Straight Line Triangle Representations

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Abstract. Which plane graphs admit a straight line representations such that all faces have the shape of a triangle? We present a characterization based on flat angle assignements, i.e., selections of angles of the graph that have size π in the representation. Another characterization is in terms of contact systems of pseudosegments. We use discrete harmonic functions to show that contact systems of pseudosegments that respect certain conditions are stretchable.

The drawback of the characterization is that we are not able to effectively check whether a given graph admits a flat angle assignment that fulfills the conditions. Hence it is still open to decide whether the recognition of graphs that admit straight line triangle representation is polynomially tractable.

1 Introduction

Our goal is to characterise planar graphs that admit a plane straight line drawing such that each face is a triangle. To achieve this each face f has exactly $\deg(f) - 3$ incident vertices that have an angle of size π in f. Each vertex has at most one angle of size π, i.e., a flat angle.

Planar graphs are a widely studied graph class, with natural subclasses such as outerplanar graphs and trees. Many different geometric representations of planar graphs have been studied. Tutte showed that 3-connected planar graphs have convex drawings (rubber band representation) [Tut63]. Koebe has shown that they can be represented as circle contact graphs [Koe36]. They also admit triangle contact representations [dFdMR94]. When the regions are required to be rectangles not all planar graphs can be represented. Graphs admitting such a representation have been characterized by Koźmiński and Kinnen [KK85] or earlier, in the dual setting, by Ungar [Ung53]. Buchsbaum et al. [BGPV08] have many pointers to the literature. It is known that all planar graphs have contact representations with convex hexagons (e.g. [DGH$^+$12]).

In this paper we study a representation of planar graphs in the classical setting, i.e., vertices are presented as points in the Euclidean plane and adjacencies as continuous curves connecting the points. We aim at classifying the class of planar graphs that admit a straight line representation in which all faces are triangles. There have been investigations of the problem in the dual setting, i.e., in the setting of side contact representations of planar graphs with triangles. Gansner, Hu and Kobourov show that outerplanar graphs, grid graphs and hexagonal grid graphs can be represented by Touching Triangle Graphs (TTG's) and they give a linear time algorithm to find the TTG [GHK11]. Alam, Fowler and Kobourov consider proper TTG's, i.e., the union of all triangles of the TTG is a triangle and there are no holes. They give a necessary condition for biconnected outerplanar graphs and a sufficient condition, however the necessary condition is not sufficient and the sufficient condition is not necessary [AFK]. Kobourov, Mondal and Nishat present construction algorithms for proper TTG's of 3-connected cubic graphs and some grid graphs and a recognition algorithm for proper TTG's [KMN12].

Here is the formal introduction of the main character for this paper.

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Definition 1.1 (SLT Representation)**.** A plane drawing of a graph such that

- all the edges are straight line segments and
- all the faces, including the outer face, bound a non-degenerate triangle

is called a *straight line triangle representation* (SLTR).

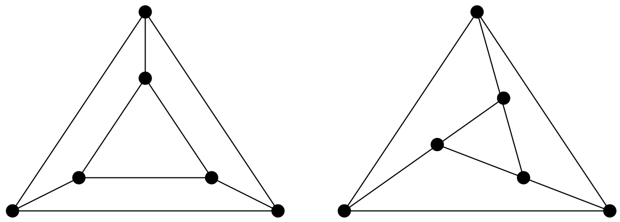

Figure 1: A graph and one of its SLT Representations.

Clearly every straight line drawing of a triangulation is an SLTR. So the class of planar graphs admitting an SLTR is rich. On the other hand, graphs admitting an SLTR can not have a cut vertex. Indeed, as shown below (Prop. 1.2), graphs admitting an SLTR are well connected. Being well connected, however, is not sufficient as shown e.g. by the cube graph.

To simplify the discussion we assume that the input graph is given with a plane embedding and a selection of three vertices of the outer face that are designated as corner vertices for the outer face. These three vertices are called *suspension vertices.*

If a degree two vertex has an angle of size π in one of its incident faces, then it also has an angle of size π in the face on the other side. Hence, this vertex and its two incident edges can be replaced by a single edge connecting the two neighbors of the vertex. Such an operation is called a *vertex reduction.* The only angles of an SLTR whose size exceeds π are the outer angles at the outer triangle. Therefore, we can use vertex reductions to eliminate all the degree two vertices except for degree two vertices that are suspensions.

A plane graph G with suspensions s_1, s_2, s_3 is said to be *internally 3-connected* when the addition of a new vertex v_∞ in the outer face, that is made adjacent to the three suspension vertices, yields a 3-connected graph.

Proposition 1.2. If a graph G admits an SLTR with s_1, s_2, s_3 as corners of the outer triangle and no vertex reduction is possible, then G is internally 3-connected.

Proof. Consider the SLTR of G. Suppose there is a separating set U of size 2. It is enough to show that each component of $G \setminus U$ contains a suspension vertex, so that $G + v_\infty$ is not disconnected by U. Since G admits no vertex reduction every degree two vertex is a suspension. Hence, if C is a component and $C \cup U$ induces a path, then there is a suspension in C. Otherwise consider the convex hull of $C \cup U$ in the SLTR. The convex corners of this hull are vertices that expose an angle of size at least π. Two of these large angles may be at vertices of U but there is at least one additional large angle. This large angle must be the outer angle at a vertex that is an outer corner of the SLTR, i.e., a suspension. □

By Proposition 1.2 we may assume that the graphs we consider are internally 3-connected since any graph that is not internally 3-connected but does admit an SLTR, is a subdivision of an internally 3-connected graph.

In Section 2 we present necessary conditions for the existence of an SLTR in terms of what we call a flat angle assignment. A flat angle assignment that fulfills the conditions is shown to induce a partition of the set of edges into a set of pseudosegments. Finally, with the aid of discrete harmonic functions we show that in our case the set of pseudosegments is stretchable. Hence, the necessary conditions are also sufficient. The drawback of the characterization is that we are not aware of an effective way of checking whether a given graph admits a flat angle assignment that fulfills the conditions.

In Section 3 we digress to prove a more general result about stretchable systems of pseudosegments with the technique based on discrete harmonic functions. The result is not new, de Fraysseix and Ossona de Mendez have investigated stretchability conditions for systems of pseudosegments in [dFdM03, dFdM04, dFdM07]. The counterpart to our theorem is Theorem 38 in [dFdM07], the proof there is based on a complicated inductive construction.

2 Necessary and Sufficient Conditions

Consider a plane, suspended, internally 3-connected graph $G = (V, E)$. Suppose that G admits an SLTR. This representation induces a set of *flat angles*, i.e., incident pairs (v, f) such that vertex v has an angle of size π in the face f.

Since G is internally 3-connected every vertex has at most one flat angle. Therefore, the flat angles can be viewed as a partial mapping of vertices to faces. Since the outer angle of suspension vertices exceeds π suspensions have no flat angle. Since each face f (including the outer face) is a triangle, each face has precisely three angles that are not flat. In other words every face f has $|f| - 3$ incident vertices that are assigned to f. This motivates the definition:

Definition 2.1 (FA Assignment)**.** A *flat angle assignment* (FAA) is a mapping from a subset U of the non-suspension vertices to faces such that

[C_v] Every vertex of U is assigned to at most one face,

[C_f] For every face f, precisely $|f| - 3$ vertices are assigned to f.

Not every FAA induces an SLTR. An example is given in Figure 2.

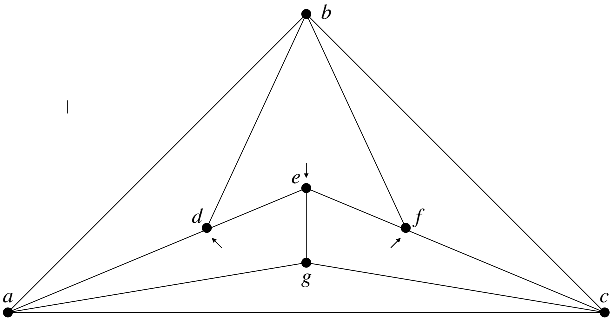

Figure 2: A graph with an FAA given by the arrows, an arrow assigns a vertex to a face. When all assigned angles are stretched, all vertices but b will be between a and c.

Hence, we have to identify another condition. To state this we need a definition. Let H be a connected subgraph of the plane graph G. The *outline cycle* $\gamma(H)$ *of* H is the closed walk corresponding to the outer face of H. An *outline cycle* of G is a closed walk that can be obtained as outer cycle of some connected subgraph of G. Outline cycles may have repeated edges and vertices, see Fig. 3. The interior $\mathsf{int}(\gamma)$ of an outline cycle $\gamma = \gamma(H)$ consists of H together with all vertices, edges and faces of G that are contained in the area enclosed by γ.

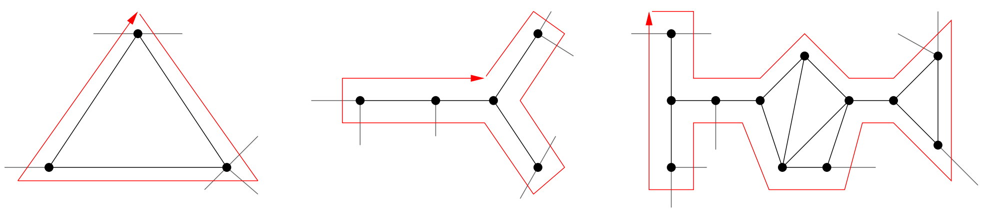

Figure 3: Three examples of outline cycles.

Proposition 2.2. An SLTR obeys the following condition C_o

[C_o] Every outline cycle that is not the outline cycle of a path, has at least three geometrically convex corners.

Proof. Consider an SLTR. Suppose that there is a connected subgraph, not a path, such that its outline cycle has less than three geometric convex corners. If the outline cycle has at most two geometric convex corners, then the subgraph is mapped to a line in the plane. The subgraph must either contain a vertex of degree more than three, or a face. If a vertex v together with three its neighbors is mapped onto a line, then the boundary of at least one of the faces incident to v is not a triangle. On the other hand if the subgraph contains a face, then this face is mapped to a line and therefore its boundary is not a triangle. In both cases the properties of an SLTR are violated. This shows that C_o is a necessary condition. □

Condition C_o has the disadvantage that it depends on a given SLTR, hence, it is useless for deciding whether a planar graph G admits an SLTR. The following definition allows to replace C_o by a combinatorial condition on a FAA.

Definition 2.3 (Combinatorial Convex Corners)**.** Given an FAA ψ. A vertex v of an outline cycle γ is a *combinatorial convex corner* for γ with respect to ψ if

- v is a suspension vertex, or
- v is not assigned to a face and there is an edge e incident to v with $e \notin \mathsf{int}(\gamma)$, or
- v is assigned to a face f, $f \notin \mathsf{int}(\gamma)$ and there exists an edge e incident to v with $e \notin \mathsf{int}(\gamma)$.

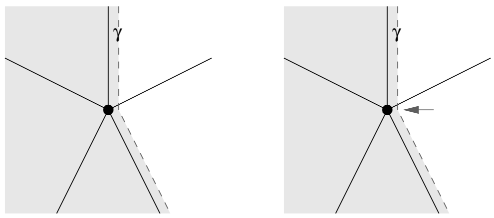

Figure 4: Combinatorially Convex Corners, the arrow in the image on the right represents the assignment of the vertex.

Proposition 2.4. Let G admit an SLTR Γ, that induces the FAA ψ and let H be a connected subgraph of G. If v is a geometrically convex corner of the outline cycle $\gamma(H)$ in Γ, then v is a combinatorially convex corner of $\gamma(H)$ with respect to ψ.

Proof. If v is a suspension vertex it is clearly geometrically and combinatorially convex.

Let v be geometrically convex and suppose that v is not a suspension and not assigned by ψ. In this case v is interior and, with respect to γ, the outer angle at v exceeds π. Therefore at least two incident faces of v are in the outside of γ. These faces can be chosen to be adjacent, hence, the edge between them is an edge e with $e \notin \mathsf{int}(\gamma)$. This shows that v is combinatorially convex.

Let v be geometrically convex and suppose that v is assigned to f by ψ. If $f \in \mathsf{int}(\gamma)$, then the inner angle of v with respect to γ is at least π. This contradicts the fact that v is geometrically convex. Hence $f \notin \mathsf{int}(\gamma)$. If there is no edge e incident to v such that $e \notin \mathsf{int}(\gamma)$, then v has an angle of size π with respect to γ. This again contradicts the fact that v is geometrically convex. Therefore, if v is geometrically convex and assigned to f, then $f \notin \mathsf{int}(\gamma)$ and there exists an edge e incident to v such that $e \notin \mathsf{int}(\gamma)$. This shows that v is a combinatorial convex corner for γ. □

The proposition enables us to replace the condition on geometrically convex corners w.r.t. an SLTR by a condition on combinatorially convex corners w.r.t. an FAA.

[C_o^*] Every outline cycle that is not the outline cycle of a path, has at least three combinatorially convex corners.

From Proposition 2.2 and Proposition 2.4 it follows that this condition is necessary for an FAA that belongs to an SLTR. Later in Theorem 2.10 we proof that if an FAA obeys C_o^* then it belongs to an SLTR. The proof is constructive. In anticipation of this result we say that an FAA obeying C_o^* is a *good flat angle assignment* and abreviate it as a *GFAA*.

Next we show that a GFAA induces a contact family of pseudosegments. This family of pseudosegments is later shown to be stretchable.

Definition 2.5 (Contact family of pseudosegments)**.** A contact family of pseudosegments is a family $\{c_i\}_i$ of simple curves

$$c_i : [0,1] \to \mathbb{R}^2, \text{ with different endpoints, i.e., } c(0) \neq c(1),$$

such that any two curves c_i and c_j $(i \neq j)$ have at most one point in common. If so, then this point is an endpoint of (at least) one of them.

A GFAA ψ on a graph G gives rise to a relation ρ on the edges: Two edges, both incident to v and f are in relation ρ if and only if v is assigned to f. The transitive closure of ρ is an equivalence relation.

Proposition 2.6. The equivalence classes of edges of G defined by ρ form a contact family of pseudosegments.

Proof. Let the equivalence classes of ρ be called arcs.

Condition C_v ensures that every vertex is interior to at most one arc. Hence, the arcs are simple curves and no two arcs cross.

Every arc has two distinct endpoints, otherwise it would be a cycle and its outline cycle has only one combinatorially convex corner. If an arc would touch itself, the outline cycle of this equivalence class has at most one combinatorially convex corner. This again contradicts C_o^*.

If two arcs share two points, the outline cycle has at most two combinatorially convex corners. This again contradicts C_o^*.

We conclude that the family of arcs satisfies the properties of a family of pseudosegments. □

Definition 2.7 (Free Point)**.** Let Σ be a family of pseudosegments and S a subset of Σ. A point p of a pseudosegment from S is a free point for S if

1. p is an endpoint of a pseudosegment in S, and
2. p is not interior to a pseudosegment in S, and
3. p is incident to the unbounded region of S, and
4. p is a suspension or p is incident to a pseudosegment that is not in S.

With Lemma 2.8 we proof that the family of pseudosegments Σ that arises from a GFAA has the following property

[C_P] Every subset S of Σ with $|S| \geq 2$ has at least three free points.

Lemma 2.8. Let ψ a GFAA on a plane, internally 3-connected graph G. For every subset S of the family of pseudosegments associated with ψ, it holds that, if $|S| \geq 2$ then S has at least 3 free points.

Proof. Let S be a subset of the contact family of pseudosegments defined by the GFAA (Proposition 2.6).

Each pseudosegment of S corresponds to a path in G. Let H be the subgraph of G obtained as union of the paths of pseudosegments in S. We assume that H is connected and leave the discusion of cases where it is not to the reader. If H itself is not a path, then by C_o^* the outline cycle $\gamma(H)$ must have at least three combinatorially convex corners. Every combinatorially convex corner of $\gamma(H)$ is a free point of S.

If S induces a path, then the two endpoints of this path are free points for S. Moreover, there exists at least one vertex v in this path which is an endpoint for two pseudosegments and not an interior point for any. Now there must be an edge e incident to v, such that $e \notin S$, therefore v is a free point for S. □

Given an internally 3-connected, plane graph G with a GFAA. To find a corresponding SLTR we aim at representing each of the pseudosegments induced by the FAA as a straight line segment. If this can be done, every assigned vertex will be between its two neighbors that are part of the same pseudosegment. This property can be modeled by requiring that the coordinates $p_v = (x_v, y_v)$ of the vertices of G satisfy a harmonic equation at each assigned vertex.

Indeed if uv and vw are edges belonging to a pseudosegment s, then the coordinates satisfy

$$x_v = \lambda_v x_u + (1 - \lambda_v)x_w \qquad \text{and} \qquad y_v = \lambda_v y_u + (1 - \lambda_v)y_w$$

the parameter λ_v is some number stricly between 0 and 1. The theory of harmonic functions applied to (plane) graphs is well explained in [Lov09].

In the SLTR every not assigned vertex is placed in a weighted barycenter of its neighbors. In terms of coordinates this can be written as

$$x_v = \sum_{u \in N(v)} \lambda_{vu} x_u \qquad y_v = \sum_{u \in N(v)} \lambda_{vu} y_u \qquad \text{with} \qquad \sum_{u \in N(v)} \lambda_{vu} = 1 \quad \text{and} \quad \lambda_{vu} > 0.$$

These are again harmonic equations. The vertices whose coordinates are not restricted by a harmonic equations are called *poles*. In this case the suspension vertices are the three of the harmonic functions for the x and y-coordinates. The coordinates for the suspension vertices are the corners of some non-degenerate triangle.

Proposition 2.9. ([Lov09, Chapter 3]) For every choice of the parameters λ_v and λ_{vu} complying with the conditions, the system has a unique solution.

Proof. The proof has three steps, first we show that every non-constant harmonic function has at least two poles. Then we show that for every map $\psi_0 : P \to \mathbb{R}^2$ from the set of poles to the plane, there is a unique extension $\psi : V \to \mathbb{R}^2$ that is harmonic in all the vertices that are not poles. Last we show that a solution of the system is an extension of a mapping of the poles.

Let f be a non-constant harmonic function on the vertices of a connected graph G and let P be the set of poles of f. Let $Q = \{v \in V : f(v) \text{ maximum}\}$ and $Q' = \{v \in Q : v \text{ has a neighbor not in } Q\}$. The set Q is not empty as f is not constant. Since G is connected also Q' is not empty and every element in Q' must be a pole. Similarly we can consider the vertices where the minimum is attained, hence there are at least two poles.

Consider $\psi_0 : P \to \mathbb{R}^2$, a map from the set of poles to the plane and suppose there are two extensions $\psi, \psi^* : V \to \mathbb{R}^2$ that satisfy the harmonic equations of all non-poles. Then the function $\omega = \psi - \psi^*$ is also harmonic in all vertices not in P. If ω is the zero function, then $\psi = \psi^*$. Otherwise, ω is a non-constant harmonic function hence has a minimum or maximum different from 0 and there is a pole. This pole can not be an element of P as ψ and ψ^* are extensions of ψ_0 hence for all poles ω is zero. But then there exists a vertex, not in P, in which ω is not harmonic, contradiction. Hence the extension is unique.

For every choice of the parameters λ_v and λ_{vu} complying with the conditions, the solution of the system extends the mapping of the poles (suspension vertices). As the functions are harmonic in each non-suspension vertex, the solution is unique for every choice of the parameters. □

Theorem 2.10. Given an internally 3-connected, plane graph G and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three free points or cardinality at most one. The unique solution of the system of equations that arrises from Σ, is an SLTR.

Note: This theorem proves that the necessary conditions are also sufficient.

Proof. The proof consists of 7 arguments, which together yield that the drawing induced from the GFAA is a non-degenerate, plane drawing. The proof has been inspired by the proof of Colin de Verdière [dV91] for convex straight line drawings of plane graphs via spring embeddings.

1. *Pseudosegments get Segments.* Let $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)$ be the set of edges of a pseudosegment defined by ψ. The harmonic conditions for the coordinates force that v_i is placed between v_{i-1} and v_{i+1}

for $i = 2, .., k-1$. Hence all the vertices of the pseudosegment are placed on the segment with endpoints v_1 and v_k.

2. *Convex Outer Face.* The outer face is bounded by three pseudosegments and the suspensions are the endpoints for these three pseudosegments. The coordinates for the suspensions (the poles of the harmonic functions) have been chosen as corners of a non-degenerate triangle and the pseudosegments are straight line segments, therefore the outer face is a triangle and in particular convex.

3. *No Concave Angles.* Every vertex, not a pole, is forced either to be on the line segment between two of its neighbors (if assigned) or in a weighted barycenter of all its neighbors (otherwise). Therefore every non-pole vertex is in the convex hull of its neighbors. This implies that these are no concave angles at non-poles.

4. *No Degenerate Vertex.* A vertex is degenerate if it is placed on a line, together with at least three of its neighbors. Suppose there exists a vertex v, such that v and at least three of its neighbors are placed on a line l. Let S be the connected component of pseudosegments that are aligned with l, such that S contains v. The set S contains at least two pseudosegments. Therefore S must have at least three free points, v_1, v_2, v_3.

By property 4 in the definition of free points, each of the free points is incident to a segment that is not aligned with l. Suppose the free points are not suspension vertices. If v_i is interior to s_i, then s_i has an endpoint on each side of l. If v_i is not assigned by the GFAA it is in the strict convex hull of its neighbors, hence, v_i is an endpoint of a segment reaching into each of the two half-planes defined by l.

Now suppose v_1 and v_2 are suspension vertices[1] and consider the third free point, v_3. It is either interior to a pseudosegment not on l, but then one endpoint of this pseudosegment lies outside the convex hull of the three suspensions, which is a contradiction. Hence it is not interior to any pseudosegment and at least one of its neighbors does not lie on l, but then v_3 should be in a weighted barycenter of its neighbors, hence again we would find a vertex outside the convex hull of the suspension vertices. Therefore at most one of the free points is a suspension and l is incident to at most one of the suspension vertices.

In either case each of v_1, v_2, v_3 has a neighbor on either side of l.

Let n^+ and $n^- = -n^+$ be two normals for line l and let p^+ and p^- be the two poles, that maximize the inner product with n^+ resp. n^-. Starting from the neighbors of the v_i in the positive halfplane of l we can always move to a neighbor with larger[2] inner product with n^+ until we reach p^+. Hence v_1, v_2, v_3 have paths to p^+ in the upper halfplane of l and paths to p^- in the lower halfplane. Since v_1, v_2, v_3 also have a path to v we can contract all vertices of the upper and lower halfplane of l to p^+ resp. p^- and all inner vertices of these paths to v to produce a $K_{3,3}$ minor of G. This is in contradiction to the planarity of G. Therefore, there is no degenerate vertex.

5. *Preservation of Rotation System.* Let $\theta(v) = \sum_f \theta(v, f)$ denote the sum of the angles around an interior vertex. Here f is a face incident to v and $\theta(v, f)$ is the (smaller!) angle between the two edges incident to v and f in the drawing obtained by solving the harmonic system. If the incident faces are oriented consistently around v, then the angles sum up to 2π, otherwise $\theta(v) > 2\pi$ (see Figure 5). We do not consider the outer face in the sums so that the b vertices incident to the outer face contribute $(b-2)\pi$ in total.

Now consider the sum $\theta(f) = \sum_v \theta(v, f)$ of the angles of a face f. At each vertex incident to f the contribution $\theta(v, f)$ is at most of size π. A closed polygonal chain with k corners, selfintersecting or not, has a sum of inner angles equal to $(k-2)\pi$. Therefore $\theta(f) \leq (|f|-2)\pi$. The sum over all vertices $\sum_v \theta(v)$ and the sum over all faces $\sum_f \theta(f)$ must be equal since they count the same angles in two different ways.

$$(|V| - b)2\pi + (b-2)\pi \leq \sum_v \theta(v) = \sum_f \theta(f) \leq ((2|E| - b) - 2(|F| - 1))\,\pi$$

This yields $|V| - |E| + |F| \leq 2$. Since G is planar Euler's formula implies equality. Therefore $\theta(v) = 2\pi$ for every interior vertex v and the faces must be oriented consistently around every vertex, i.e. the

[1] Not all three suspension vertices lie on one line, hence at least one of the three free points is not a suspension.

[2] If n^+ is perpendicular to another segment this may not be possible. In this case we can use a slightly perturbed vector n^+_ε to break ties.

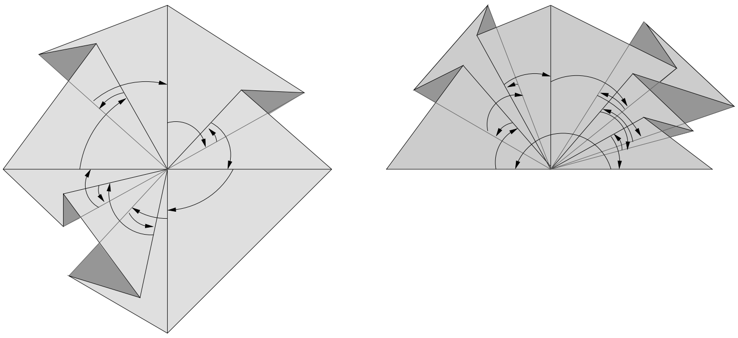

Figure 5: Examples of vertices with their surrounding faces not oriented consistently.

rotation system is preserved. Note that the rotation system could have been flipped, between clockwise and counterclockwise but then it is flipped at every vertex.

6. *No Crossings.* Suppose two edges cross. On either side of both of the edges there is a face, therefore there must be a point p in the plane which is covered by at least two faces. Outside of the drawing there is only the unbounded face. Move along a ray, that does not pass through a vertex of the graph, from p to infinity. A change of the cover number, i.e. the number of faces by which the point is covered, can only occur when crossing an edge. But then the rotation system at the vertices of that edge must be wrong. This would contradict the previous item. Therefore a crossing can not exist.

7. *No Degeneracy.* Suppose there is an edge of length zero. Since every vertex has a path to each of the three suspensions there has to be a vertex a that is incident to an edge of length zero and an edge ab of non-zero length. Following the direction of forces we can even find such a vertex-edge pair with b contributing to the harmonic equation for the coordinates of a. We now distinguish two cases.

If a is assigned, it is on the segment between b and b', together with the neighbor of the zero length edge this makes three neighbors of a on a line. Hence, a is a degenerate vertex. A contradiction.

If a is not assigned it is in the convex hull of its neighbors. However, starting from a and using only zero-length edges we eventually reach some vertex a' that is incident to an edge $a'b'$ of non-zero length, such that b' is contributing to the harmonic equation for the coordinates of a'. Vertex a' has the same position as a and is also in the convex hull of its neighbors. This makes a crossing of edges unavoidable. A contradiction. Hence, there are no edges of length zero.

Suppose there is an angle of size zero. Since every vertex is in the convex hull of its neighbors there are no angles of size larger than π. Moreover there are no crossings, hence the face with the angle of size zero is stretching along a line segment with two angles of size zero. Since there are no edges of length zero and all vertices are in the convex hull of their neighbors all but two vertices of the face must be assigned to this face. Therefore, there are two pseudosegments bounding this face, which have at least two points in common, this contradicts that Σ is a family of pseudosegments. We conclude that there is no degeneracy.

From the above arguments we conclude that the drawing is plane and thus an SLTR. $\square$

We obtained equivalence between the SLTR, the FAA satisfying C_v, C_f and C_o^*, and a stretchable system of pseudosegments that arrises from this FAA.

3 Stretchability of Systems of Pseudosegments

A contact system of pseudosegments is *stretchable* if it is homeomorphic to a contact system of straight line segments.

De Fraysseix and Ossona de Mendez characterized strechable systems of pseudosegments [dFdM03, dFdM04, dFdM07]. Their result is based on the notion of an extremal point.

Theorem 3.1 (De Fraysseix & Ossona de Mendez)**.** A contact family Σ of pseudosegments is stretchable if and only if each subset $S \subseteq \Sigma$ of pseudosegments with $|S| \geq 2$, has at least 3 extremal points.

Definition 3.2 (Extremal Point)**.** Let Σ be a family of pseudosegments and S a subset of Σ. A point p is an extremal point for S if

1. p is an endpoint of a pseudosegment in S, and
2. p is not interior to a pseudosegment in S, and
3. p is incident to the unbounded region of S.

So it seems that our notion of a free point (Def. 2.7) is more restrictive than the notion of extremal points from [dFdM07, Section 5.2]. In the following we show that there is no big difference. First in Prop. 3.3 we show that in the case of families of pseudosegments that live on a plane graph via an FAA, the two notions coincide. Then we continue by reproving Theorem 3.1 as a corroraly of our Theorem 2.10.

Proposition 3.3. Given an internally 3-connected, plane graph G and Σ a family of pseudosegments associated to an FAA, such that each subset $S \subseteq \Sigma$ has three extremal points or cardinality at most one. The unique solution of the system of equations that arrises from Σ, is an SLTR.

Proof. Note that in the proof of Theorem 2.10 the notion of free points is only used to show that there is no degenerate vertex.

Consider again the set S, we will show that there are at least three extremal points which are also free points. Let p an extremal point of S, not a free point, hence it is not an interior point for any pseudosegment of Σ and all the pseudosegments for which p is an endpoint are in S. By 3-connectivity p is incident to at least three pseudosegments, all of which lie on the line l. Since all regions are bounded by three pseudosegments, all the regions incident to p must lie on l. But then p is not incident to the unbounded region of S, hence p is not an extremal point. Therefore all extremal points of S are free points.

Proposition 3.3 now follows from Theorem 2.10. □

Alternative proof of Theorem 3.1

Proof. Let Σ a contact family of pseudosegments which is stretchable. As Σ is stretchable there exists a rubber band representation of Σ. Consider a set $S \subseteq \Sigma$ of cardinality at least two, then this set has at least three geometrically convex corners in the rubber band representation, which are incident to the unbounded region of S. These convex corners are not interior to a pseudosegment of S and hence they are extremal points of S. Therefore, each subset $S \subseteq \Sigma$ with $|S| \geq 2$, has at least 3 extremal points.

Conversely, assume that each subset $S \subseteq \Sigma$ of pseudosegments, with $|S| \geq 2$, has at least 3 extremal points. We will construct an extended system Σ^+ of pseudosegments in which every region is bounded by precisely three pseudosegments.

First we add so-called *protection points*, these points ensure that the pseudosegments of Σ will be mapped to straight lines. For each region R in Σ, for each pseudosegment s in R, we add a protection point for each visible side of s. The protection point is connected to the endpoints of s, with respect to R and the visible side of s (see Figure 6).

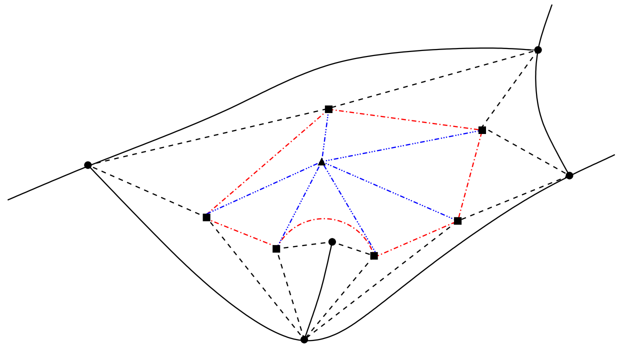

Figure 6: The extension of one region of Σ, the circles, squares and triangle represent the endpoints of pseudosegments of Σ, the protection points and the triangulation point respectively.

Now the inner part of R is bounded by an alternating sequence of endpoints of Σ and protection points. We connect two protection points if they share a neighbor in this sequence. Last we add a *triangulation point* and connect it to all protection points.

We have obtained Σ^+, a family of pseudosegments such that every region is bounded by precisely three pseudosegments and every subset $S \subseteq \Sigma^+$ has at least extremal points, unless it has cardinality one[3].

Let V be the set of points of Σ^+ and E the set of arcs connecting two points of V in Σ^+. It follows from the construction that $G = (V, E)$ is internally 3-connected.

By Proposition 3.3 the graph $G = (V, E)$ together with Σ^+ is stretchable to an SLTR. Removing the protection points, triangulation points and their incident edges yields a contact system of straight line segments homeomorphic to Σ. □

4 Conclusion and Open Problems

We have given necessary and sufficient conditions for a 3-connected planar graph to have an SLT Representation. Given an FAA and a set of rational parameters $\{\lambda_i\}_i$, the solution of the harmonic system can be computed in strongly polynomial time. Hence, a degenerate solution can be identified in polynomial time. It shows that the FAA is not good.

Checking whether an FAA satisfies C_o can be done in polynomial time, but a graph may admit different assignments of which only some are good. We are not aware of an effective way of finding a GFAA. We therefore leave the problem: Is the recognition of graphs that have an SLTR (GFAA) in P?

Given a 3-connected planar graph and a GFAA, interesting optimization problems arise, e.g. find the set of parameters $\{\lambda_i\}_i$ such that the smallest angle in the graph is maximized, or the set of parameters such that the length of the shortest edge is maximized.

[3]The new arcs have no interior points and if a new point is not in the unbounded region, the set contains at least two pseudosegments of Σ hence the set deleted the new arcs admits at least three extremal points. These points are also extremal for S.

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