, a_2, a_3\}$ is a_2b_3. In particular,

$$\chi(b_3, a_1, a_2) = +. \tag{3}$$

Now consider $\{a_1, a_2, b_1, b_3\}$. From $s_{12} = +$, equation (3), and $\chi(a_1, b_1, b_3) = -$ we obtain the pattern $- + -*$. Since $- + -+$ is forbidden we obtain

$$\chi(a_2, b_1, b_3) = -. \tag{4}$$

The set $\{a_2, a_3, b_1, b_3\}$ needs a strong separator. The candidate pair a_3b_1 is made impossible by $\chi(a_3, b_1, b_3) = +$, a_3b_3 is made impossible by $s_{32} = -$, and a_2b_3 is made impossible by (4). Hence a_2b_1 is the strong separator and, in particular, it holds

$$\chi(a_2, b_1, a_3) = +. \tag{5}$$

But now the set $\{a_1, a_2, a_3, b_1\}$ has no strong separator. The candidate pair a_1b_1 is impossible because of $s_{13} = -$, a_2b_1 does not separate because $s_{12} = +$, and (5) shows that a_3b_1 cannot separate the set. This contradiction proves the case $|A| = 3$.

For the remaining case $|A| \geq 4$ consider a counterexample (χ, A, B) minimizing the size of the smaller of the two sets. We have $4 \leq |A| \leq |B|$.

Let $a^* \in A$. By minimality $A' = A \backslash \{a^*\}$ is separable from B. Let $a \in A'$ and $b \in B$ such that $\chi(a, b, A') = +$ and $\chi(a, b, B) = -$. Hence

$$\chi(a, b, a^*) = -. \tag{6}$$

Let $b^* = b(a^*)$, i.e., $\chi(a^*, b^*, B) = -$. There is some $a' \in A'$ such that

$$\chi(a^*, b^*, a') = -. \tag{7}$$

If $a' = a$, then $b \neq b^*$ because of (6) and (7). From (6), (7), $\chi(a, b, B) = -$, and $\chi(a^*, b^*, B) = -$ it follows that the four-element set $\{a, a^*, b, b^*\}$ has the sign pattern $+ - -+$, hence there is no separator, see Table 2. This shows that $a' \neq a$.

Let $b' = b(a')$. If $b \neq b'$ we look at the four elements $\{a, b, a', b'\}$. It corresponds to $+ - *-$ so that we can conclude $\chi(a, a', b') = -$. If $b = b'$, then $a' \in A'$ implies $\chi(a, b, a') = +$ which yields $\chi(a', b', a) = -$.

Hence, regardless whether $b = b'$ or $b \neq b'$ we have

$$\chi(a', b', a) = - \ . \tag{8}$$

Since $|A| \geq 4$, we know by the minimality of the instance (χ, A, B) that the set $\{a, b, a', b', a^*, b^*\}$, which has 3 elements of A and at least 1 element of B, is separable. It follows from $\chi(a, b, B) = \chi(a', b', B) = \chi(a^*, b^*, B) = -$ that the only possible strong separators are ab, $a'b'$, and a^*b^*. They, however, do not separate because of (6), (7) and (8) respectively. This contradiction shows that there is no counterexample. $\square$

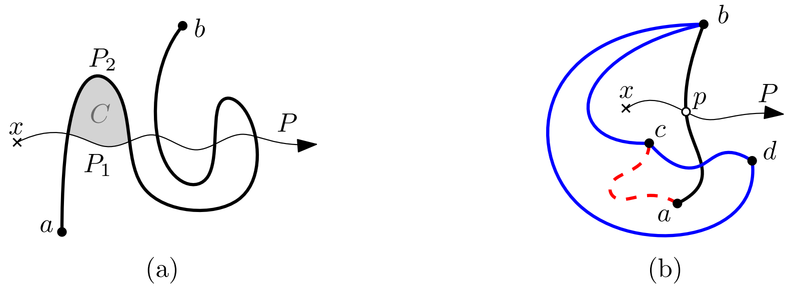

Figure 4: (a) and (b) give an illustration of the proof of Theorem 2.

4 Carathéodory's Theorem

Carathéodory's Theorem asserts that, if a point x lies in the convex hull of a point set P in $\mathbb{R}^d$, then x lies in the convex hull of at most $d+1$ points of P.

As already mentioned in Section 3, Goodman and Pollack [GP82] proved a dual of Carathéodory's Theorem, which transfers to pseudolinear drawings.

A more general version of Carathéodory's Theorem in the plane is due to Balko, Fulek, and Kynčl, who provided a generalization to topological drawings. In this section, we present a shorter proof for their theorem.

Theorem 2 (Carathéodory for Topological Drawings [BFK15, Lemma 4.7])**.** *Let D be a topological drawing of K_n and let $x \in \mathbb{R}^2$ be a point contained in a bounded connected component of $\mathbb{R}^2 - D$. Then there is a triangle in D that contains x in its interior.*

Proof. Suppose towards a contradiction that there is a pair (D, x) violating the claim. We choose D minimal with respect to the number of vertices n.

Let a be a vertex of the drawing. If we remove all incident edges of a from D, then, by minimality of the example, x becomes a point of the outer face. Therefore, if we remove the incident edges of a one by one, we find a last subdrawing D' such that x is still in a bounded face. Let ab be the edge such that in the drawing $D' - ab$ the point x is in the outer face.

There is a simple curve P connecting x to infinity, which does not cross any of the edges in $D' - ab$. By the choice of D', curve P has at least one crossing with ab. We choose P minimal with respect to the number of crossings with ab.

We claim that P intersects ab exactly once. Suppose that P crosses ab more than once. Then there is a *lense* C formed by P and ab, that is, two crossings of P and ab such that the simple closed curve ∂C, composed of a subcurve P_1 of P and a part P_2 of edge ab between the crossings, encloses a simply connected region C, see Figure 4(a).

Now consider the curve P' from x to infinity which is obtained from P by replacing the subcurve P_1 by a curve P_2' which is a close copy of P_2 in the sense that it has the same crossing pattern with all edges in D and the same topological properties, but is disjoint from ab. As P was chosen minimal with respect to the number of crossings with ab, there has to be an edge of the drawing D' that intersects P_2' (and by the choice of P_2' also P_2). This edge has no crossing with P, by construction, and crosses ab at most once, so it has one of its endpoints inside the lense C and one outside C. Depending on whether $b \in C$ or not, we choose an endpoint c_1 of that edge such that the edge bc_1 in D' intersects ∂C. But since they are adjacent, bc_1 cannot intersect ab and by the choice of P it does not intersect P. The contradiction shows that P crosses ab in a unique point p.

If a has another neighbor c_2 in the drawing D' then, since only edges incident to a have been removed there is an edge connecting b to c_2 in D'. The edges ac_2 and bc_2 do not cross P, so x is in the interior of the triangle abc_2 and we are done.

If there is no edge ac_2 in D', then $\deg(a) = 1$ in D'. As x is not in the outer face of D', there must be an edge cd in D' which intersects the partial segment of the edge ab starting in a and ending in p, in its interior. Let c be the point on the same side of ab as x; see Figure 4(b). The edges bc and bd of D' cross neither P nor ab. Consequently, the triangle bcd (drawn blue) must contain a in its interior. We claim that the edge ac in the original drawing D (drawn red dashed) lies completely inside the triangle bcd: The bounded region defined by the edges ab, cd, and bd of D' contains a and c. Since D is a topological drawing, and ac has no crossing with ab and cd, ac has no crossing with bd. This proves the claim. Now the curve P does not intersect ac, and the only edge of the triangle abc intersected by P is ab. Therefore, x lies in the interior of the triangle abc. This contradicts the assumption that (D, x) is a counterexample. □

4.1 Colorful Carathéodory Theorem

Bárány [Bár82] generalized Carathéodory's Theorem as follows: Given finite point sets $P_0, \ldots, P_d$ from $\mathbb{R}^d$ such that there is a point $x \in \text{conv}(P_0) \cap \ldots \cap \text{conv}(P_d)$, then x lies in a simplex spanned by $p_0 \in P_0, \ldots, p_d \in P_d$. Such a simplex is called *colorful*. The theorem is known as the *Colorful Carathéodory Theorem*.

A strengthening, known as the *Strong Colorful Carathéodory Theorem*, was shown by Holmsen, Pach, and Tverberg [HPT08] (cf. [Kal09]): It is sufficient if there is a point x with $x \in \text{conv}(P_i \cup P_j)$ for all $i \neq j$, to find a colorful simplex. The Strong Colorful Carathéodory Theorem was further generalized to oriented matroids by Holmsen [Hol16]. In particular, the theorem applies to pseudolinear drawings (which are in correspondence with oriented matroids of rank 3).

There are several ways to prove Colorful Carathéodory Theorem for pseudolinear drawings. Besides Holmsen's proof [Hol16], which uses sophisticated methods from topology, we have also convinced ourselves that Bárány's proof [Bár82] can be adapted to pseudoconfigurations of points in the plane. However, Bárány's proof idea does not directly generalize to higher dimensions because oriented matroids of higher ranks do not necessarily have a representation in terms of pseudoconfigurations of points in d-space (cf. [BLW$^+$99, Chapter 1.4]).

Another way to prove the Strong Colorful Carathéodory Theorem for pseudolinear drawings is by computer assistance: Since the statement of the theorem only involves 10 points and only the relative positions play a role (not the actual coordinates), one can verify the theorem by checking all combinatorially different point configurations using

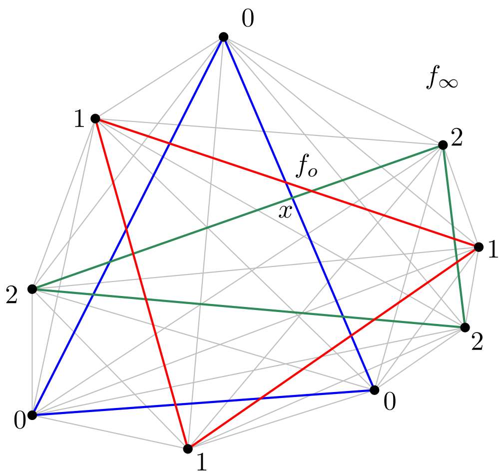

Figure 5: A face-convex drawings of K_9. If the cell f_o is chosen as the outer face, then Colorful Carathéodory Theorem does not hold for the colored triangles and x. The special cell of the pseudolinear drawing is marked f_∞.

the order type database (cf. [AAK02] and [SSS20, Section 6.1]). Alternatively, one can – similar as in [Sch20] – formulate a SAT instance that models the statement of the Strong Colorful Carathéodory Theorem. Using modern SAT solvers one can then verify that there is no 10-point configuration that violates the theorem.

The following result shows that in the convexity hierarchy of topological drawings of K_n the Colorful Carathéodory Theorem is not valid beyond the class of pseudolinear drawings.

Proposition 2. *The Colorful Carathéodory Theorem does not hold for the face-convex drawing of Figure 5.*

Proof. The drawing depicted in Figure 5 is face-convex because it is obtained from a straight-line drawing by choosing f_o as outer face. The point x is contained in the three colored triangles. This point is separated from the outer face only by three colored edges. Therefore, there is no triangle containing x with a vertex of each of the three colors. □

5 Helly's Theorem

The *Helly number* of a family of sets $\mathcal{F}$ with empty intersection is the size of the smallest subfamily of $\mathcal{F}$ with empty intersection. *Helly's Theorem* asserts that the Helly number of a family of n convex sets $S_1, \ldots, S_n$ from $\mathbb{R}^d$ is at most $d+1$, i.e., the intersection of $S_1, \ldots, S_n$ is non-empty if the intersection of every subfamily of size $d+1$ is non-empty.

In the following we discuss the Helly number in the context of topological drawings, where the sets S_i are triangles of the drawing.

From the results of Goodman and Pollack [GP82] it follows that Helly's Theorem generalizes to pseudoconfigurations of points in two dimensions, and thus for pseudolinear drawings. A more general version of Helly's Theorem was shown by Bachem and Wanka [BW88]. They prove Helly's and Radon's Theorem for oriented matroids with the "intersection property". Since all oriented matroids of rank 3 have the intersection property (cf. [BW88] and [BW89]) and oriented matroids of rank 3 correspond to pseudo-configurations of points, which in turn yield pseudolinear drawings, the two theorems are valid for pseudolinear drawings.

We show that Helly's Theorem does not hold for face-convex drawings, moreover, the Helly number can be arbitrarily large in face-convex drawings. Note that the following proposition does not contradict the Topological Helly Theorem [Hel30] (cf. [GPP$^+$17]) because there are triangles whose intersection is disconnected.

Proposition 3. *Helly's Theorem does not generalize to face-convex drawings. Moreover, for every integer $n \geq 3$, there exists a face-convex drawing of K_{3n} with Helly number at least n, i.e., there are n triangles such that the bounded sides of any $n-1$ triangles have a common interior point, but the intersection of the bounded sides of all n triangles is empty.*

Proof. Consider a straight-line drawing D of K_{3n} with n triangles T_i as shown for the case $n = 7$ in Figure 6. With D' we denote the drawing obtained from D by making the gray cell f_o the outer face. Let O_i be the side of ∂T_i that is bounded in D'. For $1 \leq i < n$ the set O_i corresponds to the outside of ∂T_i in D while O_n corresponds to the inside of ∂T_n.

In D' we have $\bigcap_{i=1}^{n-1} O_i \neq \emptyset$, indeed any point p_n which belongs to the outer face of D is in this intersection. Since $T_n \subset \bigcup_{i=1}^{n-1} T_i$, we have $T_n \cap \bigcap_{i=1}^{n-1} O_i = \emptyset$, i.e., $\bigcap_{i=1}^{n} O_i = \emptyset$. For each $i \in \{1, \ldots, n-1\}$ there is a point $p_i \in T_i \cap T_n$ which is not contained in any other T_j. Therefore, $p_i \in \bigcap_{j=1; j \neq i}^{n} O_i$.

In summary, the intersection of any $n-1$ of the n sets $O_1, \ldots, O_n$ is non-empty but the intersection of all of them is empty. □

6 Generalized Signotopes: Structure and Enumeration

In this section we discuss the connection between generalized signotopes and topological drawings. We show that the number of generalized signotopes on n elements is of order $2^{\Theta(n^3)}$. By introducing a notion of flips for generalized signotopes, we show that generalized signotopes indeed are a proper generalization of topological drawings and estimate how many generalized signotopes can be represented by a topological drawing. From the known estimates for the asymptotic number of topological drawings, it then follows that most generalized signotopes do not come from topological drawings.

6.1 Flip-equivalent Generalized Signotopes

Let χ be a generalized signotope on $[n]$. A pair (i, j) of distinct elements of $[n]$ is said to be *flippable* in χ if inverting the signs of the triples containing i and j yields a generalized

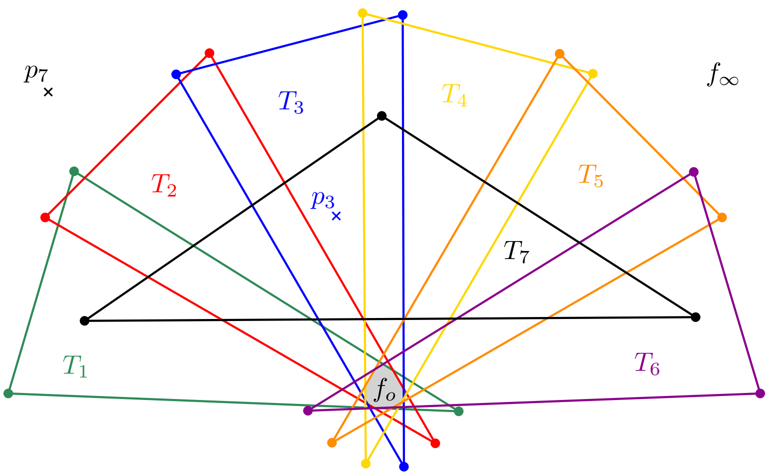

Figure 6: A drawing D of K_{21} is obtained by adding the remaining edges as straight-line segments. Making the gray cell f_o the outer face, we obtain a face-convex drawing with Helly number 7.

signotope. If χ comes from a topological drawing and (i, j) is an edge incident to the outer face, then (i, j) is flippable in χ. Moreover, the generalized signotope χ' obtained by inverting the triples containing i and j again comes from a drawing. Indeed, if D is a drawing corresponding to χ and the edge $e = (i, j)$ is incident to the outer face c_1, then there is a second cell c_2 which is separated from c_1 only by e. Using stereographic projections, one can wrap the edge e around the drawing to make c_2 the outer face. The drawing D' obtained this way corresponds to χ'. Type-I_a and type-I_b of the topological drawings of K_4 (see Figure 3) differ by such a flip operation applied to the edge $(3, 4)$.

Two generalized signotopes χ, χ' are *flip-equivalent* if there is a sequence (i_1, j_1), $\ldots$, (i_k, j_k) of pairs and a sequence $\chi_0, \ldots, \chi_k$ of generalized signotopes with $\chi = \chi_0$, $\chi' = \chi_k$, and χ_ℓ is obtained from $\chi_{\ell-1}$ by flipping the pair (i_ℓ, j_ℓ). This flip-equivalence relation partitions the set of all generalized signotopes into *flip classes*, which we further consider to be closed under relabeling of the elements.

In the following, we show that two weakly isomorphic drawings yield flip-equivalent generalized signotopes. In fact, the following lemma will be the key to show that most generalized signotopes do not come from topological drawings.

Lemma 3. *Two weakly isomorphic drawings D and D' of K_n yield flip-equivalent generalized signotopes.*

Proof. According to [Gio05], we can transform D into D' using triangle-flips (Figure 2) only. Suppose we have $D = D_0, D_1, \ldots, D_m = D'$, where D_i is transformed into D_{i+1} by a triangle-flip. We have to show that $\chi(D_i)$ and $\chi(D_{i+1})$ are flip-equivalent generalized signotopes. A crucial point is that generalized signotopes come from drawings in the plane while weak isomorphism is a property of spherical drawings. Hence, we have to allow triangle-flips with the triangle being the outer face.

	Gen.Sig.	Relabeling Cl.	Flip Cl.	Weak Isom. Cl.
3	2	1	1	1
4	14	2	2	2
5	544	6	3	5
6	173 128	167	16	102
7	630 988 832	63 451	442	11 556
8	?	?	?	5 370 725
9				7 198 391 729
$\vdots$				
n	$2^{\Theta(n^3)}$	$2^{\Theta(n^3)}$	$2^{\Theta(n^3)}$	$2^{\Theta^*(n^2)}$

Table 3: The first three columns show the number of generalized signotopes on n elements, equivalence classes up to relabeling, and flip classes, respectively. The last column shows the number of weak isomorphism classes of topological drawings of K_n from [ÁAFM+15] (cf. [Pam14] and OEIS/A276110). The asymptotic bounds are provided in Theorem 7 and [Kyn13, PT06], respectively.

Let $\chi(D_i)$ be the generalized signotope of the drawing D_i and let $\triangle_i$ be the triangular cell in D_i which is flipped to obtain D_{i+1}. If $\triangle_i$ is a bounded cell, we are done because of $\chi(D_i) = \chi(D_{i+1})$. Otherwise, if $\triangle_i$ is the outer face, apply an flip of an edge bounding the outer face to obtain an isomorphic drawing D'_i in which another face is the outer face. Because of the edge-flip, $\chi(D'_i)$ is flip-equivalent to $\chi(D_i)$. □

6.2 Small Configurations

To get a better understanding of which generalized signotopes come from topological drawings, we have enumerated all generalized signotopes and flip classes up to $n = 7$ elements using a simple computer program; see Table 3. Moreover, since drawings from the same weak isomorphism class induce flip-equivalent generalized signotopes (Lemma 3), Table 3 also restates the number of weak isomorphism classes from [ÁAFM+15] for means of comparison.

In Section 2.2, we have seen that there are precisely two weak isomorphism classes of topological drawings of K_4. Via relabeling and mirroring, all 14 generalized signotopes on $n = 4$ elements are realized by type-I and type-II; cf. Figure 3. These 14 generalized signotopes partition into two relabeling classes and two flip classes. One of the classes corresponds to drawings of type-I (1 crossing) and the other one to drawings of type-II (0 crossings). In particular, the flip operation for generalized signotopes preserves the number of crossings in a 4-tuple. Therefore, we can define the *crossing number* of a generalized signotope χ on n elements as the number of induced 4-tuples which belong to the flip class of the type-I drawing.

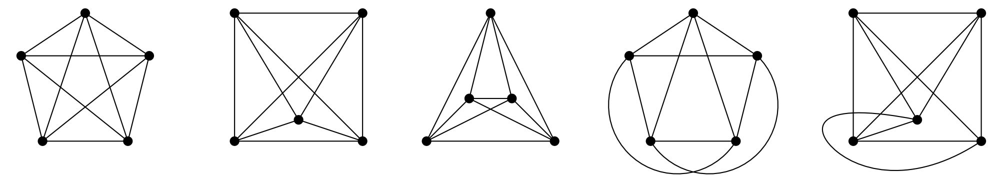

Figure 7: The five types of topological drawings of K_5.

For $n = 5$, there are 544 generalized signotopes, which belong to 6 relabeling classes and 3 flip classes, respectively. There are five weak isomorphism classes of topological drawings of K_5, see Figure 7. We have verified by computer that each of the 544 generalized signotope on $n = 5$ elements is realized by a topological drawing of K_5. Since we can read whether a 4-tuple of vertices induces a crossing from the generalized signotope, it is clear that drawings with different number of crossings do not correspond to a common class of generalized signotopes. Indeed, the class with 24 generalized signotopes corresponds to type I and type V (both 5 crossings), the class with 280 generalized signotopes corresponds to type II and type IV (both 3 crossings), and the class with 240 generalized signotopes corresponds to type III (1 crossing). We conclude that generalized signotopes are not able to encode the weak isomorphism class. Also convexity is not encoded: type I and type V induce the same generalized signotope but type I is face-convex while type V is non-convex.

For $n = 6$, there are 173 128 generalized signotopes, 167 relabeling classes, and 16 flip classes. We have verified by computer that 151 of the 167 relabeling classes are realized by a topological drawing of K_6. However, from each of the 16 flip classes there is a representative which can be realized by a topological drawing of K_6.

The non-realizable generalized signotopes on $n = 6$ belong to three flip classes, which have 3, 4, and 5 crossings, respectively. Note that there is a unique flip class with 3 crossings, a unique flip class with 4 crossings, and two flip classes with 5 crossings.

We now consider the flip class F of generalized signotopes on $n = 6$ elements with 3 crossings. There is, up to strong isomorphism, a unique[2] topological drawing D of K_6 which has the minimum of 3 crossings; see Figure 8. Therefore, every drawing realizing a generalized signotope from F is isomorphic to D. Since the drawing D is highly symmetric, there are up to isomorphism only 3 choices for the outer face, and hence only 3 of the 10 generalized signotopes from the flip class F are realized; cf. Listing 1. The remaining 7 generalized signotopes of that flip class are not realizable; cf. Listing 2. Note that in Listings 1 and 2 we encode a generalized signotope χ on the elements $\{1, \ldots, 6\}$ only by its $+$-triples, that is, the pre-image $\chi^{-1}(+)$.

Listing 1: Three realizable generalized signotopes on the elements $\{1, 2, 3, 4, 5, 6\}$ from the flip class F encoded by its $+$-triples.

```
{235,236,245,246,345,346,356,456}
```

[2] Uniqueness can easily be shown using a computer but also with classical proving techniques; cf. Exercise 1.4 in Marcus Schaefer's book [Sch18] and Lemma 10 in [EHLP11].

```
{235,236,245,246,256,345,346,356}
{234,235,245,246,256,346,356,456}
```

Listing 2: Seven non-realizable generalized signotopes on the elements $\{1,2,3,4,5,6\}$ from the flip class F encoded by its $+$-triples.

```
{234,235,236,245,256,346,356,456}
{234,235,236,246,256,345,356,456}
{234,235,246,256,345,346,356,456}
{136,234,245,256,345,456}
{234,236,245,246,256,345,356,456}
{234,236,245,256,345,346,356,456}
{235,236,245,246,256,345,346,456}
```

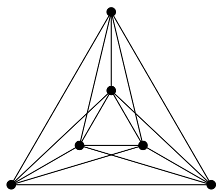

Figure 8: The unique topological drawing of K_6 which has the minimum of 3 crossings.

To lift the non-representable examples to higher number of elements we use the all-plus-extension of a generalized signotope.

Lemma 4 (All-plus-extension)**.** *Let χ be a generalized signotope on n elements and let $n' \geq n$ be an integer. Then the mapping $\chi' : [n']_3 \to \{+,-\}$*

$$\chi'(x,y,z) = \begin{cases} \chi(x,y,z) & \text{if } x,y,z \in [n] \\ + & \text{otherwise.} \end{cases}$$

is a generalized signotope on n' elements.

Proof. Consider four elements $x,y,z,w \in [n']$. If $x,y,z,w \in [n]$, then the sequence $\chi'(xyz), \chi'(xyw), \chi'(xzw), \chi'(yzw)$ avoids the forbidden patterns $+-+-$ and $-+-+$ because χ is a generalized signotope. Otherwise, the sequence $\chi'(xyz), \chi'(xyw), \chi'(xzw), \chi'(yzw)$ contains at least three $+$-entries. □

Corollary 5. *For $n \leq 5$ all generalized signotopes on n elements are realizable as topological drawing of K_n. For $n \geq 6$ there exist non-realizable generalized signotopes on n elements.*

Proof. The first part follows from earlier discussions in this subsection.

For the second part, consider the non-realizable generalized signotope χ on 6 elements from above. Now, for every integer n' with $n' \geq 6$, the all-plus-extension of χ (Lemma 4) is also non-realizable since it contains χ as an induced subconfiguration. □

Another interesting example is the generalized signotope on $n = 7$ elements shown in Listing 3. This configuration is not representable by any topological drawing of K_7 because it has crossing number 7 while every drawing of K_7 has at least 9 crossings [Guy72].

Listing 3: A generalized signotope on the elements $\{1, 2, 3, 4, 5, 6, 7\}$ with only 7 crossings encoded by its +-triples.

```
{235,236,237,245,246,247,257,267,345,346,347,356,367,456,457,567}
```

It would be interesting to have good bounds for the minimum number of crossings of generalized signotopes on n elements.

6.3 The Asymptotic Number

In this subsection we show that the number $g(n)$ of generalized signotopes on n elements is of order $2^{\Theta(n^3)}$. This bound also applies to the numbers of relabeling classes and flip classes, respectively, because reflections and relabelings only give a factor of at most $2 \cdot n!$ and the number of elements in a flip class is at most $2^{\binom{n}{2}}$. Last but not least, we show that most generalized signotopes are not induced by a topological drawing.

Upper Bound for $g(n)$: To eventually show that $g(n) \leq g(t)^{\binom{n}{t}/\binom{n-3}{t-3}}$ is an upper bound on the number of generalized signotopes on n elements, we make use of Shearer's Entropy Lemma [CGFS86].

Lemma 6 (Shearer's Entropy Lemma, [CGFS86])**.** *Let S be a finite set and $A_1, \ldots, A_m$ be subsets of S such that every element of S is contained in at least k of the sets $A_1, \ldots, A_m$. If $\mathcal{F}$ is a collection of subsets of S and $\mathcal{F}_i = \{F \cap A_i \colon F \in \mathcal{F}\}$ for $1 \leq i \leq m$. Then*

$$|\mathcal{F}|^k \leq \prod_{i=1}^{m} |\mathcal{F}_i|.$$

Let $t \leq n$. We consider the set $S = \binom{[n]}{3}$ of all triples from $[n]$ and, for each t-subset I of $[n]$, let $A_I = \binom{I}{3}$ be the set of triples of I. There are $m = \binom{n}{t}$ choices for I and as many sets A_I. Each triple in S belongs to $k = \binom{n-3}{t-3}$ sets A_I.

A generalized signotope on n elements is uniquely encoded by its +-triples, which form a subset of S. Let $\mathcal{F}$ be the family of all generalized signotopes on n elements given by their +-triples. For every I, let $\mathcal{F}_I = \{F \cap A_I \colon F \in \mathcal{F}\}$. Note that $\mathcal{F}_I$ is a family of generalized signotopes on I, whence $|\mathcal{F}_I| \leq g(t)$.

Lemma 6 implies

$$g(n)^k = |\mathcal{F}|^k \leq \prod_{I \in \binom{[n]}{t}} |\mathcal{F}_I| \leq g(t)^m,$$

with $m = \binom{n}{t}$ and $k = \binom{n-3}{t-3}$. Therefore,

$$g(n) \leq g(t)^{m/k} = 2^{c(t)\binom{n}{3}} \quad \text{with} \quad c(t) = \log_2(g(t))/\binom{t}{3}.$$

Using $g(7) = 630\,988\,832$ (cf. Table 3), we obtain that the number $g(n)$ of generalized signotopes on n elements is at most $2^{c_2 \cdot \binom{n}{3}} + o(1)$ where $c_2 = c(7) \approx 0.8352$.

Note that the above shows that $c(n) \leq c(t)$, that is, c is non-increasing. Thus, the factor c_2 can be expected to decrease if a value of $c(t')$ with $t' > 7$ becomes available.

Lower Bound for $g(n)$: First, we give a recursive construction of a set $\mathcal{X}_{3n}$ of generalized signotopes on $3n$ elements. The set $\mathcal{X}_3$ consists of the two generalized signotopes on $\{1, 2, 3\}$.

For the step, we construct $\mathcal{X}_{3n}$ based on $\mathcal{X}_n$: Let $A = \{1, \ldots, n\}$, $B = \{n+1, \ldots, 2n\}$, and $C = \{2n + 1, \ldots, 3n\}$. Pick three generalized signotopes χ_A, χ_B, χ_C from $\mathcal{X}_n$ and an arbitrary mapping $M : A \times B \times C \to \{+, -\}$. We define χ by the following rule: for $x < y < z$ we set

$$\chi(x, y, z) = \begin{cases} \chi_A(x, y, z) & \text{if } x, y, z \in A \\ \chi_B(x, y, z) & \text{if } x, y, z \in B \\ \chi_C(x, y, z) & \text{if } x, y, z \in C \\ M(x, y, z) & \text{if } x \in A, y \in B, z \in C \\ + & \text{otherwise.} \end{cases}$$

An easy case distinction shows that χ is a generalized signotope on n elements: For any four elements $x < y < z < w$, at least two are from the same class $S \in \{A, B, C\}$. We look at the signs of the sequence xyz, xyw, xzw, yzw.

If all four elements are from S, then we use that χ_S is a generalized signotope. If exactly three of the elements are from S, then there are at least three $+$ signs in the sequence, whence, the forbidden patterns $+-+-$ and $-+-+$ do not occur. Now if exactly two of the elements are from S, then if the two elements are x, y the triples xyz and xyw map to plus and we have $++**$, where $* \in \{+, -\}$ is arbitrary, if y, z are from S, we have $+**+$, and if z, w are from S, we have $**++$. In any case, the forbidden patterns $+-+-$ and $-+-+$ cannot occur, and hence χ is a generalized signotope.

Since there are $|\mathcal{X}_n|^3 \cdot 2^{n^3}$ possibilities to choose $\chi_A, \chi_B, \chi_C, M$, and no two such selections yield the same χ, we have

$$|\mathcal{X}_{3n}| = |\mathcal{X}_n|^3 \cdot 2^{n^3}.$$

Now, using all-plus-extensions (cf. Lemma 4), we obtain sets $\mathcal{X}_{3n+1}$ and $\mathcal{X}_{3n+2}$ of generalized signotopes on $3n+1$ and $3n+2$ elements, respectively, with $|\mathcal{X}_{3n}| = |\mathcal{X}_{3n+1}| = |\mathcal{X}_{3n+2}|$. Hence, for $f(n) = \log_2 |\mathcal{X}_n|$ we have

$$f(n) = 3f(\lfloor n/3 \rfloor) + \lfloor n/3 \rfloor^3.$$

Inductively assuming $f(n) \geq \frac{1}{24}n^3 - \frac{3}{8}n^2$, which is easy to check for $n = 1$ and $n = 2$, we obtain

$$\begin{aligned} f(n) &= 3f(\lfloor n/3 \rfloor) + \lfloor n/3 \rfloor^3 \\ &\geq 3\left(\frac{1}{24} \cdot \lfloor n/3 \rfloor^3 - \frac{3}{8} \cdot \lfloor n/3 \rfloor^2\right) + \lfloor n/3 \rfloor^3 \\ &\geq 3\left(\frac{1}{24} \cdot \left(\frac{n-2}{3}\right)^3 - \frac{3}{8} \cdot \left(\frac{n-2}{3}\right)^2\right) + \left(\frac{n-2}{3}\right)^3 \\ &= \frac{1}{24}n^3 - \frac{3}{8}n^2 + n - \frac{5}{6} \;\geq\; \frac{1}{24}n^3 - \frac{3}{8}n^2 \end{aligned}$$

for every $n \geq 3$.

We summarize the results in the following theorem.

Theorem 7. *The number $g(n)$ of generalized signotopes on n elements is between $2^{c_1 \cdot \binom{n}{3} + o(n^3)}$ and $2^{c_2 \cdot \binom{n}{3}} + o(1)$ for constants $c_1 = 0.25$ and $c_2 \approx 0.8352$.*

Last but not least, we investigate how many generalized signotopes come from topological drawings. There are at most $2^{O^*(n^2)}$ weak isomorphism classes of drawings of the complete graph K_n [Kyn13] (cf. [PT06]) and, by Lemma 3, each weak isomorphism classes is contained in a flip-equivalence class of generalized signotopes. Since the number of generalized signotopes in a flip-equivalence class is at most $2^{\binom{n}{2}}$, we conclude that at most $2^{O^*(n^2)} \cdot 2^{\binom{n}{2}} = 2^{O^*(n^2)}$ generalized signotopes come from topological drawings of K_n.

7 Discussion

We conclude this article with three further classical theorems from Convex Geometry.

Lovász (cf. Bárány [Bár82]) generalized Helly's Theorem as follows: Let $\mathcal{C}_0, \ldots, \mathcal{C}_d$ be families of compact convex sets from $\mathbb{R}^d$ such that for every "colorful" choice of sets $C_0 \in \mathcal{C}_0, \ldots, C_d \in \mathcal{C}_d$ the intersection $C_0 \cap \ldots \cap C_d$ is non-empty. Then, for some k, the intersection $\bigcap \mathcal{C}_k$ is non-empty. This result is known as the *Colorful Helly Theorem*. Kalai and Meshulam [KM05] presented a topological version of the Colorful Helly Theorem, which, in particular, carries over to pseudolinear drawings. Since Helly's Theorem does not generalize to face-convex drawings (cf. Proposition 3), neither does the Colorful Helly Theorem.

The (p, q)*-Theorem* (conjectured by Hadwiger and Debrunner, proved by Alon and Kleitman [AK92], cf. [KST18]) says that for any $p \geq q \geq d + 1$ there is a finite number $c(p, q, d)$ with the following property: If $\mathcal{C}$ is a family of convex sets in $\mathbb{R}^d$, with the property that among any p of them, there are q that have a common point, then there are $c(p, q, d)$ points that cover all the sets in $\mathcal{C}$. The case $p = q = d + 1$ is Helly's Theorem, i.e., $c(d + 1, d + 1, d) = 1$. We are not aware whether a (p, q)-Theorem for triangles in topological drawings exists.

Last but not least, we would like to mention *Tverberg's Theorem*, which asserts that every set V of at least $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into $V = V_1 \dot\cup \ldots \dot\cup V_r$ such that $\mathrm{conv}(V_1) \cap \ldots \cap \mathrm{conv}(V_r)$ is non-empty. A generalization of Tverberg's Theorem applies to pseudolinear drawings [Rou88] and to drawings of K_{3r-2} if r is a prime-power [Öza87] (cf. [BSS81]). Also a generalization of Birch's Theorem, a weaker version of Tverberg's Theorem, was recently proven for topological drawings of complete graphs [FS20]. The general case, however, remains unknown. For a recent survey on generalizations of Tverberg's Theorem, we refer to [BS18].

Besides the mentioned characterization of pseudolinear drawings [BFK15, Theorem 3.2], Balko, Fulek, and Kynčl also provide a characterization of which generalized signotopes can be drawn as x-monotone topological drawings and which can be drawn as x-monotone semisimple drawings by forbidding finitely many subconfigurations [BFK15, Theorem 3.1]. In the spirit of their results, Kynčl's Theorem [Kyn20], and the hereditary-convex and convex classification by Arroyo et al. [AMRS17a], we pose the following question on characterizing drawable generalized signotopes:

Question 1. *Is there a finite number k such that, given any generalized signotope, if all k-tuples are drawable, then the generalized signotope is drawable?*

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